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On Symmetric Bi-derivations in Rings

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Summary. - *Let R be a ring with centre $Z(R)$. A bi-additive symmetric mapping $D(\cdot,\cdot) : R \times R \longrightarrow R$ is called symmetric bi-derivation if for any fixed $y \in R$, $x \mapsto D(x,y)$ is a derivation. The main result of the present paper states that if R is a semiprime ring of characteristic different from two and three which admits a symmetric bi-derivation D such that $[[D(x,x),x],x] \in Z(R)$ holds for all $x \in R$, then $[D(x,x),x] = 0$, for all $x \in R$. Further, some commutativity results are also obtained.*

1. Introduction

Throughout, R will represent an associative ring with centre $Z(R)$. A ring R is prime if $aRb = (0)$ implies that $a = 0$ or $b = 0$, and is semiprime if $aRa = (0)$ implies $a = 0$. The commutator $xy - yx$ will be written as $[x,y]$. We shall make the extensive use of commutator identities $[xy,z] = [x,z]y + x[y,z]$ and $[x,yz] = y[x,z] + [x,y]z$. An additive mapping $d : R \longrightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x,y \in R$. A mapping $F : R \longrightarrow R$ is said to be centralizing on R if $[F(x),x] \in Z(R)$ for all $x \in R$. In the special case where $[F(x),x] = 0$, for all $x \in R$, the mapping F is called commuting on R. A mapping $B(\cdot,\cdot) : R \times R \longrightarrow R$ will be called symmetric if $B(x,y) = B(y,x)$ for all pairs $x,y \in R$. A mapping $f : R \longrightarrow R$ defined by $f(x) = B(x,x)$, where $B(\cdot,\cdot) : R \times R \longrightarrow R$ is a symmetric mapping will be called the trace of B. It is obvious that in case $B(\cdot,\cdot) : R \times R \longrightarrow R$ is a symmetric mapping which

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is also bi-additive (i.e. additive in both arguments), the trace of B satisfies $f(x+y) = f(x) + f(y) + 2B(x,y)$; $x, y \in R$. A symmetric bi-additive mapping $D(\cdot,\cdot) : R \times R \longrightarrow R$ is called symmetric bi-derivation if $D(xy, z) = D(x,z)y + xD(y,z)$ holds for $x, y, z \in R$.

The study of centralizing and commuting mappings was initiated by a well-known theorem due to Posner [9] which states that the existence of a non-zero centralizing derivation on a prime ring R implies that R is commutative. A number of authors have extended the Posner's theorem in several ways (cf. [1],[2],[3],[4],[7],[11] & [13], where further references can be found). The notion of additive commuting mapping is closely connected with the notion of bi-derivation. Every additive commuting mapping $F : R \longrightarrow R$ gives rise to a bi-derivation on R. Namely, linearizing $[F(x), x] = 0$, we get $[F(x), y] = [x, F(y)]$; $x, y \in R$ and hence we note that the map $(x, y) \mapsto [F(x), y]$ is a bi-derivation. The concept of bi-derivation was introduced by G. Maksa [5]. It is shown in [6] that symmetric bi-derivations are related to general solution of some functional equations. Some results concerning symmetric bi-derivations in prime rings can be found in [10] and [12]. Our main theorem in the present paper can be regarded as a contribution to the theory of centralizing and commuting mappings in semiprime rings. Further, we investigate commutativity of ring R which admits a symmetric bi-derivation $D(\cdot,\cdot) : R \times R \longrightarrow R$ such that either $xy - D(xy, xy) = yx - D(yx, yx)$ or $xy + D(xy, xy) = yx + D(yx, yx)$, for all $x, y \in R$. Finally, it is shown that under rather a weak assumption, R turns out to be commutative.

2. Main results

THEOREM 2.1. *Let R be a 2-torsion free and 3-torsion free semiprime ring. Suppose that there exists a symmetric bi-derivation $D(\cdot,\cdot) : R \times R \longrightarrow R$ such that the mapping $x \mapsto [f(x), x]$ is centralizing on R, where f denotes the trace of D. Then f is commuting on R.*

Proof. First we shall show that the mapping $x \mapsto [f(x), x]$ is commuting on R. By our hypothesis, we have

$$[[f(x), x], x] \in Z(R), \quad \text{for all } x \in R. \tag{1}$$

Linearize (1), to get

$$\begin{aligned}[][[f(y), x], x] + 2[[D(x, y), x], x] + [[f(x), y], x] + [[f(y), y], x] +\\ 2[[D(x, y), y], x] + [[f(x), x], y] + [[f(y), x], y] + 2[[D(x, y), x], y] +\\ [[f(x), y], y] + 2[[D(x, y), y], y] \in Z(R)\end{aligned}$$

Now replacing x by $-x$ and comparing the relation so obtained with the above, we get by 2-torsionfreeness of R

$$\begin{aligned}&2[[D(x, y), x], x] + [[f(x), y], x] + [[f(y), y], x] + [[f(x), x], y]\\ &+[[f(y), x], y] + 2[[D(x, y), y], y] \in Z(R), \quad \text{for all } x, y \in R.\end{aligned} \tag{2}$$

Substituting $2x$ instead of x in (2), comparing the relation so obtained with (2) and using the fact that characteristic of R is different from 2 and 3, we obtain

$$\begin{aligned}[][[f(x), y], x] + [[f(x), x], y] + 2[[D(x, y), x], x] \in Z(R),\\ \text{for all } x, y \in R.\end{aligned} \tag{3}$$

Replace y by x^2 in (3) to get

$$[[f(x), x^2], x] + [[f(x), x], x^2] + 2[[f(x)x + xf(x), x], x] \in Z(R).$$

This yields that

$[[f(x), x], x]x + x[[f(x), x], x] + [[f(x), x], x]x + x[[f(x), x], x] + 2[[f(x), x], x]x + 2x[[f(x), x], x] \in Z(R)$. Since $[[f(x), x], x] \in Z(R)$ and characteristic of R is different from 2 and 3, the later relation reduces to $[[f(x), x], x]x \in Z(R)$ for all $x \in R$. Thus we obtain

$$[[f(x), x], x][y, x] = 0, \quad \text{for all } x, y \in R. \tag{4}$$

Replacing y by $y[f(x),x]$ in (4) and using (4), we have

$$[[f(x),x],x]y[[f(x),x],x]=0$$

, and the semiprimeness of R yields that

$$[[f(x),x],x]=0, \quad \text{for all } x\in R. \tag{5}$$

Now, using similar techniques as used to get (3) from (1), we arrive at

$$[[f(x),x],y]+[[f(x),y],x]+2[[D(x,y),x],x]=0, \quad \text{for all } x,y\in R. \tag{6}$$

Substituting yz for y in (6), we get

$$\begin{aligned}
&y[[f(x),x],z]+[[f(x),x],y]z+[y[f(x),z]+[f(x),y]z,x]+\\
&\qquad 2[[yD(x,z)+D(x,y)z,x],x]\\
&=y[[f(x),x],z]+[[f(x),x],y]z+y[[f(x),z],x]+[y,x][f(x),z]\\
&\qquad +[f(x),y][z,x]\\
&\qquad +[[f(x),y],x]z+2y[[D(x,z),x],x]+4[y,x][D(x,z),x]\\
&\qquad +2[[y,x],x]D(x,z)+\\
&\qquad 4[D(x,y),x][z,x]+2D(x,y)[[z,x],x]+2[[D(x,y),x],x]z\\
&=0
\end{aligned}$$

Now, application of (6) yields that

$$\begin{aligned}
&[f(x),y][z,x]+[y,x][f(x),z]+4[D(x,y),x][z,x]+2D(x,y)[[z,x],x]+\\
&\qquad 2[[y,x],x]D(x,z)+4[y,x][D(x,z),x]=0, \quad \text{for all } x,y,z\in R.
\end{aligned} \tag{7}$$

Replacing of z by x in (7) gives that

$$5[y,x][f(x),x]+2[[y,x],x]f(x)=0, \quad \text{for all } x,y\in R. \tag{8}$$

Again replace y by x and z by y in (7), to get

$$5[f(x),x][y,x]+2f(x)[[y,x],x]=0, \quad \text{for all } x,y\in R. \tag{9}$$

Now replacing y by yz in (8), we have

$$5[y,x]z[f(x),x] + 5y[z,x][f(x),x] + 2[[y,x],x]zf(x) + 2y[[z,x],x]f(x) + 4[y,x][z,x]f(x) = 0,$$

and application of (8) gives that

$$5[y,x]z[f(x),x] + 2[[y,x],x]zf(x) + 4[y,x][z,x]f(x) = 0, \quad \text{for all } x,y,z \in R. \tag{10}$$

Now putting $f(x)$ for z in (10), we get

$$5[y,x]f(x)[f(x),x] + 2[[y,x],x]f(x)^2 + 4[y,x][f(x),x]f(x) = 0, \quad \text{for all } x,y \in R. \tag{11}$$

We also conclude from (8) that $5[y,x][f(x),x]f(x) + 2[[y,x],x]f(x)^2 = 0$, for all $x,y \in R$. Combining of the last equation with (11) yields that

$$[y,x]\,(5f(x)[f(x),x] - [f(x),x]f(x)) = 0, \quad \text{for all } x,y \in R. \tag{12}$$

Replacing y by zy in (12), we get $[z,x]y\,(5f(x)[f(x),x] - [f(x),x]f(x)) = 0$, for all $x,y,z \in R$, and in particular if $z = f(x)$, then we arrive at

$$[f(x),x]y\,(5f(x)[f(x),x] - [f(x),x]f(x)) = 0, \quad \text{for all } x,y \in R. \tag{13}$$

Left multiplication of (13) by $5f(x)$ gives

$$5f(x)[f(x),x]y\,(5f(x)[f(x),x] - [f(x),x]f(x)) = 0, \quad \text{for all } x,y \in R. \tag{14}$$

Putting in (13) $f(x)y$ for y, we arrive at

$$[f(x),x]f(x)y\,(5f(x)[f(x),x] - [f(x),x]f(x)) = 0, \quad \text{for all } x,y \in R. \tag{15}$$

Subtracting (15) from (14), we arrive at

$$(5f(x)[f(x),x] - [f(x),x]f(x))\, y\, (5f(x)[f(x),x] - [f(x),x]f(x)) = 0, \text{ for all } x, y \in R$$

and the semiprimeness of R yields that

$$5f(x)[f(x),x] - [f(x),x]f(x) = 0. \tag{16}$$

Now replacing zy for y in (9) and using similar techniques as used to get (12) from (8), we have $(5[f(x),x]f(x) - f(x)[f(x),x])\,[y,x] = 0$. Further, repetition of arguments which led to (16) from (12) yields that

$$5[f(x),x]f(x) - f(x)[f(x),x] = 0. \tag{17}$$

Combining (16) and (17) and using the fact that characteristic of R is different from 2 and 3, one obtains

$$f(x)[f(x),x] = 0, \quad \text{for all } x \in R. \tag{18}$$

Using similar arguments as used to get (2) from (1), the above relation yields that

$$f(x)[f(y),x] + f(y)[f(x),x] + f(y)[f(y),x] + 4D(x,y)[D(x,y),x] + 2f(x)[D(x,y),y] + 2f(y)[D(x,y),y] + 2D(x,y)[f(x),y] + 2D(x,y)[f(y),y] = 0, \quad \text{for all } x, y \in R. \tag{19}$$

Now, replacing x by $2x$ in the above relation and comparing the new relation so obtained with the above, we get

$$f(y)[f(y),x] + 2f(y)[D(x,y),y] + 2D(x,y)[f(y),y] = 0, \quad \text{for all } x, y \in R. \tag{20}$$

Substituting xy for x in (20), we have

$$f(y)[f(y),x]y + f(y)x[f(y),y] + 2f(y)[D(x,y),y]y + 2f(y)[x,y]f(y) + 2f(y)x[f(y),y] + 2D(x,y)y[f(y),y] = 0, \quad \text{for all } x,y \in R. \quad (21)$$

In view of (20) the above relation at once yields that

$$3f(y)x[f(y),y] + 2f(y)[x,y]f(y) - 2D(x,y)[[f(y),y],y] = 0$$

, and application of (5), gives

$$3f(y)x[f(y),y] + 2f(y)[x,y]f(y) = 0, \quad \text{for all } \ x,y \in R. \quad (22)$$

Replace x by yx in (22), to get

$$3f(y)yx[f(y),y] + 2f(y)y[x,y]f(y) = 0, \quad \text{for all } \ x,y \in R. \quad (23)$$

Left multiplication of (22) by y gives

$$3yf(y)x[f(y),y] + 2yf(y)[x,y]f(y) = 0, \quad \text{for all } \ x,y \in R. \quad (24)$$

Subtracting (24) from (23), we obtain

$$3[f(y),y]x[f(y),y] + 2[f(y),y][x,y]f(y) = 0, \quad \text{for all } \ x,y \in R. \quad (25)$$

In particular, if $y = f(x)$ in (10), and changing x by y and z by x, then we have

$$5[f(y),y]x[f(y),y] + 4[f(y),y][x,y]f(y) = 0, \quad \text{for all } \ x,y \in R. \quad (26)$$

Now from (25) and (26), it follows that $[f(y),y]x[f(y),y] = 0$, and the semiprimeness of R yields that $[f(y),y] = 0$, for all $y \in R$ - i.e. f is commuting on R. □

REMARK 2.2. *If $f(x)$ is centralizing on R -i.e. $[f(x),x] \in Z(R)$ for all $x \in R$, then a stronger result has been obtained by Bresar ([2], Theorem 2). In fact it has been shown that if R is a semi-prime ring with characteristic different from 2 and 3 and $D(\cdot,\cdot) : R \times R \longrightarrow R$*

a symmetric and bi-additive mapping such that $[f(x), x] \in Z(R)$ *for all* $x \in R$*, then* $[f(x), x] = 0$*, for all* $x \in R$*. In view of this result, it would be interesting to generalize further Theorem 2.1 in the case when the underlying mapping* $D(\cdot,\cdot) : R \times R \longrightarrow R$ *is only symmetric and bi-additive(not a bi-derivation).*

REMARK 2.3. *Theorem 2.1 also leads to the following conjecture: Let* R *be a semiprime ring with suitable characteristic restriction and let* $D(\cdot,\cdot) : R \times R \longrightarrow R$ *be a symmetric biderivation. Suppose that for some integer* $n \geq 1, f_n(x) = 0$*, for all* $x \in R$*, where* $f_1(x) = f(x)$ *and* $f_{k+1}(x) = [f_k(x), x]$*. In this case* $f_2(x) = 0$*. We feel that the proof of this conjecture requires a different approach than those used in the proof of Theorem 2.1*

REMARK 2.4. *It is shown in ([10], Theorem 1) that the existence of a non-zero symmetric bi-derivation* $D(\cdot,\cdot) : R \times R \longrightarrow R$*, where* R *is a prime ring of characteristic different from 2, with the property* $[D(x, x), x] = 0,\ x \in R$ *forces* R *to be commutative. Hence combining this result with the above theorem we find the following theorem due to Vukman ([12], Theorem 2).*

COROLLARY 2.5. *Let* R *be a 2-torsion free and 3-torsion free prime ring. If there exists a non-zero symmetric bi-derivation* $D(\cdot,\cdot) : R \times R \longrightarrow R$ *such that the mapping* $x \mapsto [f(x), x]$ *is centralizing on* R*, where* f *denotes the trace of* D*, then* R *is commutative.*

Recently Daif and Bell ([8], Theorem 2) proved that if a semiprime ring R admits a derivation d such that either $xy - d(xy) = yx - d(yx)$ for all $x, y \in R$, or $xy + d(xy) = yx + d(yx)$ for all $x, y \in R$, then R is commutative. Motivated by this observation we prove the following.

THEOREM 2.6. *Let* R *be a 2-torsion free ring. Suppose that there exists a symmetric bi-derivation* $D(\cdot,\cdot) : R \times R \longrightarrow R$ *such that* $xy - f(xy) = yx - f(yx)$ *for all* $x, y \in R$*, where* f *is the trace of* D*. Then* R *is commutative.*

Proof. By our hypothesis, we have $[x, y] = f(xy) - f(yx)$ for all $x, y \in R$. This can be rewritten as

$$[x, y] = [x^2, f(y)] + [f(x), y^2] + 2xD(x, y)y - 2yD(x, y)x, \quad \text{for all } x, y \in R. \tag{27}$$

Replace x by $x+y$ in (27), to get

$$[x,y] = [x^2, f(y)] + [xy, f(y)] + [yx, f(y)] + [f(x), y^2] + 2[D(x,y), y^2] + 2xD(x,y)y + 2xf(y)y - 2yD(x,y)x - 2yf(y)x.$$

This in view of (27) yields that

$$0 = [xy, f(y)] + [yx, f(y)] + 2[D(x,y), y^2] + 2xf(y)y - 2yf(y)x \quad \text{for all} x, y \in R. \tag{28}$$

Replacing y by $x+y$ in (28) and using (28), we have

$$2([x^2, f(y)] + [f(x), y^2] + 2xD(x,y)y - 2yD(x,y)x) = 0$$

for all $x, y \in R$. Since the characteristic of R is different from two, the last equation implies that

$$[x,y] = [x^2, f(y)] + [f(x), y^2] + 2xD(x,y)y - 2yD(x,y)x = 0$$

for all $x, y \in R$, and hence R is commutative. □

Using similar techniques as above, one can prove the following.

THEOREM 2.7. *Let R be a 2-torsion free ring. Suppose that there exists a symmetric bi-derivation $D(\cdot,\cdot) : R \times R \longrightarrow R$ such that $xy + f(xy) = yx + f(yx)$ for all $x, y \in R$, where f is the trace of D. Then R is commutative.*

THEOREM 2.8. *Let R be a 2-torsion free ring. Suppose that there exists a symmetric bi-additive mapping $B(\cdot,\cdot) : R \times R \longrightarrow R$ for which either $xy - B(x,x) = yx - B(y,y)$, for all $x, y \in R$ or $xy + B(x,x) = yx + B(y,y)$, for all $x, y \in R$. Then R is commutative.*

Proof. Suppose that $xy - B(x,x) = yx - B(y,y)$, for all $x, y \in R$. This can be rewritten as $[x,y] = f(x) - f(y)$, where f is the trace of B. Repalcing x by $x+y$, we get

$$[x,y] = f(x) + 2B(x,y), \quad \text{for all } x,y \in R. \tag{29}$$

Now substituting $-x$ for x in (29) and comparing the relation so obtained with (29), we get $2f(x) = 0$. This implies that $f(x) = 0$ for all $x \in R$. Now linearizing $f(x) = 0$, one obtains $2B(x,y) = 0$ for all $x, y \in R$, and hence in view of (29), we get the required result. □

In the event if R satisfies $xy + B(x,x) = yx + B(y,y)$, for all $x, y \in R$, then by using similar arguments as above one can prove the result.

In view of the above theorems it is natural to question that : what can we say about the commutativity of a ring R if it satisfies rather a weak condition namely $[x,y] - f(xy) + f(yx) \in Z(R)$ for all $x, y \in R$ or $[x,y] + f(xy) - f(yx) \in Z(R)$ for all $x, y \in R$? The following theorems deal with the commutativity of such rings.

THEOREM 2.9. *Let R be a 2-torsion free semiprime ring. Suppose that there exists a symmetric bi-derivation $D(\cdot,\cdot) : R \times R \longrightarrow R$ such that either $[x,y] - f(xy) + f(yx) \in Z(R)$ for all $x, y \in R$ or $[x,y] + f(xy) - f(yx) \in Z(R)$, for all $x, y \in R$, where f is the trace of D. Then R is commutative.*

Proof. Let R satisfy $[x,y] - f(xy) + f(yx) \in Z(R)$, for all $x, y \in R$. Using similar arguments as used to get (28), we have $[xy, f(y)] + [yx, f(x)] + 2[D(x,y), y^2] + 2xf(y)y - 2yf(y)x \in Z(R)$, for all $x, y \in R$. Further replacing y by $x+y$ in the last relation and using the fact that characteristic of R is different from 2, we find that $f(xy) - f(yx) \in Z(R)$. Combining this with our hypothesis, we get $[x,y] \in Z(R)$. Now repalce y by yx, to get $[x,y]x \in Z(R)$ i.e. $[x,y][x,r] = 0$, for all $x, y, r \in R$. Substituting ry for r, we have $[x,y]r[x,y] = 0$ and the semiprimeness of R gives that $[x,y] = 0$ for all $x, y \in R$. Use similar arguments if R satisfies the property $[x,y] + f(xy) - f(yx) \in Z(R)$. □

Similarly, one can prove the following.

THEOREM 2.10. *Let R be a 2-torsion free semiprime ring. Suppose that there exists a symmetric bi-additive mapping $B(\cdot,\cdot) : R\times R \longrightarrow R$ such that either $[x,y]-B(x,x)+B(y,y)\in Z(R)$, for all $x,y\in R$ or $[x,y]+B(x,x)-B(y,y)\in Z(R)$, for all $x,y\in R$. Then R is commutative.*

REMARK 2.11. *It is equally easy to prove the commutativity of a 2-torsion free ring R (resp. 2-torsion free semiprime ring R) satisfying the property $[x,y]=B(x,y)$, for all $x,y\in R$ (resp. $[x,y]-B(x,y)\in Z(R)$, for all $x,y\in R$), where $B(\cdot,\cdot) : R\times R \longrightarrow R$ is a symmetric bi-additive mapping.*

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