Rend. Istit. Mat. Univ. Trieste
Suppl. Vol. XXX, 111–121 (1999)

Fuzziness in Chang's Fuzzy Topological Spaces

Valentín Gregori and Anna Vidal (*)

Summary. - *It is known that fuzziness within the concept of openness of a fuzzy set in a Chang's fuzzy topological space (fts) is absent. In this paper we introduce a gradation of openness for the open sets of a Chang fts* $(X, \mathcal{T})$ *by means of a map* $\sigma : I^X \longrightarrow I$ $(I = [0,1])$*, which is at the same time a fuzzy topology on* X *in Shostak's sense. Then, we will be able to avoid the fuzzy point concept, and to introduce an adequate theory for* α*-neighbourhoods and* $\alpha - T_i$ *separation axioms which extend the usual ones in General Topology. In particular, our* α*-Hausdorff fuzzy space agrees with* α^**-Rodabaugh Hausdorff fuzzy space when* $(X, \mathcal{T})$ *is interpreservative or* α*-locally minimal.*

1. Introduction

In 1968 C. Chang [1] introduced the concept of a fuzzy topology on a set X as a family $\mathcal{T} \subset I^X$, where $I = [0,1]$, satisfying the well-know axioms, and he referred to each member of $\mathcal{T}$ as an open set. So, in his definition of a fuzzy topology some authors notice fuzziness in the

---

(*) Authors' addresses: Valentín Gregori, Departamento de Matemática Aplicada, Universidad Politécnica de Valencia Escola Universitària de Gandia Carretera Nazaret-Oliva S.N. 46730-Grau de Gandia, Valencia, Spain, e-mail: vgregori@mat.upv.es,avidal@mat.upv.es
Anna Vidal, Departamento de Matemática Aplicada, Universidad Politécnica de Valencia Escola Universitària de Gandia Carretera Nazaret-Oliva S.N. 46730-Grau de Gandia, Valencia, Spain
AMS Classification: 54A40
Keywords: fuzzy topology, gradation of openness, fuzzy point.
While working on this paper the first-listed author has been partially supported by a grant from Ministerio de Educación y Ciencia DGES PB 95-0737.

concept of opennes of a fuzzy set has not been considered. Keeping this in view, A.P. Shostak [8], began the study of fuzzy structures of topological type.

The idea of this paper is to allow open sets of a Chang's fuzzy topology to be open to some degree by means of a particular Shostak's fuzzy topology (or gradation of openness [2]) on X (Proposition 3.1). This gradation of openness will enable us to introduce fuzzy topological concepts which are generalitation of the corresponding ones in General Topology and to work with points of X instead of fuzzy points (the idea of a fuzzy point and fuzzy point belonging is a rather problematic, see Gottwald [4] for a discussion). After preliminary section, in section 3 we define the concept of an α-set and study some definitions and properties relative to it. In particular we show the family of all α-neighborhoods of $x \in X$, have similar properties to the classic cases. In section 4 we define and study the families of interpreservative and α-locally minimal spaces. In section 5 we define the concept of an α-T_i space $(i = 0, 1, 2)$ and show that the concept of an α-T_2 space coincides with the α^*-Hausdorff concept due to S.E. Rodabaugh [7] in the spaces mentioned in section 4. Our study may be thought to be just the beginning of this subjet which is far from being completed.

2. Preliminary notions

Let X be a nonempty set and I the closed unit interval. A fuzzy set of X is a map $M : X \longrightarrow I$. $M(x)$ is interpreted as the degree of membership of a point $x \in X$ in a fuzzy set M, while an ordinary subset $A \subset X$ is identified with its characteristic function and, in consequence $\emptyset$ and X are identified with the constant functions on X, $\mathbf{0}$ and $\mathbf{1}$ respectively. As usual in fuzzy sets, we write $A \subset B$ if $A(x) \leq B(x)$, $x \in X$. We define the union, intersection and complement of fuzzy sets as follows: $\left(\bigcup_i A_i\right)(x) = \bigvee_i A_i(x),\ x \in X$

$$\left(\bigcap_i A_i\right)(x) = \bigwedge_i A_i(x),\ x \in X.$$

$$A^c(x) = 1 - A(x),\ x \in X$$

A.P. Shostak [8] defined a fuzzy topology on X as a function $\tau : I^X \longrightarrow I$ satisfying the following axioms:

(i) $\tau(\mathbf{0}) = \tau(\mathbf{1}) = 1$

(ii) $\mu, \nu \in I^X$ implies $\tau(\mu \cap \nu) \geq \tau(\mu) \wedge \tau(\nu)$

(iii) $\mu_i \in I^X$ for all $i \in J$ implies $\tau(\bigcup_i \mu_i) \geq \bigwedge_i \tau(\mu_i)$

K.C. Chattopadhyay et al. [2] rediscovered the Shostak's fuzzy topology concept and called *gradation of openness* the function τ. Also, they called *gradation of closedness* on X [2], a function $\mathcal{F} : I^X \longrightarrow I$ satisfying the above axioms (i)-(iii) but interchanging the intersection with the union and vice-versa. From now, a fuzzy topology in Shostak's sense will be called gradation of openness, and we define a fuzzy topological space, or fts for short, as a pair $(X, \mathcal{T})$ where $\mathcal{T}$ is a fuzzy topology in Chang's sense, on X, i.e., $\mathcal{T}$ is a collection of fuzzy sets of X, closed under arbitrary unions and finite intersections. A set is called open if it is in $\mathcal{T}$, and closed if its complement is in $\mathcal{T}$. The interior of a fuzzy set A is the largest open fuzzy set contained in A. If confusion is not possible we say X is a space instead of a fts. We will denote inf B, the infimum of a set B of real numbers.

Recall the support of a fuzzy set A is supp $A = \{x \in X : A(x) > 0\}$. We denote $x \,\hat{\in}\, A$ whenever $x \in$ supp A, and we say A contains the point x or that x is in A.

The next definition was given by Pu Pao-Ming et al. [6].

Definition 2.1. *A* **fuzzy point** *is a fuzzy set p_x which takes the value 0 for all $y \in X$ except one, that is $x \in X$. The fuzzy point p_x is said to belong to the fuzzy set A, denoted by $p_x \tilde{\in} A$, iff $p_x(x) \leq A(x)$.*

We notice $x \hat{\in} A$ if $p_x \tilde{\in} A$.

3. Gradation of openness

The proof of the following proposition can be seen in [5]

Proposition 3.1. *Let X be a nonempty set. Then the map $\sigma : I^X \longrightarrow I$ given by $\sigma(\mathbf{0}) = 1$ and $\sigma(A) = \inf\{A(x) : x \in supp\ A\}$ if $A \neq \mathbf{0}$, satisfies both the axioms of gradation of openness and the axioms of gradation of closedness.*

The real number $\sigma(A)$ is the degree of openness [8] of the fuzzy set A; clearly, $\sigma(A) = \alpha$ implies the degree of membership of each point in the support of A, in the fuzzy set A, is at least α. We notice $\sigma(A) = 1$ iff A is an ordinary subset of X and $\sigma(A) = 0$ iff there is a sequence $\{x_n\}$ in X such that $A(x_n) > 0$, $\forall n \in \mathbb{N}$ and $\lim_n A(x_n) = 0$. With this terminology we give the following definitions.

DEFINITIONS 3.2. *The fuzzy set A of X is an* **α-set** *if $\sigma(A) \geq \alpha$; moreover, if A was open (closed) we will say A is* **α-open** *(***α-closed***).*

Clearly, each $A \in I^X$ is a 0-set and the 1-sets are only the ordinary subsets of X.

Since σ is a gradation of openness and closedness, we have the following proposition.

PROPOSITION 3.3. *The union and intersection of α-sets is an α-set.*

The following example shows that if A is an α-set and $A \subset B$, then B is not an α-set necesarily.

EXAMPLE 3.4. *Let X be a set with at least two points and $\alpha \in]0,1]$. Let $\{M, N\}$ be a partition of X. We define the following fuzzy sets A and B:*

$A(x) = \alpha$ if $x \in M$ and $A(x) = 0$ if $x \in N$

$B(x) = \alpha$ if $x \in M$ and $B(x) = \alpha/2$ if $x \in N$

We have that A is an α-set and $A \subset B$, but B is not.

Nevertheless we have the following proposition.

PROPOSITION 3.5. *Let A, B be fuzzy sets. If A is an α-set, $A \subset B$ and supp $B \subset$supp A, then B is an α-set.*

Proof. It is obvious. □

DEFINITIONS 3.6. *Let $(X, \mathcal{T})$ be a fts and let $\alpha \in I$. The fuzzy topology $\mathcal{T}_\alpha = \{A \in \mathcal{T} : \sigma(A) \geq \alpha\}$ is called the* **α -level of openness** *of the fuzzy topology $\mathcal{T}$.*

Clearly $\{\mathcal{T}_\alpha : \alpha \in I\}$ is a descending family, (i.e., $\alpha > \beta$ implies $\mathcal{T}_\alpha \subset \mathcal{T}_\beta$), where $\mathcal{T}_0 = \mathcal{T}$ and $\mathcal{T}_1$ is an ordinary topology on X.

We call **α-interior** *of the fuzzy set A, denoted $int_\alpha(A)$, the largest α-open contained in A, i.e.,*

$int_\alpha(A) = \cup\{G \in \mathcal{T}_\alpha : G \subset A\}$.

Clearly, $int_\alpha(A)$ is welldefined, since $\mathbf{0} \in \mathcal{T}_\alpha$ $\forall\alpha \in I$, and $int_\alpha(A) \subset A$ for each $A \in I^X$. Note, $int_0(A)$ is the interior of A in Chang's sense and the α-interior of a fuzzy set A is just its interior in the α-level fuzzy topology $\mathcal{T}_\alpha$.

We say that the fuzzy set A is an **α-neighborhood**, *or α-nbhd for short, of $p \in X$ if there exists $G \in \mathcal{T}_\alpha$ such that $p \hat{\in} G \subset A$. Equivalently, a point in (the support of) $int_\alpha(A)$ will be called an* **α-interior point** *of A. The α-nbhd system of a point $p \in X$, is the family $\mathcal{N}_\alpha(p)$ of all α-nbhd's of the point p. Obviously, if $\alpha > \beta$ then $\mathcal{N}_\alpha(p) \subset \mathcal{N}_\beta(p)$. With this notation we have the following proposition.*

PROPOSITION 3.7. *Let $(X, \mathcal{T})$ be a fts, $A \in I^X$, $\alpha \in I$ and $p \in X$. Then,*

(i) A is α-open if and only if $A = int_\alpha(A)$.

(ii) $p \hat{\in} int_\alpha(A)$ if and only if $A \in \mathcal{N}_\alpha(p)$.

Proof. It is obvious. □

If A is an α-open set, then A is an α-neighbourhood of all points of its support, but the converse is not true as shows the following example.

EXAMPLE 3.8. *Let (X, T) be a topological space,$\alpha \in]0, 1[$ and let $\mathcal{T} = \{\mathbf{0}, \mathbf{1}\} \cup \{\alpha \cdot U \ : \ U \in T\}$. Then any $U \in T$ is an α-neighbourhood for any point $x \in U$, i.e. for any point of its support, but obviously U fails to be α-open in $(X, \mathcal{T})$.*

In the next proposition we show that the family $\mathcal{N}_\alpha(p)$ satisfies similar properties to the corresponding ones in General Topology.

PROPOSITION 3.9. *Let $(X, \mathcal{T})$ be a fts and let $\alpha \in I$. For each point $p \in X$ let $\mathcal{N}_\alpha(p)$ be the family of all α-nbhd's of p. Then*

1. *If $M \in \mathcal{N}_\alpha(p)$, then $p \hat{\in} M$.*
2. *If $M, N \in \mathcal{N}_\alpha(p)$, then $M \cap N \in \mathcal{N}_\alpha(p)$.*

3. If $M \in \mathcal{N}_\alpha(p)$ and $M \subset N$, then $N \in \mathcal{N}_\alpha(p)$.

4. If $M \in \mathcal{N}_\alpha(p)$, then there is $N \in \mathcal{N}_\alpha(p)$ such that $\sigma(N) \geq \alpha$, $N \subset M$ and $N \in \mathcal{N}_\alpha(q)$, $\forall q \,\hat{\in}\, N$.

Proof. 1. If $M \in \mathcal{N}_\alpha(p)$, then there exists $G \in \mathcal{T}_\alpha$ such that $p \,\hat{\in}\, G \subset M$, and therefore $p \,\hat{\in}\, M$.

2. If $M, N \in \mathcal{N}_\alpha(p)$, then there are two α-open fuzzy sets G_1 and G_2 such that $p \,\hat{\in}\, G_1 \subset M$, $p \,\hat{\in}\, G_2 \subset N$. Since $G_1 \cap G_2 \in \mathcal{T}_\alpha$ and $p \,\hat{\in}\, G_1 \cap G_2 \subset M \cap N$, we have $M \cap N \in \mathcal{N}_\alpha(p)$.

3. If $M \in \mathcal{N}_\alpha(p)$, there exists $G \in \mathcal{T}_\alpha$ such that $p \,\hat{\in}\, G \subset M \subset N$ and therefore $N \in \mathcal{N}_\alpha(p)$.

4. Suppose $M \in \mathcal{N}_\alpha(p)$. Then there exists $G \in \mathcal{T}_\alpha$ such that $p \,\hat{\in}\, G \subset M$. Let $N = G$. We have $\sigma(N) \geq \alpha$ and $N \in \mathcal{N}_\alpha(q)$, $\forall q \,\hat{\in}\, N$.

□

PROPOSITION 3.10. *Let $\alpha \in I$. If $\mathcal{N}_\alpha$ is a function which assigns to each $p \in X$ a nonempty family $\mathcal{N}_\alpha(p)$ of fuzzy sets satisfying properties 1, 2 and 3 of the above proposition, then the family*

$$\mathcal{T}_\alpha = \{M \in I^X : \sigma(M) \geq \alpha, M \in \mathcal{N}_\alpha(p), \ \forall p \,\hat{\in}\, M\}$$

is a fuzzy topology on X. If property 4 of the above proposition is also satisfied, then $\mathcal{N}_\alpha(p)$ is precisely the α-nbhd system of p relative to the topology $\mathcal{T}_\alpha$.

Proof. First, we will show that $\mathcal{T}_\alpha$ is a fuzzy topology on X.

Obviously $\mathbf{0} \in \mathcal{T}_\alpha$ and since $M \subset \mathbf{1}$, $\forall M \in \mathcal{N}_\alpha(p)$, according to property 3, $\mathbf{1} \in \mathcal{N}_\alpha(p)$, $\forall p \in X$, i.e, $\mathbf{1} \in \mathcal{T}_\alpha$.

Let $M, N \in \mathcal{T}_\alpha$. If $p \,\hat{\in}\, M \cap N$, clearly $p \,\hat{\in}\, M$ and $p \,\hat{\in}\, N$, therefore $M, N \in \mathcal{N}_\alpha(p)$ and according to property 2 we have $M \cap N \in \mathcal{N}_\alpha(p)$. Now, by Proposition 3.3, $\sigma(M \cap N) \geq \alpha$ and then $M \cap N \in \mathcal{T}_\alpha$.

Let $\{M_i\}_{i \in J}$ be a family of sets of $\mathcal{T}_\alpha$ and let $M = \bigcup_i M_i$. If $p \,\hat{\in}\, M$, then we have $0 < \left(\bigcup_i M_i\right)(p)$, and therefore there exists

$j \in J$ such that $0 < M_j(p)$, i.e, $p \hat{\in} M_j$. Therefore, $M_j \in \mathcal{N}_\alpha(p)$ and according to property 3, $M \in \mathcal{N}_\alpha(p)$. Now, by Proposition 3.3, $\sigma(M) \geq \alpha$ and then $M \in \mathcal{T}_\alpha$.

Now, we suppose property 4 is also satisfied. We will see that the α-neighbourhood system of p, $\mathcal{V}_\alpha(p)$, relative to the fuzzy topology $\mathcal{T}_\alpha$, is the family $\mathcal{N}_\alpha(p)$.

If $M \in \mathcal{V}_\alpha(p)$, then there exists an α-open, G of $\mathcal{T}_\alpha$ with $p \hat{\in} G \subset M$. Therefore, $G \in \mathcal{N}_\alpha(p)$ and, according to property 3, we have $M \in \mathcal{N}_\alpha(p)$.

If $M \in \mathcal{N}_\alpha(p)$, according to property 4, there exists $N \in \mathcal{N}_\alpha(p)$, with $N \subset M$ and such that $\sigma(N) \geq \alpha$ and $N \in \mathcal{N}(q)$, $\forall q \hat{\in} N$. Then we have $N \in \mathcal{T}_\alpha$ such that $p \hat{\in} N \subset M$, i.e., $M \in \mathcal{N}_\alpha(p)$. □

OBSERVETION 3.11: In [6], the authors defined the concept of neighborhood of a fuzzy point and they showed similar results, but in [9] Shostak remarks that there are inaccuracies in the formulation of these authors. In fact, the family constructed by the authors is a base for a fuzzy topology, but it is not a fuzzy topology.

4. Interpreservative and locally minimal fts

We begin with the following definitions.

DEFINITIONS 4.1. *Let* $(X, \mathcal{T})$ *be a fts. We say* X *is* **interpreservative** *if the intersection of each family of open sets is an open set, or equivalently, if the family of closed sets is a fuzzy topology on* X*. We say* X *is* **locally minimal** *if* $\cap\{G \in \mathcal{T} : x \hat{\in} G\}$ *is open for each* $x \in X$*, i.e., each* $x \in X$ *admits a smallest nbhd. We say* X *is* **α-locally minimal**, $\alpha \in]0,1]$, *if* $\cap\{G \in \mathcal{T} : x \hat{\in} G\}$ *is* α*-open, for each* $x \in X$.

Clearly, for $\alpha > \beta > 0$, α-locally minimal implies β-locally minimal. Also, an α-locally minimal space is locally minimal but the converse is false as shows the next example.

EXAMPLE 4.2. *Let* X *the real interval* $]1, +\infty[$ *with the fuzzy topology* $\mathcal{T} = \{\mathbf{0}, \mathbf{1}, G\}$ *where* $G(x) = 1/x$, $x \in X$*. Obviosly* G *is the smallest nbhd of each point of* X *and so,* X *is locally minimal but* $\sigma(G) = 0$ *and then* X *is not* α*-locally minimal for any* $\alpha \in]0,1]$.

In the following proposition we study the relationship between interpreservative and locally minimal spaces.

PROPOSITION 4.3. *Let* $\alpha \in]0,1]$ *and let* $(X,\mathcal{T})$ *be an interpreservative fts where* $\sigma(G) \geq \alpha$, *for each* $G \in \mathcal{T}$. *Then* $\mathcal{T}$ *is* α- *locally minimal (therefore locally minimal).*

Proof. Let $x \in X$ and $\mathcal{A}_x = \{G \in \mathcal{T} : x\hat{\in}G\}$. We consider $G_x = \bigcap_{G\in\mathcal{A}_x} G$. Since $\mathcal{T}$ is interpreservative we have $G_x \in \mathcal{T}$. It is sufficient to prove $G_x \neq \mathbf{0}$. Now, for each $G \in \mathcal{A}_x$ we have $G(x) \geq \alpha > 0$. Therefore $G_x(x) = \bigwedge_{G\in\mathcal{A}_x} G(x) \geq \alpha > 0$ and $x \mathrel{\hat{\in}} G_x$, i.e., $G_x \neq \mathbf{0}$. Then the α-open G_x is the smallest nbhd of x. In the following example we will see that we cannot remove the condition $\sigma(G) \geq \alpha > 0$, for each $G \in \mathcal{T}$, in the above proposition. □

EXAMPLE 4.4. *Let* X *be the unit interval [0,1]. For each* $h \in]0,1]$, *we consider the following functions*

$$f_h(x) = \begin{cases} 2hx, & x \in [0,1/2] \\ 2h(1-x), & x \in [1/2,1] \end{cases}$$

The family $\mathcal{A} = \{f_h : 0 < h \leq 1\} \cup \{\mathbf{0}\} \cup \{\mathbf{1}\}$ *is an interpreservative fuzzy topology, however there does not exist the smallest nbhd for any* $x \in X$.

Also, we can find a locally minimal space that is not an interpreservative space.

EXAMPLE 4.5. *Consider in the real line* $\mathbb{R}$ *the laminated indiscrete fuzzy topology* $\mathcal{L}$, *i.e.,* $\mathcal{L}$ *is constituted by the constant functions from* $\mathbb{R}$ *to the unit interval* I. *We denote* $f_c : \mathbb{R} \longrightarrow I$ *the constant function* $f_c(x) = c$ *for each* $x \in \mathbb{R}$. *Take* $\alpha \in]0,1]$ *and consider the fuzzy topology* $\mathcal{T} = C_\alpha \cup \{f_{\alpha/2}\}$ *with* $C_\alpha = \{f_c \in \mathcal{L} : c > \alpha\} \cup \{\mathbf{0}\}$. *Then,* $f_{\alpha/2}$ *is the smallest nbhd of* x, *for all* $x \in \mathbb{R}$ *and therefore* $(\mathbb{R},\mathcal{T})$ *is locally minimal. However*

$$\bigcap_{c>\alpha} f_c = f_\alpha \notin \mathcal{T}.$$

We have seen that in general the two concepts interpreservative and locally minimal are not equivalent, but they are for ordinary topologies.

PROPOSITION 4.6. *Let $(X, \mathcal{T})$ be a topological space. Then X is interpreservative if and only if it is locally minimal.*

Proof. It is obvious. □

PROPOSITION 4.7. *Let $(X, \mathcal{T})$ be a locally minimal fts. Then each nonempty intersection of open sets contains a nonempty open set.*

Proof. Let $G = \bigcap_{i \in J} G_i$ with $G_i \in \mathcal{T}$, $\forall i \in J$. If $G \neq \mathbf{0}$, then there exists $x \hat{\in} G$ and therefore $x \hat{\in} G_i$, $\forall i \in J$. For this $x \in X$ let G_x be the smallest nbhd of x. We have $G_x \subset G_i$, $\forall i \in J$ and therefore $G_x \subset \bigcap_{i \in J} G_i$. G_x is the required open set. □

5. Separation axioms in fts

We will define new separation axioms for fts.

DEFINITION 5.1. *Let $\alpha \in I$. We say the fts $(X, \mathcal{T})$ is* **$\boldsymbol{\alpha}$-Hausdorff**, *or* **$\boldsymbol{\alpha}$-$\mathbf{T_2}$**, *if for all points of space $x, y \in X$ with $x \neq y$, there are $G, H \in \mathcal{T}_\alpha$ such that $x \hat{\in} G$, $y \hat{\in} H$ and $G \cap H = \mathbf{0}$.* **$\boldsymbol{\alpha}$-$\mathbf{T_1}$** *if for all $x, y \in X$ with $x \neq y$ there are $G, H \in \mathcal{T}_\alpha$ such that $x \hat{\in} G$, $y \hat{\in} H$, $x \notin$ supp H and $y \notin$ supp G.* **$\boldsymbol{\alpha}$-$\mathbf{T_0}$** *if for all $x, y \in X$ with $x \neq y$ there is $G \in \mathcal{T}_\alpha$ such that $x \hat{\in} G$, and $y \notin$ supp G.*

Clearly, the following implications are satisfied.

$$\alpha\text{-T}_2 \longrightarrow \alpha\text{-T}_1 \longrightarrow \alpha\text{-T}_0$$

Also, for $\beta > \alpha$ we have β-$\text{T}_i \longrightarrow \alpha$-$\text{T}_i$, for $i = 0, 1, 2$. The following definition is due to S.E. Rodabaugh [7].

DEFINITION 5.2. *A fts $(X, \mathcal{T})$ is α^*-Hausdorff if for all $x, y \in X$ with $x \neq y$, there are $G, H \in \mathcal{T}$ such that $G(x) \geq \alpha$, $H(y) \geq \alpha$ and $G \cap H = \mathbf{0}$.*

Clearly an α-Hausdorff space is α^*-Hausdorff.

We will see in the next two propositions that the α-Hausdorff and α^*-Hausdorff concepts agree in interpreservative and α-locally minimal spaces.

PROPOSITION 5.3. *Let $(X,\mathcal{T})$ be an interpreservative fts and let $\alpha \in]0,1]$. Then $(X,\mathcal{T})$ is α-Hausdorff if and only if it is α^*-Hausdorff.*

Proof. We only see the converse. Assume that $(X,\mathcal{T})$ is interpreservative and α^*-Hausdorff and let $x,z \in X$, $x \neq z$. Further, let the open fuzzy sets $U_x = \wedge\{U : U \in \mathcal{T}, U(x) \geq \alpha\}$, $V_z = \wedge\{V : V \in \mathcal{T}, V(z) \geq \alpha\}$. Then, obviously, $U_x \wedge V_z = \mathbf{0}$ and $\sigma(U_x) \geq \alpha$, $\sigma(V_z) \geq \alpha$ (notice that supp $U_x = \{x\}$ and supp $V_z = \{z\}$, since X is α^*-Hausdorff) and hence X is α-Hausdorff. □

PROPOSITION 5.4. *Let $\alpha \in]0,1]$ and let $(X,\mathcal{T})$ be an α-locally minimal space. Then $(X,\mathcal{T})$ is α-Hausdorff if and only if it is α^*-Hausdorff.*

Proof. We only see the converse.

Suppose X is α^*-Hausdorff. For each $a \in X$ we denote G_a the smallest nbhd of a, which is α-open by the hypothesis. Now consider $U_a = \wedge\{U : U \in \mathcal{T}, U(a) \geq \alpha\}$. We have $G_a \subset U_a$, and therefore supp $G_a = \{a\}$, since X is α^*-Hausdorff. Finally, let $x,z \in X$ with $x \neq z$. Then, $G_x \cap G_z = \mathbf{0}$ and thus $(X,\mathcal{T})$ is α-Hausdorff. □

There are some definitions of Hausdorfness depending on fuzzy points. One of these was given by D. Adnajevic.

DEFINITION 5.5. *The fts $(X,\mathcal{T})$ is Hausdorff (denoted Adn-H_2, here) if for all fuzzy points $p_x, q_y \in I^X$ with $x \neq y$, there are $G, H \in \mathcal{T}$, such that $p_x \hat{\in} G$, $q_y \hat{\in} H$ and $G \cap H = \mathbf{0}$.*

OBSERVETION 5.6: In [3] there is the following diagram which relates various fuzzy Hausdorff conditions:

$$\text{Adn-H}_2 \iff \text{GSW-H} \implies \text{SLS-H} \iff \text{LP-FT}_2 \implies \alpha^*\text{-Hausdorff}$$

Clearly a fts X is 1^*-Hausdorff if and only if it is Adn-H_2. Now, as a consequence of Propositions 5.3 and 5.4 we can complete and particularize the above diagram. In fact, the conditions Adn-H_2, GSW-H, SLS-H, LP-FT2, 1*-Hausdorff and 1-Hausdorff are equivalent for interpreservative fts , 1-locally minimal fts or locally minimal (ordinary) topological space.

Acknowledgement The authors are grateful to the referee for his valuable suggestions.

References

[1] C.L. Chang, *Fuzzy topological spaces*, J. Math. Anal. Appl. **24** (1968), 182–190.

[2] K.C. Chattopadhyay, R.N. Hazra, and S.K. Samanta, *Gradation of openness: fuzzy topology*, Fuzzy sets and Systems **49** (1992), 237–242.

[3] D.R. Cutler and I.L. Reilly, *A comparison of some Hausdorff notions in fuzzy topological spaces*, Computers Math. Applic. **19** (1990), no. 11, 97–104.

[4] S. Gottwald, *Fuzzy points and local properties of fuzzy topological spaces*, Fuzzy Sets and Systems **5** (1981), 199–201.

[5] V. Gregori and A. Vidal, *Gradations of openness and Chang's fuzzy topologies*, to appear.

[6] Pu Pao-Ming and Liu Ying-Ming, *Fuzzy topology. I. Neighbourhood structure of a fuzzy point and Moore-Smith convergence*, Journal of Mathematical Analysis and Applications **76** (1980), 571–599.

[7] S.E. Rodabaugh, *The Hausdorff separation axiom for fuzzy topological spaces*, Topology and its Applications **11** (1980), 319–334.

[8] A.P. Shostak, *On a fuzzy topological structure*, Rend. Circ. Mat. Palermo Ser II (1985), no. 11, 89–103.

[9] A.P. Shostak, *Two decades of fuzzy topology: basic ideas, notions, and results*, Uspeki Mat. Nau. **44** (1989), no. 6, 99–147.

Received October 24, 1997.