

A Note on Sylvester's Problem for Random Polytopes in a Convex Body

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SUMMARY. - *For any d -dimensional convex body K of unit volume, $d \geq 2$, let $M_r(K; n)$, $r \geq 1$, $n \geq d + 1$, be the r -th order moment of the volume of the convex hull of n random points from K . The paper deals with the problem of determining maximizers of $M_r(K; n)$ in the class of all d -dimensional convex bodies of unit volume.*

A method for selecting possible solutions, which is based on special continuous movements of convex bodies, is presented. The results obtained by this method support the conjecture that, for every r and n , the only maximizers of $M_r(K; n)$ are simplices.

1. Introduction

The present paper deals with a famous problem posed in 1864 by J. J. Sylvester [17].

In the d -dimensional setting, Sylvester's problem can be stated as follows. Let K be a convex body in \mathbb{R}^d , i.e. a compact convex set with non-empty interior, and take $d + 2$ random points from K .

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If $P(K)$ denotes the probability that one of these points falls in the convex hull of the remainders, for which bodies $P(K)$ attains its extremal values?

In order to handle this problem, it is convenient to notice that for a body K of unit volume, $P(K)$ is proportional to the expected value of the volume of a random simplex contained in K . We denote by $M(K)$ such an expected value divided by the volume of K .

The first significant results on extrema of $M(K)$ were obtained by Blaschke in [4]. There, it is proved that for $d = 2$ the minimum of $M(K)$ is attained if and only if K is an ellipse, and that triangles are the only maximizers of $M(K)$.

The techniques employed by Blaschke involve two distinct types of transformations acting on convex bodies. Namely, for the characterization of ellipses the well-known Steiner symmetrization is used, while the ‘‘Schüttelung’’ process comes in for the maximum problem. We recall that the Steiner symmetral of K along the direction v is obtained moving each chord parallel to v on a straight line so that its midpoint lie on v^\perp . The Schüttelung (or shakedown) process along v moves each chord of K parallel to v so that one of its endpoints lies on v^\perp and all the chords stay on the same side of v^\perp .

Blaschke’s argument relies on the following two facts. First, the functional $M(K)$ does not increase (resp. decrease) under the Steiner (resp. Schüttelung) process. Second, every plane convex set K can be reduced to a disk (resp. a triangle) via countably many Steiner symmetrizations (resp. Schüttelung) along suitably selected directions.

The latter property is true for both processes also in any dimension (see [10] and [3]), whereas the same does not hold for the first one. In fact, in higher dimensions $M(K)$ is still not increasing under Steiner symmetrizations, as Groemer proved in [8], while it has not a monotonic behaviour with respect to the Schüttelung process when $d > 2$ (see the paper by Pfieler [13] for a discussion regarding this aspect). Consequently, the d -dimensional Sylvester’s problem for the minimum case is solved, and the only solutions are ellipsoids. The maximum problem is still open for $d > 2$, and it is conjectured that the simplices are the only maximizers.

In this paper new efforts are provided to such a conjecture. We

develop a method which enables one to restrict the class of possible maximizers of $M(K)$. Our approach is based on continuous movements of convex bodies, i.e. families K_t of convex bodies depending continuously on a parameter t . The movements considered here can be described as follows. Fix a convex body K_0 and a direction v and move each chord of K_0 parallel to v with speed αv , where α is independent of t . The speed function α is required to be such that K_t is convex as t ranges in some interval containing $t = 0$. For particular choices of the speed function, these movements reduce to the Steiner and Schüttelung processes.

We emphasize that the movements we are dealing with are special cases of the linear parameter systems studied by Rogers and Shephard in [14] (see also [16]).

The method developed here relies on the fact that $M(K_t)$ is a convex function of t and it is strictly convex unless K_t is an affine image of K_0 for every t (see Theorem 3.1).

This implies that no maximizer of $M(K)$ can be an interior point of a continuous movement K_t unless the speed function is linear.

This result has several consequences. We show that if K is a maximizer, then its boundary does not contain any open smooth subset with positive Gaussian curvature.

A further consequence regards convex bodies of the following type. Let K be the union of two convex bodies, with disjoint interiors, K_1 and K_2 , such that each point from $\partial K \cap K_1 \cap K_2$ is a singular point for K . Then K can be excluded as a possible maximizer.

The method works as well in other special cases. For instance, let K be symmetric with respect to the hyperplane π and let v be a direction orthogonal to π . If K is not affinely equivalent to its Schüttelung with respect to v , then K cannot be a maximizer. From this fact it follows that no Platonic polyhedron other than the simplex can give the maximum of the Sylvester functional $M(K)$.

Our argument applies also to generalizations of Sylvester's functional $M(K)$. Namely, all our results can be extended to the r -th order moment $M_r(K; n)$ of the volumes of polytopes which are convex hulls of n random points from K , $n \geq d + 1$, $r \geq 1$.

Extrema of the functionals $M_r(K; n)$ were studied by several authors. In particular, Groemer [9] showed that ellipsoids are still the

only minimizers of $M_r(K; n)$, for every $n \geq d + 1$, $r \geq 1$. For $r = 1$, Dalla and Larman [6] and Giannopoulos [7] proved that in \mathbb{R}^2 triangles are the unique solutions of the maximum problem. Results concerning the asymptotic behaviour of $M_1(K; n)$, as n tends to infinity, were obtained by Bárány and Larman in [2] and by Bárány and Buchta in [1].

The method of selection that we present is not sufficient by itself to prove the simplex conjecture for Sylvester's problem; for instance, it does not exclude a cylinder in \mathbb{R}^3 whose base is a triangle (see Example 2.6). On the other hand the use of continuous movements reveals itself as a unifying approach to extremum problems for Sylvester's functionals. Partial applications of the method employed here were already contained, as underlying features, in [8], [9] and [6].

2. RS-decomposability

We begin by recalling the notion of *linear parameter system* used by Rogers and Shephard in [14]. Let \mathcal{K}^d be the class of all convex bodies in \mathbb{R}^d . For a given $K \in \mathcal{K}^d$ and for a fixed direction v , we set

$$K_t = \text{conv}\{x + t\alpha(x)v : x \in K\}, t \in \mathbb{R},$$

where $\alpha(x)$ is a real valued function (the *speed function*) defined on K and $\text{conv}[I]$ denotes the convex hull of a set I . The family K_t , which is a continuous movement of the body K , is called a linear parameter system.

The main feature of these movements is expressed by the following theorem (proofs are given by Rogers and Shephard [14], Shephard [16] and Hadwiger [11]):

THEOREM 2.1. *The d -dimensional volume $V_d(K_t)$ is a convex function of t .*

As pointed out by Rogers and Shephard in [14], the orthogonal projection of a linear parameter system onto any linear subspace is also a linear parameter system. Therefore, Theorem 2.1 ensures that not only the d -dimensional volume but also every quermassintegral of K_t is a convex function of t . Indeed, by Kubota's formulas (see

Schneider [15, Section 5.3]), for $i = 1, 2, \dots, d - 1$, the i -th quermassintegral $W_i(K)$ of K is the average of the $(d - i)$ -dimensional volumes of $(d - i)$ -dimensional projections of K .

Here we are interested in the particular case of linear parameter systems, whose speed function is constant on each chord of K parallel to v .

Let $\pi_v : \mathbb{R}^d \rightarrow v^\perp = \{x \in \mathbb{R}^d : \langle x, v \rangle = 0\}$ be the orthogonal projection along v and let $K|_{v^\perp} = \pi_v(K)$.

DEFINITION 2.2. *The family of sets*

$$K_t = \{x + t\beta(\pi_v(x))v : x \in K\}, t \in [a, b], \quad (1)$$

where β is a real valued function on $K|_{v^\perp}$ and $0 \in [a, b]$, is said to be an RS-movement of K if $K_t = \text{conv}[K_t]$, for every $t \in [a, b]$.

Simple examples of RS-movements of a body $K \in \mathcal{K}^d$ along v are obtained letting β be any linear function defined on v^\perp . Indeed, in this case, K_t is an affine image of K for every $t \in \mathbb{R}$.

Other examples can be obtained in connection with the Steiner symmetrization and the Schüttelung process respectively. Namely, for every direction v there exist two convex functions f_v and g_v , defined on $K|_{v^\perp}$, such that

$$K = \{(x, y) \in \mathbb{R}^{d-1} \times \mathbb{R} : x \in K|_{v^\perp}, f_v(x) \leq y \leq -g_v(x)\}. \quad (2)$$

Setting $\beta = g_v - f_v$ in (1), we obtain the RS-movement

$$K_t = \{(x, y) \in \mathbb{R}^{d-1} \times \mathbb{R} : x \in K|_{v^\perp}, [f_v + t(g_v - f_v)](x) \leq y \leq [-g_v + t(g_v - f_v)](x)\}, t \in [0, 1]. \quad (3)$$

Clearly $K_0 = K$ and K_1 is the reflection of K with respect to v^\perp . Moreover $K_{1/2}$ is the Steiner symmetral of K with respect to v^\perp .

If we choose $\beta = g_v$, the corresponding RS-movement is

$$K_t = \{(x, y) \in \mathbb{R}^{d-1} \times \mathbb{R} : x \in K|_{v^\perp}, f_v(x) + tg_v(x) \leq y \leq (t - 1)g_v(x)\}, t \in [0, 1]. \quad (4)$$

In this case $K_0 = K$ and K_1 is the Schüttelung of K along v .

As $W_i(K_0) = W_i(K_1)$, $i = 1, 2, \dots, d - 1$, for the RS-movement (3), from the convexity of $W_i(K_t)$ with respect to t we deduce that for every i the i -th quermassintegral does not increase via a Steiner symmetrization.

On the other hand it is well-known (see [10]) that every convex body can be reduced to a ball through repeated Steiner symmetrizations along suitably chosen directions.

These two facts imply the following isoperimetric-type inequalities

$$\frac{[W_i(K)]^d}{[V_d(K)]^{d-i}} \geq \frac{[W_i(B)]^d}{[V_d(B)]^{d-i}}, \quad i = 1, 2, \dots, d - 1,$$

where B is a ball.

As far as the RS-movement (4) is concerned, the Schüttelung of K is an endpoint of the family. Thus the convexity of the quermassintegrals is not sufficient to obtain comparison results of the same type as in the case of Steiner symmetrization. As a matter of facts simple examples show that the surface area $W_1(K)$ has not in general a monotonic behaviour with respect to the Schüttelung process. If we know in addition that $\frac{d}{dt}W_i(K_t)|_{t=0+}$ is non-negative, then clearly $W_i(K)$ does not decrease via the Schüttelung process. In the case of regular bodies, a computation of such a derivative, involving integrals of symmetric functions of principal curvatures, was performed by Voss [19] and Tso [18].

The above considerations can help to understand the following general facts.

Let \mathcal{F} be a continuous functional on \mathcal{K}^d (equipped with the Hausdorff metric) invariant under reflections with respect to hyperplanes. Assume that \mathcal{F} is a convex function of the parameter of any RS-movement. Then among all the convex bodies of prescribed d -dimensional volume, the minimum of \mathcal{F} exists and it is attained at a ball.

The existence of the maximum of \mathcal{F} requires some additional assumptions. Let \mathcal{F} be invariant under affine volume-preserving maps. Then we can restrict ourselves to the subclass of bodies contained in a ball of sufficiently large radius; this follows via standard arguments from the classical result by John [12]. Therefore our assumptions guarantee that \mathcal{F} has a maximum. Notice that the Sylvester

functional we shall deal with in the next section fulfils all the above requests.

We also observe that the condition on \mathcal{F} to be invariant under affine volume-preserving maps can be proved to be necessary for the existence of the maximum.

RS-movements turn out to be useful to search for maximizers of \mathcal{F} . Indeed, let K_t be an RS-movement, $t \in [a, b]$, such that the speed function is not linear. Then no K_t , for $t \in (a, b)$, can give the maximum of \mathcal{F} , unless $\mathcal{F}(K_t)$ is constant.

Consequently, we introduce the following definition.

DEFINITION 2.3. *A body K is said to be RS-indecomposable if, for every RS-movement K_t , $t \in [-1, 1]$, such that $K_0 = K$, the speed function of the movement is linear.*

A body is called RS-decomposable if it is not RS-indecomposable.

EXAMPLE 2.4. *Every triangle is RS-indecomposable.*

Let T be a triangle and let v be a direction of \mathbb{R}^2 . At least one of the functions f_v and g_v (see formula (2)) is linear on $T|_{v^\perp}$. Therefore, the speed function of any RS-movement T_t , $t \in [-1, 1]$, with $T_0 = T$, has to be linear. \square

EXAMPLE 2.5. *Every square is RS-decomposable.*

Let Q be the square with vertices at $(0, 0)$, $(1, 0)$, $(1, 1)$, $(0, 1)$. We consider the family of sets

$$Q_t = \text{conv}\{(0, 0), (0, 1), (1, 0), (1 + t, 1 - t)\}, \quad t \in [-1, 1].$$

It is easy to see that Q_t is an RS-movement of Q keeping fixed $\text{conv}\{(0, 0), (0, 1), (1, 0)\}$; thus the relevant speed function is not linear. \square

EXAMPLE 2.6. *Let $K \in \mathcal{K}^d$ be either a cylinder or a cone. Then K is RS-decomposable if and only if the base is.*

Let K be a cylinder in \mathcal{K}^d . Up to an affine map, we may assume that

$$K = \{(x, y) \in \mathbb{R}^{d-1} \times \mathbb{R} : x \in H, 0 \leq y \leq 1\},$$

where $H \in \mathcal{K}^{d-1}$ is the base of K .

If H is RS-decomposable, then there exists an RS-movement H_t , $t \in [-1, 1]$, such that $H_0 = H$ and the speed function is not linear. Clearly the family

$$K_t = \{(x, y) \in \mathbb{R}^{d-1} \times \mathbb{R} : x \in H_t, 0 \leq y \leq 1\},$$

is an RS-movement of K , whose speed function is not linear.

Conversely, assume that H is RS-indecomposable and let K_t be an RS-movement of K along the direction v . We distinguish two cases.

If v is parallel to H then the speed function has to be linear on the projection of H onto v^\perp . If not, H would be RS-decomposable. Without loss of generality, we may suppose that the speed function vanishes identically on $H|_{v^\perp}$. For each $h \in [0, 1]$, consider the section $K_{t,h}$ of K_t , parallel to H , at height h . The $(d-1)$ -dimensional volume of $K_{t,h}$ is independent of t and h . Therefore, by the Brunn-Minkowski theorem (see [15, Section 6.1]), $K_{t,h}$ is a translate of H , for every t and h . Moreover, the speed function must be linear also on any vertical line segment of the form $\{(x, y) : 0 \leq y \leq 1\}$, where $x \in H|_{v^\perp}$. Hence the speed function is linear on $K|_{v^\perp}$.

Let us turn to the case when the direction v of the movement K_t is not parallel to H . Arguing as above, we know that the speed function is linear on the projection onto v^\perp of each vertical chord of K ; furthermore, we may assume that the speed function vanishes identically on $H|_{v^\perp}$. Taking into account that $H|_{v^\perp}$, as a $(d-1)$ -dimensional convex body, has nonempty interior leads to the conclusion that K is RS-indecomposable.

An even simpler proof dispensing with the Brunn-Minkowski theorem, can be repeated in the case when K is a cone. \square

Further considerations on classes of RS-decomposable sets will be made in connection with Sylvester's functionals later on.

3. Sylvester's functionals

Let $K \in \mathcal{K}^d$. We consider the normalized mean expected value $M(K)$ of the volume of the simplex whose vertices are randomly, independently and uniformly chosen from K . Such a functional can

be expressed as

$$M(K) = \frac{1}{[V_d(K)]^{d+2}} \underbrace{\int \dots \int_K V_d(\text{conv}[x_i]_{i=1}^{d+1})}_{(d+1 \text{ times})} dx_1 \dots dx_{d+1}, \quad (5)$$

where $\text{conv}[x_i]_{i=1}^{d+1} = \text{conv}[x_1, \dots, x_{d+1}]$.

The classical Sylvester problem is equivalent to search for the maximum and the minimum of $M(K)$.

It is easy to check that $M(K)$ is continuous with respect to the Hausdorff metric and it is invariant under affine transformations. Therefore the existence of maximizers and minimizers of $M(K)$ is ensured. As proved by Blaschke [4] and Groemer [8], ellipsoids are the only minimizers of $M(K)$. The problem of finding maximizers is still open, except for the case $d = 2$; in the latter case Blaschke [4] showed that $M(K)$ is maximum if and only if K is a triangle.

Extensions of $M(K)$ to other functionals which are continuous and affinely invariant were considered by several authors. A quite natural one is given by the normalized mean expected value of the volume of a random polytope with at most n vertices from K :

$$M(K; n) = \frac{1}{[V_d(K)]^{n+1}} \underbrace{\int \dots \int_K V_d(\text{conv}[x_i]_{i=1}^n)}_{(n \text{ times})} dx_1 \dots dx_n; \quad (6)$$

obviously here $n \geq d + 1$.

For $d = 2$, $M(K; n)$ is maximum if and only if K is a triangle (see Dalla and Larman [6] for the *if* part and Giannopoulos [7] for the *only if* part).

For $d \geq 2$, Groemer [9] proved that ellipsoids are still the only minimizers of $M(K; n)$ as well as of its generalization

$$M_r(K; n) = \frac{1}{[V_d(K)]^{n+r}} \underbrace{\int \dots \int_K (V_d(\text{conv}[x_i]_{i=1}^n))^r}_{(n \text{ times})} dx_1 \dots dx_n, \quad (7)$$

where $r \geq 1$. Functionals (7) express the normalized higher order moments of the volume of a random polytope in K .

A change of variables in the above integral shows that $M_r(K; n)$ is invariant under affine transformations.

Our purpose is to establish a general property of functionals (7), which in particular can be used to recover most of the quoted results about extremum problems for such functionals.

THEOREM 3.1. *If K_t , $t \in [-1, 1]$, is an RS-movement, then $M_r(K_t; n)$ is a convex function with respect to t , for every $r \geq 1$ and $n \geq d + 1$.*

Furthermore $M_r(K_t; n)$ is strictly convex if and only if the speed of K_t is not a linear function.

Proof. Let v be the direction of the movement K_t and β be the relevant speed function; denote by f_v and g_v the convex functions such that

$$K_0 = \{(x, y) \in \mathbb{R}^{d-1} \times \mathbb{R} : x \in K_0|_{v^\perp}, f_v(x) \leq y \leq -g_v(x)\}.$$

By Fubini's theorem, functionals (7) can be rewritten as

$$\begin{aligned} M_r(K_t; n) &= \frac{1}{[V_d(K_0)]^{n+r}} \underbrace{\int_{K_0|_{v^\perp}} \cdots \int_{K_0|_{v^\perp}}}_{(n\text{-times})} \left(\int_{f_v+t\beta(x_1)}^{(-g_v+t\beta)(x_1)} \cdots \int_{f_v+t\beta(x_n)}^{(-g_v+t\beta)(x_n)} \right. \\ &\quad \left. [V_d(\text{conv}[x_i + y_i v]_{i=1}^n)]^r dy_1 \cdots dy_n \right) dx_1 \cdots dx_n \\ &= \frac{1}{[V_d(K_0)]^{n+r}} \underbrace{\int_{K_0|_{v^\perp}} \cdots \int_{K_0|_{v^\perp}}}_{(n \text{ times})} \left(\int_{f_v(x_1)}^{-g_v(x_1)} \cdots \int_{f_v(x_n)}^{-g_v(x_n)} \right. \\ &\quad \left. [V_d(\text{conv}[x_i + (t\beta(x_i) + y_i)v]_{i=1}^n)]^r dy_1 \cdots dy_n \right) dx_1 \cdots dx_n. \quad (8) \end{aligned}$$

Thus $M_r(K_t; n)$ is the repeated integral of a function which is the r -th power of a convex function, according to Theorem 2.1. Therefore $M_r(K_t; n)$ is convex itself.

As far as the strict convexity is concerned, if β is a linear function then $M_r(K_t; n)$ is constant in $[-1, 1]$. Indeed, K_t is an affine image of K_0 , for every $t \in [-1, 1]$.

In order to prove the converse, we start with the case $r = 1$, $n = d + 1$. Assume that β is not a linear function. It is sufficient to prove that

$$M(K_{-1}) + M(K_1) > 2M(K_0). \quad (9)$$

Indeed, if $t_1, t_2 \in [-1, 1]$, by suitably rescaling t we obtain a new RS-movement \tilde{K}_t such that $\tilde{K}_{-1} = K_{t_1}$, $\tilde{K}_1 = K_{t_2}$. Thus (9) implies that $M(K_t)$ is not linear in any subinterval of $[-1, 1]$; hence it is strictly convex.

If $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_d$ are arbitrary fixed affinely independent vectors from the interior of $K_0|_{v^\perp}$, then, up to adding a linear function to β , we may suppose that $\beta(\bar{x}_i) = 0$, $i = 1, 2, \dots, d$. Since β is not linear, there exists at least one vector from $K_0|_{v^\perp}$ with not vanishing speed.

Firstly, assume that we can find a vector \bar{x}_{d+1} from $\text{conv}[\bar{x}_1, \bar{x}_2, \dots, \bar{x}_d]$ such that $\beta(\bar{x}_{d+1}) \neq 0$. For every \bar{x}_i , $i = 1, 2, \dots, d$, let $\bar{y}_i \in \mathbb{R}$ be such that $\bar{z}_i = \bar{x}_i + \bar{y}_i v \in K_0$. Denote by τ the hyperplane through $\bar{z}_1, \bar{z}_2, \dots, \bar{z}_d$ and by \bar{y}_{d+1} the number such that $\bar{x}_{d+1} + \bar{y}_{d+1} v \in \tau$.

The function

$$\begin{aligned} \varphi(x_1, \dots, x_{d+1}, y_1, \dots, y_{d+1}) &= V_d \left(\text{conv}[x_i + (y_i + \beta(x_i))v]_{i=1}^{d+1} \right) + \\ &V_d \left(\text{conv}[x_i + (y_i - \beta(x_i))v]_{i=1}^{d+1} \right) - 2V_d \left(\text{conv}[x_i + y_i v]_{i=1}^{d+1} \right) \end{aligned}$$

is non-negative by Theorem 2.1.

Moreover $\varphi(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_{d+1}, \bar{y}_1, \bar{y}_2, \dots, \bar{y}_{d+1}) > 0$. Taking into account the continuity of φ , we deduce inequality (9).

In case that $\beta(x) = 0$ for every $x \in \text{conv}[\bar{x}_1, \bar{x}_2, \dots, \bar{x}_d]$, let \bar{x}_{d+1} be a vector from the interior of $K_0|_{v^\perp}$ such that $\beta(\bar{x}_{d+1}) \neq 0$. Denote by $\bar{z}_{d+1} \in K_0$ a vector such that its projection onto v^\perp is \bar{x}_{d+1} . Let τ be a hyperplane through \bar{z}_{d+1} not containing v and intersecting the interior of the set $I = \{z \in K_0 : z|_{v^\perp} \in \text{conv}[\bar{x}_1, \bar{x}_2, \dots, \bar{x}_d]\}$. Choose $\tilde{z}_1, \tilde{z}_2, \dots, \tilde{z}_d \in \tau \cap I$ in such a way that their projections $\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_d$ onto v^\perp are affinely independent vectors and set $\tilde{z}_{d+1} = \bar{z}_{d+1}$. The function φ is strictly positive at $(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_{d+1}, \tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_{d+1})$, where \tilde{y}_i satisfies $\tilde{z}_i = \tilde{x}_i + \tilde{y}_i v$, $i = 1, 2, \dots, d + 1$. Therefore (9) holds in this case too.

Now let us establish the strict convexity of $M_r(K_t; n)$, for arbitrary $r \geq 1$ and $n \geq d + 1$. We have that

$$\begin{aligned} & M_r(K_{-1}; n) + M_r(K_1; n) - 2M_r(K_0; n) = \\ &= \frac{1}{[V_d(K_0)]^{n+r}} \underbrace{\int \dots \int_{K_0 \quad K_0}}_{(n \text{ times})} \psi(z_1, \dots, z_n) dz_1 \dots dz_n, \end{aligned} \quad (10)$$

where

$$\begin{aligned} \psi(z_1, \dots, z_n) &= [V_d(\text{conv}[z_i + \beta(z_i|_{v^\perp})v]_{i=1}^n)]^r \\ &+ [V_d(\text{conv}[z_i - \beta(z_i|_{v^\perp})v]_{i=1}^n)]^r - 2[V_d(\text{conv}[z_i]_{i=1}^n)]^r. \end{aligned}$$

Notice that ψ is a continuous function, which is non-negative by virtue of Theorem 2.1.

As shown in the first part of the proof, there exist $\bar{z}_1, \bar{z}_2, \dots, \bar{z}_{d+1}$ such that

$$\psi(\bar{z}_1, \bar{z}_2, \dots, \bar{z}_{d+1}, \bar{z}_{d+1}, \dots, \bar{z}_{d+1}) > 0.$$

Hence, the strict convexity of $M_r(K_t; n)$ is obtained by using (10) and arguing as in the previous case.

This concludes the proof. \square

As a first application, Theorem 3.1 gives Groemer's result: *Ellipsoids are the only minimizers of $M_r(K_t; n)$.*

Indeed, let \bar{K} be such that $M_r(\bar{K}; n) \leq M_r(K; n)$, for every $K \in \mathcal{K}^d$. If \bar{K} is not an ellipsoid, then there exists a direction v such that \bar{K} is not affinely equivalent to its Steiner symmetral with respect to v^\perp (see [5, §70]). Hence $M_r(K_t; n)$ is strictly convex in t , being K_t the RS-movement defined as in (3), with $K_0 = \bar{K}$.

Let us turn now to the problem of finding maximizers of $M_r(K; n)$.

In the two-dimensional setting, Theorem 3.1 implies that *if T is a triangle, then*

$$M_r(K; n) \leq M_r(T; n), \quad (11)$$

for every $K \in \mathcal{K}^2$, $r \geq 1$, $n \geq 3$.

We prove (11) in the case when K is a polygon; by virtue of the continuity of the functional $M_r(K; n)$, inequality (11) holds in the general case.

Let K be a polygon with vertices a_1, a_2, \dots, a_m , $m > 3$, ordered clockwise. Choose v as a direction parallel to the line joining a_1 and a_3 . Set

$$K_t = \text{conv}[a_1, a_2 + tv, a_3, \dots, a_m].$$

There exist $\delta_1, \delta_2 > 0$ such that K_t , $t \in [-\delta_1, \delta_2]$, is an RS-movement and both $K_{-\delta_1}$, K_{δ_2} have $m - 1$ vertices. By Theorem 3.1, $M_r(K; n)$ is strictly convex; therefore

$$M_r(K; n) < \max\{M_r(K_{-\delta_1}; n), M_r(K_{\delta_2}; n)\}.$$

Iterating the above procedure yields $M_r(K; n) < M_r(T; n)$, where T is a triangle.

In higher dimensions coupling Theorem 3.1 and Definition 2.2 gives the following necessary condition.

THEOREM 3.2. *Every maximizer of $M_r(K; n)$ is RS-indecomposable.*

Our next step is to employ Theorem 3.2 as a selection criterion and to identify classes of RS-decomposable convex bodies.

The first case we consider is the class \mathcal{A}^d of all convex bodies in \mathbb{R}^d such that a non empty open subset of the boundary is of class $C^{2,+}$, i.e. is C^2 and all the principal curvatures are positive.

THEOREM 3.3. *Every convex body from \mathcal{A}^d is RS-decomposable.*

Proof. Let K be in \mathcal{A}^d and choose a coordinate system $(O; x_1, \dots, x_d)$ so that the interior of K has nonempty intersection with the hyperplane $x_d = 0$ and $\partial K \cap \{x \in \mathbb{R}^d : x_d \geq 0\}$ is $C^{2,+}$.

Let v be a direction parallel to the x_1 -axis and let f_v and g_v be the convex functions defined as in (2). There exists a number $\delta > 0$ such that the least eigenvalue of both the Hessian matrices $D^2 f_v(y)$ and $D^2 g_v(y)$ is greater than δ , for every $y \in K|_{v^\perp} \cap \{x \in \mathbb{R}^d : x_d \geq 0\}$. Let $\beta \in C^2(K|_{v^\perp})$, $\beta \not\equiv 0$, be a function vanishing in $K|_{v^\perp} \cap \{x \in \mathbb{R}^d : x_d \leq 0\}$ such that all the eigenvalues of $D^2 \beta(y)$ are in $(-\delta, \delta)$. Since $f_v \pm \beta$ and $g_v \pm \beta$ are still convex functions, the movement K_t , $t \in [-1, 1]$, along v with speed function β (see (1)) and $K_0 = K$, is an RS-movement. Clearly β is not a linear function, therefore K is RS-decomposable. \square

Theorem 3.3 suggests to confine the search of maximizers of $M_r(K; n)$ within the class of convex polytopes. The same hint comes from the results proved by Bárány and Larman in [2]. There an upper bound for $M(K; n)$ is found; moreover it is showed that $M(P; n)$ behaves asymptotically, as n tends to infinity, like such a bound when P is a polytope.

In fact, the simplex was repeatedly conjectured to be the only maximizer of $M(K; n)$ (see, for instance, [6] and [13]). Such a conjecture is supported by the result of Bárány and Buchta [1], who proved that for every $K \in \mathcal{K}^d$ there exists \bar{n} , depending on K , such that $M(K; n) \leq M(T; n)$, for all $n \geq \bar{n}$, where T is a simplex.

Another contribution to the conjecture was given by Dalla and Larman in [6]. They characterize simplices in \mathcal{K}^d as the only maximizers of $M(K; n)$ in the class of all the convex polytopes with at most $d + 2$ vertices. This result can be extended to the functional $M_r(K; n)$ as a consequence of Theorem 3.2.

THEOREM 3.4. *If $P \in \mathcal{K}^d$ is a polytope with at most $d + 2$ vertices, then*

$$M_r(P; n) \leq M_r(T; n)$$

for every $r \geq 1$ and $n \geq d + 1$, where $T \in \mathcal{K}^d$ is a simplex. Equality holds if and only if P is a simplex.

Proof. Standard compactness arguments show that $M_r(P; n)$ attains a maximum in the class of all convex polytopes in \mathcal{K}^d with at most $d + 2$ vertices. Furthermore, it is clear that Theorem 3.2 applies also in such a restricted class.

Let us suppose that $P \in \mathcal{K}^d$ is not a simplex. Let us fix $d + 1$ vertices of P which do not lie on the same hyperplane. Among all the hyperplanes containing d of the fixed vertices, let π be a hyperplane containing interior points of P . Thus π divides P in two simplices, T_1 and T_2 . One can easily build an RS-movement of P which keeps T_1 unchanged and transforms affinely T_2 . Therefore P is RS-decomposable. \square

The proof of Theorem 3.4 can be adapted to a wider class of convex bodies, which consequently have to be excluded as maximizers of $M_r(K; n)$.

THEOREM 3.5. *Let $K \in \mathcal{K}^d$. Assume that there exists a hyperplane π intersecting the interior of K , and such that each point from $\partial K \cap \pi$ is a singular point for K . Then K is RS-decomposable.*

Proof. By the assumptions of the theorem, the hyperplane π splits K into two convex bodies K_1 and K_2 . Then the proof is the same as the one of the previous theorem, with T_1 and T_2 replaced by K_1 and K_2 . \square

Our final result deals with convex bodies having a hyperplane of symmetry.

THEOREM 3.6. *If $K \in \mathcal{K}^d$ is symmetric with respect to a hyperplane $\pi = v^\perp$, and the corresponding functions f_v and g_v (see (2)) are not linear, then K is RS-decomposable.*

Proof. Formula (4) with $t \in [-1, 1]$, i.e. the Schüttelung processes along v and $-v$, gives an RS-movement of K whose speed function is not linear. \square

It follows in particular from this theorem that in \mathbb{R}^3 no Platonic polyhedron other than the regular tetrahedron can give the maximum for the Sylvester problem.

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