Countable Covers
and Uniform Closure

I. Garrido and F. Montalvo $^{(*)}$

Summary. - *In this paper we present, in a unified way, several re-
sults of uniform approximation for real-valued continuous and
uniformly continuous functions on a space X. We obtain all of
them by applying a general method of proof that involves certain
kind of countable covers of X, the so-called 2-finite covers. For
instance, if X is endowed with the weak uniformity given by a
vector lattice $\mathfrak{F}$ of real-valued functions on X containing all the
real constant functions then, using that method, we characterize
the uniform density of $\mathfrak{F}$ only in terms of the family $\mathfrak{F}$, improving
a previous result in this line.*

Throughout this paper X will denote either a completely regular
space or a uniform space, and $\mathfrak{F}$ will be a vector lattice of real-
valued continuous functions or uniformly continuous functions on X
containing all the real constant functions.

We shall use $C(X)$ (respectively $U(X)$) to denote the set of all-
real valued continuous functions (resp. uniformly continuous func-
tions) on X. Cozero-sets (resp. uniform cozero-sets) in X will be
sets of the form $\text{coz}(f) = \{x : f(x) \neq 0\}$ where $f \in C(X)$ (resp.
$f \in U(X)$)). For further notations and basic results about com-
pletely regular spaces and uniform spaces we refer to Gillman–Jerison
[6] and Engelking [1].

$^{(*)}$ Authors' addresses: I. Garrido, Departamento de Matemáticas. Universidad
de Extremadura, 06071 Badajoz, Spain, e-mail: igarrido@unex.es
F. Montalvo, Departamento de Matemáticas. Universidad de Extremadura,
06071 Badajoz, Spain, e-mail: montalvo@unex.es

The central notion here will be the concept of 2-*finite cover* of X. Recall that a countable cover $\{C_m\}_{m \in \mathbb{Z}}$ of X is said to be 2-finite if $C_m \cap C_{m'} = \emptyset$ when $|m - m'| > 1$.

This kind of covers appear in a natural way in the study of problems on uniform approximation for unbounded functions. Indeed, by means of them, we have obtained some uniform density results on $C(X)$ and $U(X)$ ([3], [5]), and we have also found internal conditions on the family $\mathfrak{F}$ making its uniform closure $\overline{\overline{\mathfrak{F}}}$ to have good algebraic properties ([4]). We have observed that all of these results can be derived following a similar scheme of proof involving suitable 2-finite covers. Such scheme is the following:

1. First, we consider a certain class $\mathcal{C}$ of 2-finite covers of X by cozero-sets or uniform cozero-sets.

2. Associated to this class $\mathcal{C}$, we define the following property on $\mathfrak{F}$:

$$\text{For every } \{C_m\}_{m \in \mathbb{Z}} \in \mathcal{C}, \text{ there is some } h \in \mathfrak{F} \qquad (\star)$$
$$\text{such that } |h(x) - m| < 2 \text{ when } x \in C_m.$$

3. Finally, we derive properties of $\overline{\overline{\mathfrak{F}}}$ whenever $\mathfrak{F}$ fulfills $(\star)$.

The aim of this paper is to exploit this scheme in order to present, in a unified way, the above mentioned results as well as some new ones.

1. Partitions of unity subordinated to 2-finite covers

First of all we need two lemmas about the existence of partitions of unity subordinated to 2-finite covers.

LEMMA 1.1. *Let X be a completely regular space. Then, for every 2-finite cover $\{C_m\}_{m \in \mathbb{Z}}$ of X by cozero-sets, there exists a continuous partition of unity $\{v_m\}_{m \in \mathbb{Z}}$ with $\mathrm{coz}(v_m) = C_m$, $m \in \mathbb{Z}$. In addition, there exists a function $\varphi \in C(X)$ such that $C_m = \{x : m - 1 < \varphi(x) < m + 1\}$, $m \in \mathbb{Z}$.*

Proof. Suppose that for every $m \in \mathbb{Z}$, $C_m = \text{coz}(g_m)$, where $g_m \in C(X)$ and $g_m \geq 0$. From the hypothesis on the cover $\{C_m\}_{m \in \mathbb{Z}}$, it follows that the functions $v_m = g_m / \sum_k g_k$, $m \in \mathbb{Z}$, define a continuous partition of unity.

Moreover if we take $\varphi = \sum_k k v_k$, then φ is a continuous function satisfying $C_m = \{x : m - 1 < \varphi(x) < m + 1\}$, $m \in \mathbb{Z}$, as we wanted. $\square$

Next we shall state the uniform version of last lemma.

LEMMA 1.2. *Let X be a uniform space. Then, for every uniform 2-finite cover $\{C_m\}_{m \in \mathbb{Z}}$ of X by uniform cozero-sets, there exists a uniformly continuous partition of unity $\{v_m\}_{m \in \mathbb{Z}}$ with $\text{coz}(v_m) = C_m$, $m \in \mathbb{Z}$. In addition, there exists a function $\varphi \in U(X)$ such that $C_m = \{x : m - 1 < \varphi(x) < m + 1\}$, $m \in \mathbb{Z}$.*

Proof. Here, we shall consider two cases.

Case 1. Suppose that the uniform space X is pseudometrizable that is, the uniformity on X is given by a unique pseudometric ρ that, without loss of generality, can be considered $\rho \leq 1$.

In this case every open set $G \subset X$ is the cozero-set of some uniformly continuous function, namely $G = \{x : \rho(x, X \setminus G) \neq 0\}$. Thus, if $\{C_m\}_{m \in \mathbb{Z}}$ is a 2-finite cover by cozero-sets in X, then we can write $C_m = \text{coz}(g_m)$ where $g_m(x) = \rho(x, X \setminus C_m)$, for every $x \in X$ and $m \in \mathbb{Z}$.

Now, as in last lemma, set $v_m = g_m / \sum_k g_k$, $m \in \mathbb{Z}$. In order to see that $\{v_m\}_{m \in \mathbb{Z}}$ is a uniformly continuous partition of unity, it is enough to check that the function $\sum_k g_k$ is uniformly continuous and has a positive lower bound. Indeed, since $\{C_m\}_{m \in \mathbb{Z}}$ is a uniform cover then, there is $r > 0$ such that for every $x \in X$, the ρ-ball with center x and radius r is contained in some C_m. Thus, for that m, we have $g_m(x) \geq r$ and then $\sum_k g_k(x) \geq r$. Moreover, since $\{C_m\}_{m \in \mathbb{Z}}$ is also a 2-finite cover then, if $\rho(x, y) \leq r$, we have $|\sum_k g_k(x) - \sum_k g_k(y)| = |(g_{m-1} + g_m + g_{m+1})(x) - (g_{m-1} + g_m + g_{m+1})(y)| \leq |g_{m-1}(x) - g_{m-1}(y)| + |g_m(x) - g_m(y)| + |g_{m+1}(x) - g_{m+1}(y)| \leq 3\rho(x, y)$. And therefore, $\sum_k g_k$ is a uniformly continuous function on X.

Finally, let $\varphi = \sum_k k v_k$. Again, φ is a continuous function with $C_m = \{x : m - 1 < \varphi(x) < m + 1\}$, $m \in \mathbb{Z}$. We finish as soon as we prove that φ is also uniformly continuous. And, this follows from the following facts:

1. For all $m \in \mathbb{Z}$, $|v_m(x) - v_m(y)| \leq (5/r^2)\rho(x, y)$ whenever $\rho(x, y) \leq r$.

2. When $\rho(x, y) \leq r$, we have $|\varphi(x) - \varphi(y)| \leq (10/r^2)\rho(x, y)$.

We shall see, for instance, the first statement because the other one follows in a similar way. Suppose $\rho(x, y) \leq r$, then

$$|v_m(x) - v_m(y)| = \left|(g_m/\sum_k g_k)(x) - (g_m/\sum_k g_k)(y)\right|$$

$$= \left|\frac{g_m(x) \cdot \sum_k g_k(y) - g_m(y) \cdot \sum_k g_k(x)}{\sum_k g_k(x) \cdot \sum_k g_k(y)}\right| \leq 1/r^2 \left|g_m(x) \cdot \sum_k g_k(y)\right.$$

$$\left. -g_m(y) \cdot \sum_k g_k(y) + g_m(y) \cdot \sum_k g_k(y) - g_m(y) \cdot \sum_k g_k(x)\right|$$

$$\leq 1/r^2 |g_m(x) - g_m(y)| \cdot \left|\sum_k g_k(y)\right| + 1/r^2 |g_m(y)| \cdot \left|\sum_k g_k(x)\right.$$

$$\left. -\sum_k g_k(y)\right| \leq (2/r^2)\rho(x, y) + (3/r^2)\rho(x, y).$$

Case 2. Suppose now that X is a uniform space not necessarily pseudometrizable.

Let $\{C_m\}_{m \in \mathbb{Z}}$ be a uniform 2-finite cover of X by uniform cozero-sets in X. From the uniformity of the cover, it follows the existence of a uniformly continuous pseudometric ρ and $r > 0$ such that, for every $x \in X$, the ρ-ball with center x and radius r is contained in some C_m.

Now, for each $m \in \mathbb{Z}$, set $C_m = \text{coz}(g_m)$ where $g_m \in U(X)$, and let ρ_m be the uniformly continuous pseudometric defined by

$$\rho_m(x, y) = |g_m(x) - g_m(y)|.$$

If we consider on X the uniformity generated by the countable family of pseudometrics $\{\rho_m, m \in \mathbb{Z}\} \cup \{\rho\}$, then this uniformity is pseudometrizable and we are, in fact, in case 1. Thus, we get the desired uniformly continuous partition of unity $\{v_m\}_{m \in \mathbb{Z}}$ and the uniformly continuous function φ, since every uniformly continuous function with this new uniformity is also uniformly continuous with the original one on X. $\square$

2. Uniform closure and 2-finite covers

Now we are going to apply the above mentioned general scheme to different classes of 2-finite covers of X. We start with covers that are given by a finite number of real-valued functions. And we shall see that for them, the property $(\star)$ on the family $\mathfrak{F}$ will be, in some sense, an internal characterization of the fact that a wide class of continuous functions belong to $\overline{\overline{\mathfrak{F}}}$.

THEOREM 2.1. *Let X be a completely regular space, $\mathfrak{F}$ a vector sub-lattice of $C(X)$ containing all the real constant functions and $g_1, \ldots,$ $g_n \in C(X)$. Let Y be a subset of $\mathbb{R}^n$ containing $(g_1, \ldots, g_n)(X) = \{(g_1(x), \ldots, g_n(x)) : x \in X\}$. Then,*

(a) $\varphi \circ (g_1, \ldots, g_n) \in \overline{\overline{\mathfrak{F}}}$ *for every $\varphi \in C(Y)$ if and only if $\mathfrak{F}$ sat-isfies the property $(\star)$ for the class of 2-finite covers $\{C_m\}_{m \in \mathbb{Z}}$ where $C_m = (g_1, \ldots, g_n)^{-1}(D_m)$ and $\{D_m\}_{m \in \mathbb{Z}}$ is an open 2-finite cover of Y;*

(b) $\varphi \circ (g_1, \ldots, g_n) \in \overline{\overline{\mathfrak{F}}}$ *for every $\varphi \in U(Y)$ if and only if $\mathfrak{F}$ satisfies the property $(\star)$ for the class of 2-finite covers $\{C_m\}_{m \in \mathbb{Z}}$ where $C_m = (g_1, \ldots, g_n)^{-1}(D_m)$ and $\{D_m\}_{m \in \mathbb{Z}}$ is an open uniform 2-finite cover of Y.*

Proof. (a) If $\{D_m\}_{m \in \mathbb{Z}}$ is an open 2-finite cover of Y, then from lemma 1.1 there is a function $\varphi \in C(Y)$ such that $D_m = \{y : m-1 < \varphi(y) < m+1\}$. Thus, if $x \in C_m$, then $(g_1, \ldots, g_n)(x) \in D_m$ and so, $|\varphi \circ (g_1, \ldots, g_n)(x) - m| \leq 1$.

Since $\varphi \circ (g_1, \ldots, g_n) \in \overline{\overline{\mathfrak{F}}}$, then there exists a function $h \in \mathfrak{F}$ with $|h(x) - \varphi \circ (g_1, \ldots, g_n)(x)| < 1$, for all $x \in X$. Thus,

$$|h(x) - m| < 2 \quad \text{if } x \in C_m$$

and hence $\mathfrak{F}$ satisfies the property $(\star)$.

To the converse, let $\epsilon > 0$ and $C_m = \{x : (m-1)\epsilon < \varphi \circ (g_1, \ldots, g_n)(x) < (m+1)\epsilon\}$, that is, $C_m = (g_1, \ldots, g_n)^{-1}(D_m)$ where $D_m = \{y : (m-1)\epsilon < \varphi(y) < (m+1)\epsilon\}$. Since φ is a continuous function on Y, then $\{D_m\}_{m \in \mathbb{Z}}$ is an open 2-finite cover of Y. Now, from the hypothesis, let $h \in \mathfrak{F}$ such that $|h(x) - m| < 2$ when $x \in C_m$. So, the function $\epsilon h \in \mathfrak{F}$ satisfies

$$|\epsilon h(x) - \epsilon m| < 2\epsilon \quad \text{if } x \in C_m.$$

Hence $|\epsilon h - \varphi \circ (g_1, \ldots, g_n)| < 3\epsilon$, and then $\varphi \circ (g_1, \ldots, g_n) \in \overline{\overline{\mathfrak{F}}}$.

(b) By mimicry the above case (a), we can obtain (b) but we have to apply lemma 1.2 instead of lemma 1.1.   □

Theorem 2.1 can be seen as a collection of results on uniform approximation given in a unified way. That is, we could obtain different results only taking different functions $g_1, \ldots, g_n$ or different subsets Y of $\mathbb{R}^n$. For instance, we have seen in [5] that if the functions $g_1, \ldots, g_n$ are in $\mathfrak{F}$ and $Y = (g_1, \ldots, g_n)(X)$, then (b) is an intermediate step to get a density theorem in $U(X)$ when X is endowed with the weak uniformity given by $\mathfrak{F}$.

Next, we are going to study another interesting particular case of theorem 2.1, namely when we have only one function g, and Y is the smallest real interval containing $g(X)$. We see how in this case it is possible to reduce and simplify the family of covers.

THEOREM 2.2. *Let X be a completely regular space, $\mathfrak{F}$ a vector sublattice of $C(X)$ containing all the real constant functions and $g \in C(X)$. Let Y be the smallest real interval containing $g(X)$. Then,*

(a) *$\varphi \circ g \in \overline{\overline{\mathfrak{F}}}$ for every $\varphi \in C(Y)$ if and only if $\mathfrak{F}$ satisfies the property $(\star)$ for the class of 2-finite covers $\{C_m\}_{m \in \mathbb{Z}}$ where $C_m = \{x : \alpha_{m-1} < g(x) < \alpha_{m+1}\}$ and $\{\alpha_m\}_{m \in \mathbb{Z}}$ is a nondecreasing sequence in $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ satisfying $\alpha_m - \alpha_{m-1} > 0$ provided α_m is a real number.*

(b) *$\varphi \circ g \in \overline{\overline{\mathfrak{F}}}$ for every $\varphi \in U(Y)$ if and only if $\mathfrak{F}$ satisfies the property $(\star)$ for the class of 2-finite covers $\{C_m\}_{m \in \mathbb{Z}}$ where $C_m = \{x : \alpha_{m-1} < g(x) < \alpha_{m+1}\}$ and $\{\alpha_m\}_{m \in \mathbb{Z}}$ is a nondecreasing sequence in $\overline{\mathbb{R}}$ for which there is $r > 0$ satisfying $\alpha_m - \alpha_{m-1} \geq r$ provided α_m is a real number.*

Proof. (a) From theorem 2.1, only one implication has to be proved. Indeed, we have to see that $\varphi \circ g \in \overline{\overline{\mathfrak{F}}}$ for all $\varphi \in C(Y)$, if $\mathfrak{F}$ satisfy property $(\star)$ for these covers.

Firstly, notice that we can restrict ourselves to the functions $\varphi \in C(Y)$ which are nondecreasing, because the linear subspace generated by them is uniformly dense in $C(Y)$ (see e.g. lemma 2.28 in [2]).

Thus, let φ be a nondecreasing function and let $\epsilon > 0$. For every $m \in \mathbb{Z}$, set $C_m = \{x : (m-1)\epsilon < \varphi \circ g(x) < (m+1)\epsilon\}$. Then, we can construct a sequence $\{\alpha_m\}_{m \in \mathbb{Z}}$ in $\overline{\mathbb{R}}$ such that,

1. If $m\epsilon \in \varphi(Y)$ then α_m satisfies $\varphi(\alpha_m) = m\epsilon$.

2. if $m\epsilon \notin \varphi(Y)$ and $m\epsilon \leq \inf \varphi(Y)$, then $\alpha_m = -\infty$.

3. if $m\epsilon \notin \varphi(Y)$ and $m\epsilon \geq \sup \varphi(Y)$, then $\alpha_m = +\infty$.

Thus, $\{\alpha_m\}_{m \in \mathbb{Z}}$ is a nondecreasing sequence defining a 2-finite cover $\{\widetilde{C}_m\}_{m \in \mathbb{Z}}$ of X. Now, by applying the hypothesis to this cover, there is a function $h \in \mathfrak{F}$ such that $|h(x) - m| < 2$ when $x \in \widetilde{C}_m$. Since for every $m \in \mathbb{Z}$, $C_m \subset \widetilde{C}_m$ then,

$$|\epsilon h(x) - \varphi \circ g(x)| < 3\epsilon \quad \text{if } x \in C_m$$

and therefore $\varphi \circ g \in \overline{\overline{\mathfrak{F}}}$.

(b) Again this follows easily by mimicry the above arguments. But, in this case, it is necessary to take into account that the linear subspace generated by the nondecreasing uniformly continuous functions on Y is uniformly dense in $U(Y)$ (see e.g. lemma 2.13 in [2]).   □

REMARK 2.3. *The condition in theorem 2.2(a), $\varphi \circ g \in \overline{\overline{\mathfrak{F}}}$ for every $\varphi \in C(Y)$ where Y is the smallest real interval containing $g(X)$, is equivalent to $\varphi \circ g \in \overline{\overline{\mathfrak{F}}}$ for every $\varphi \in C(I)$ and for every open real interval containing $g(X)$. This equivalence follows because every continuous functions on Y can be continuously extended to some open interval.*

REMARK 2.4. *Similarly, the condition in theorem 2.2(b), $\varphi \circ g \in \overline{\overline{\mathfrak{F}}}$ for every $\varphi \in U(Y)$ where Y is the smallest interval containig $g(X)$, is equivalent to $\varphi \circ g \in \overline{\overline{\mathfrak{F}}}$ for every $\varphi \in U(\mathbb{R})$. This follows now because every real interval is U-embedded in $\mathbb{R}$, that is, every uniformly continuous function on a real interval can be extended to a uniformly continuous function on $\mathbb{R}$. An interesting reference about U-embedding is the paper by Levy and Rice [7].*

The next part of the paper is devoted to apply our general scheme to some classes of 2-finite covers that are given in a more general

way, not restricted to a finite number of real-valued functions. More precisely we shall consider, for a completely regular space X, the class of all 2-finite covers by cozero-sets in X and the class of all 2-finite covers by cozero-sets of functions in $\mathfrak{F}$. And also we shall consider the corresponding uniform classes when X is a uniform space.

THEOREM 2.5 ([3]). *Let X be a completely regular space, $\mathfrak{F}$ a vector sublattice of $C(X)$ containing all the real constant functions. Then, $\mathfrak{F}$ is uniformly dense in $C(X)$ if, and only if, $\mathfrak{F}$ satifies the property $(\star)$ for the class of all 2-finite covers of X by cozero-sets.*

Proof. Suppose that $\mathfrak{F}$ is uniformly dense in $C(X)$. Let $\{C_m\}_{m \in \mathbb{Z}}$ be a 2-finite cover of X by cozero-sets, then from lemma 1.1, there exists $\varphi \in C(X)$ such that $|\varphi(x) - m| < 1$ if $x \in C_m$, $m \in \mathbb{Z}$.

Now, from the uniform density of $\mathfrak{F}$, there is $h \in \mathfrak{F}$ with $|h(x) - \varphi(x)| < 1$, for every $x \in X$. So,

$$|h(x) - m| < 2 \quad \text{if } x \in C_m$$

and hence $\mathfrak{F}$ satisfies the property $(\star)$.

The converse follows at once from theorem 2.2(a).   □

The uniform version of theorem 2.5 is the following result whose proof is completely analogous to the preceding one. Only note that now we have to apply lemma 1.2 and theorem 2.2(b).

THEOREM 2.6. *Let X be a uniform space, $\mathfrak{F}$ a vector sublattice of $U(X)$ containing all the real constant functions. Then, $\mathfrak{F}$ is uniformly dense in $U(X)$ if, and only if, $\mathfrak{F}$ satifies the property $(\star)$ for the class of all uniform 2-finite covers of X by uniform cozero-sets.*

We consider next the class of 2-finite covers of X by cozero-sets of functions in $\mathfrak{F}$. Thus, any property of $\overline{\mathfrak{F}}$ that we can derive just like that, can be considered as given in an internal way, because it will be obtained only in terms of the family $\mathfrak{F}$.

THEOREM 2.7 ([4]). *Let X be a completely regular space, $\mathfrak{F}$ a vector sublattice of $C(X)$ containing all the real constant functions. Then $\overline{\mathfrak{F}}$ is an inverse-closed subring of $C(X)$ if, and only if, $\mathfrak{F}$ satifies the property $(\star)$ for the class of all 2-finite covers of X by cozero-sets of functions in $\mathfrak{F}$.*

The proof of this result can be seen in [4] where we also show a further equivalent condition, namely, $\overline{\mathfrak{F}}$ is closed under composition with real-valued and continuous functions defined on open intervals, that is, $\mathfrak{F}$ satisfies theorem 2.2(a), for every function $f \in \overline{\overline{\mathfrak{F}}}$. Although this means that a wide number of continuous functions are in $\overline{\mathfrak{F}}$ this does not imply that $\overline{\mathfrak{F}}$ is all $C(X)$. Indeed, it is well-known that when X is not a Lindelöf nor almost compact space, then there exists a uniformly closed and inverse-closed subring of $C(X)$ which is not all $C(X)$ (see e.g. Hager and Johnson [9]).

However, we shall see that the uniform version of last theorem provides in fact a new result of uniform density in $U(X)$, for some special uniform spaces. We are talking about the spaces whose uniformity is the weak given by $\mathfrak{F}$, that is, the uniformity defined by the family of pseudometrics $\rho_g(x, y) = |g(x) - g(y)|$, $g \in \mathfrak{F}$.

THEOREM 2.8. *Let $\mathfrak{F}$ be a vector lattice of real-valued functions on X containing all the real constant functions. Suppose that X is endowed with the weak uniformity given by $\mathfrak{F}$. Then, $\mathfrak{F}$ is uniformly dense in $U(X)$ if, and only if, $\mathfrak{F}$ satifies the property $(\star)$ for the class of all uniform 2-finite covers of X by cozero-sets of functions in $\mathfrak{F}$.*

Proof. From theorem 2.6, it is clear that if $\mathfrak{F}$ is uniformly dense in $U(X)$ then $\mathfrak{F}$ satisfies the property $(\star)$ for this class of 2-finite covers of X.

To the converse, note that we can restrict ourselves to the approximation of nonnegative functions, since every $f \in U(X)$ is the difference of two nonnegative functions in $U(X)$, namely, $f^+ = \sup\{f, 0\}$ and $f^- = \sup\{-f, 0\}$.

Now, we shall use a uniform approximation result due to Hager ([8]). Namely, if X has the weak uniformity given by $\mathfrak{F}$, then every nonnegative function in $U(X)$ can be uniformly approximated by functions of the form $\sup_{n \in \mathbb{N}} h_n$, where $\{h_n\}_{n \in \mathbb{N}}$ is a finitely-equiuniform sequence of nonnegative functions in $\mathfrak{F}$, and the family $\{\text{coz}(h_n)\}_{n \in \mathbb{N}}$ is a uniform 2-finite cover of X. Recall that, in this context, a sequence $\{h_n\}_{n \in \mathbb{N}}$ is finitely-equiuniform if there exist $g_1, \ldots, g_k \in \mathfrak{F}$ such that for each $\epsilon > 0$ there exist $\delta > 0$ such that $|g_i(x) - g_i(y)| < \delta$, $i = 1, \ldots, k$, implies $|h_n(x) - h_n(y)| < \epsilon$, for all $n \in \mathbb{N}$.

Let $f = \sup_{n \in \mathbb{N}} h_n$ one of these functions. Then, from the hypothesis, there exists $u_1 \in \mathfrak{F}$ such that

$$|u_1(x) - n| < 2 \quad \text{if } x \in C_n = \operatorname{coz} h_n$$

(note that we are taking for $n < 0$, $h_n = 0$ and then $C_n = \operatorname{coz} h_n = \emptyset$).

In order to see that $f \in \overline{\overline{\mathfrak{F}}}$, it is enough to prove that $f + u_1 \in \overline{\overline{\mathfrak{F}}}$. So, let $\epsilon > 0$ and

$$\widetilde{C}_m = \{x : (m-1)\epsilon < (f + u_1)(x) < (m+1)\epsilon\}, \ m \in \mathbb{Z}.$$

Since $f + u_1$ is a uniformly continuous function, then $\{\widetilde{C}_m\}_{m \in \mathbb{Z}}$ is a uniform 2-finite cover of X. So, if each $\widetilde{C}_m$, $m \in \mathbb{Z}$, was a cozero-set of some function in $\mathfrak{F}$, then from the hypothesis, there would exist $u_2 \in \mathfrak{F}$ with

$$|u_2(x) - m| < 2 \quad \text{if } x \in \widetilde{C}_m.$$

Thus $u_1 + f \in \overline{\overline{\mathfrak{F}}}$, since $\epsilon u_2 \in \mathfrak{F}$ satisfies $|\epsilon u_2(x) - (u_1 + f)(x)| < 3\epsilon$.

We claim that every $\widetilde{C}_m$ is the cozero-set of some function in $\mathfrak{F}$ and then the proof is complete. Indeed, it is easy to check that for each $m \in \mathbb{Z}$, $\widetilde{C}_m$ meets only a finite number of members of $\{C_n\}_{n \in \mathbb{N}}$. Thus, if $\widetilde{C}_m$ meets $C_1, \ldots, C_N$ then,

$$\widetilde{C}_m = \bigcup_{i=1}^{N} (C_i \cap \widetilde{C}_m)$$

$$= \bigcup_{i=1}^{N} \{x \in \operatorname{coz}(h_i) : (m-1)\epsilon < (f + u_1)(x) < (m+1)\epsilon\}$$

$$= \bigcup_{i=1}^{N} \{x \in \operatorname{coz}(h_i) : (m-1)\epsilon$$
$$< (\sup\{h_{i-1}, h_i, h_{i+1}\} + u_1)(x) < (m+1)\epsilon\}$$

$$= \bigcup_{i=1}^{N} (\operatorname{coz}(h_i) \cap \{x : (m-1)\epsilon$$
$$< (\sup\{h_{i-1}, h_i, h_{i+1}\} + u_1)(x) < (m+1)\epsilon\}).$$

Since $\mathfrak{F}$ is a vector lattice containing all the real constant functions then the set

$$\{x : (m-1)\epsilon < (\sup\{h_{i-1}, h_i, h_{i+1}\} + u_1)(x) < (m+1)\epsilon\}$$

is a cozero-set in $\mathfrak{F}$. Finally, note that the family of cozero-sets in $\mathfrak{F}$ is closed under finite unions and finite intersections and therefore $\widetilde{C}_m$ is also a cozero-set in $\mathfrak{F}$. $\square$

As we said after theorem 2.1, we have proved in [5] another uniform density result in the same context of the above theorem. Namely, if X is endowed with the weak uniformity given by $\mathfrak{F}$, then $\mathfrak{F}$ is uniformly dense in $U(X)$ if and only if $\mathfrak{F}$ satisfies theorem 2.1(b) for every $g_1, \dots, g_n \in \mathfrak{F}$, $n \in \mathbb{N}$, where $Y = (g_1, \dots, g_n)(X)$. Thus, we can consider that our last theorem 2.8 improves the result in [5], since we characterize now the uniform density of $\mathfrak{F}$ only in terms of the family $\mathfrak{F}$.

On the other hand, we could ask if for this kind of uniform spaces theorem 2.6 and theorem 2.8 are in fact the same. That is if, in this case, the family of cozero-sets of functions in $\mathfrak{F}$ coincides with the family of all the uniform cozero-sets in X. The answer is "no" as easy examples show. For instance, let $X = \mathbb{R}$ and let $\mathfrak{F}$ be the vector lattice of the usual uniformly continuous polygonals for which there exists $r > 0$ so that the first coordinates of their vertices are contained in $\{nr : n \in \mathbb{Z}\}$. Then $\mathfrak{F}$ gives the usual uniformity on $\mathbb{R}$, it is uniformly dense in $U(\mathbb{R})$ and not every open set in $\mathbb{R}$ is the cozero-set of some function in $\mathfrak{F}$.

In closing, we note that we do not know whether an analogue of theorem 2.8 can be obtained for general uniform spaces.

References

[1] R. Engelking, *General topology*, PWN-Polish Scientific, Warsaw, 1977.

[2] I. Garrido, *Aproximación uniforme en espacios de funciones continuas*, Ph.D. thesis, Dpto. Matemáticas, Uex, 24, 1990.

[3] I. Garrido and F. Montalvo, *Uniform approximation theorems for real-valued continuous functions*, Topology Appl. **45** (1992), 145–155.

[4] I. Garrido and F. Montalvo, *Algebraic properties of the uniform closure on spaces of continuous functions*, Papers on General Topology and Applications, Ann. New York Acad. Sci., vol. 788, 1996, pp. 101–107.

[5] I. Garrido and F. Montalvo, *Some density results for uniformly continuous functions*, Topology Appl. **77** (1997), 131–137.

[6] L. GILLMAN AND M. JERISON, *Rings of continuous functions*, Springer-Verlag, New York, 1976.

[7] R. LEVY AND M. D. RICE, *Tecniques and examples in U-embedding*, Topology Appl. **22** (1986), 157–174.

[8] A. W. HAGER, *Approximation theorems for uniformly continuous functions*, Trans. Amer. Math. Soc. **236** (1978), 263–273.

[9] A. W. HAGER AND D. G. JOHNSON, *A note on certain subalgebras of* $C(X)$, Canad. J. Math. **20** (1968), 389–393.