Rend. Istit. Mat. Univ. Trieste
Suppl. Vol. XXX, 75–85 (1999)

Old and New Results on Quasi-uniform Extension

Ákos Császár (*)

Summary. - *According to [17] or [12],* $\mathcal{U}$ *is a quasi-uniformity on a set* X *iff it is a filter on* $X \times X$*, the diagonal* $\Delta = \{(x,x) : x \in X\} \subset U$ *for* $U \in \mathcal{U}$ *(i.e.* $\mathcal{U}$ *is composed of entourages on* X*), and, for each* $U \in \mathcal{U}$*, there is* $U' \in \mathcal{U}$ *such that* ${U'}^2 = U' \circ U' = \{(x,z) : \exists y \text{ with } (x,y),(y,z) \in U'\} \subset U$*.*

The restriction $\mathcal{U} \mid X_0$ *to* $X_0 \subset X$ *of the quasi-uniformity* $\mathcal{U}$ *on* X *is composed of the sets* $U \mid X_0 = U \cap (X_0 \times X_0)$ *for* $U \in \mathcal{U}$*; it is a quasi-uniformity on* X_0*.*

Let $Y \supset X$*,* $\mathcal{W}$ *be a quasi-uniformity on* Y*;* $\mathcal{W}$ *is an extension of the quasi-uniformity* $\mathcal{U}$ *on* X *if* $\mathcal{W} \mid X = \mathcal{U}$*.*

The purpose of the present paper is to give a survey on results, due mainly to Hungarian topologists, concerning extensions of quasi-uniformities.

1. Preliminaries

In the following, $\mathcal{U}$ and $\mathcal{W}$ will always denote quasi-uniformities on X and $Y \supset X$, respectively. We shall write $Z = Y - X$.

The *conjugate* of $\mathcal{U}$ is the quasi-uniformity $\mathcal{U}^- = \{U^{-1} : U \in \mathcal{U}\}$ where $U^{-1} = \{(x,y) : (y,x) \in U\}$.

The quasi-uniformity $\mathcal{U}$ *induces* a topology $\mathcal{T} = \mathcal{U}^{tp}$ on X for which the neighbourhood filter of $x \in X$ is composed of the sets $U(x)$ for $U \in \mathcal{U}$; here $U(A) = \{y \in X \exists x \in A \text{ with } (x,y) \in U\}$

(*) Author's address: H-1052 Budapest, Pàrizsi utca 6/a
Research supported by Hungarian Foundation for Scientific Research, grant no. T 016094

whenever $A \subset X$ and $U \subset X \times X$, and $U(x) = U(\{x\})$ if $x \in X$. We write $\mathcal{U}^{-tp}$ for $(U^-)^{tp}$.

A quasi-uniformity U' on X is *finer* than $\mathcal{U}$ if $\mathcal{U} \subset \mathcal{U}'$; the finest $\mathcal{U}$ is *discrete* (i.e. $\Delta \in \mathcal{U}$). $\mathcal{U}$ is *coarser* than $\mathcal{U}'$ iff $\mathcal{U}'$ is finer than $\mathcal{U}$. Each topology admits a finest quasi-uniformity inducing it, its *fine quasi-uniformity.*

For an extension $\mathcal{W}$ on Y of $\mathcal{U}$, let us write

$$\mathcal{T} = \mathcal{W}^{tp}, \quad \mathcal{T}^- = \mathcal{W}^{-tp}, \quad \mathcal{V} = \mathcal{W} \mid Z.$$

Let $\mathfrak{s}(p)$ be the *trace* $\mathfrak{b}(p) \mid X$ on X of the $\mathcal{T}$-neighbourhood filter $\mathfrak{b}(p)$ of $p \in Z$ ($\varnothing \in \mathfrak{s}(p)$ may happen), $\mathfrak{s}^-(p)$ be the same for $\mathcal{T}^-$ instead of $\mathcal{T}$, $\mathfrak{t}(x)$ be the trace on Z of $\mathfrak{b}(x)$ for $x \in X$.

Suppose X, Y, $\mathcal{U}$ and some combination of $\mathcal{T}$, $\mathcal{T}^-$, $\mathfrak{s}$, $\mathfrak{s}^-$, $\mathcal{V}$ are given. An extension $\mathcal{W}$ is said to be *compatible* with this combination iff it induces the given elements of the combination. If $\mathcal{T}$ is given, it is always a topology on Y; similarly, a given $\mathcal{T}^-$ is a topology on Y, $\mathfrak{s}$ and $\mathfrak{s}^-$ are mappings from Z to the collection $\mathrm{Fil}(X)$ of all (proper or improper) filters in X, and $\mathcal{V}$ is a quasi-uniformity on Z. If $\mathcal{T}$ or $\mathcal{T}^-$ is given, $\mathfrak{b}(a)$ and $\mathfrak{b}^-(a)$ are the $\mathcal{T}$- and $\mathcal{T}^-$-neighbourhood filter of $a \in Y$, respectively, and $\mathfrak{s}(p)$ ($\mathfrak{s}^-(p)$) is the trace on X of $\mathfrak{b}(p)$ ($\mathfrak{b}^-(p)$) for $p \in Z$.

2. The case $(\mathcal{U}, \mathfrak{s})$

We look for an extension $\mathcal{W}$ compatible with $(\mathcal{U}, \mathfrak{s})$, i.e. such that $\mathcal{W} \mid X = \mathcal{U}$ and the trace of the $\mathcal{W}^{tp}$-neighbourhood filter of $p \in Z$ is a given filter $\mathfrak{s}(p)$ in X. A filter $\mathfrak{r}$ in X is said to be *$\mathcal{U}$-round* iff, for $R \in \mathfrak{r}$ there are $U \in \mathcal{U}$ and $R' \in \mathfrak{r}$ such that $U(R') \subset R$.

THEOREM 2.1. *[4]. There is an extension compatible with $(\mathcal{U}, \mathfrak{s})$ iff each filter $\mathfrak{s}(p)$ $(p \in Z)$ is $\mathcal{U}$-round.*

A topology on Y is a *loose* extension iff X is open for this topology and the subspace Z is discrete.

COROLLARY 2.2. *[5]. If the above condition is fullfilled, then there is a finest extension compatible with $(\mathcal{U}, \mathfrak{s})$ for which $\mathcal{T}$ is a loose extension of $\mathcal{U}^{tp}$ and $\mathcal{V}$ is discrete.*

In the case considered now, there is, in general, no coarsest compatible extension ([9]).

3. The case $(\mathcal{U}, \mathfrak{s}^-, \mathfrak{s})$

Now, besides $\mathcal{U}$, mappings $\mathfrak{s}, \mathfrak{s}^- : Z \to \mathrm{Fil}(X)$ are given and we look for a $\mathcal{W}$ such that $\mathcal{W} \mid X = \mathcal{U}$ and $\mathfrak{s}(p)$ $(\mathfrak{s}^-(p))$ is the trace on X of the $\mathcal{W}^{tp}$ $(\mathcal{W}^{-tp})$-neighbourhood filter of $p \in Z$.

A pair $(\mathfrak{r}^-, \mathfrak{r})$ of filters in X is said to be $(\mathcal{U}-)$*Cauchy* iff,for $U \in \mathcal{U}$, there are sets $R^- \in \mathfrak{r}^-$ and $R \in \mathfrak{r}$ satisfying $R^- \times R \subset U$.

THEOREM 3.1. *[8]. There is an extension compatible with $(\mathcal{U}, \mathfrak{s}^-, \mathfrak{s})$ iff each $\mathfrak{s}^-(p)$ is $\mathcal{U}^-$-round, each $\mathfrak{s}(p)$ is $\mathcal{U}$-round, and each pair $(\mathfrak{s}^-(p), \mathfrak{s}(p))$ is Cauchy ($p \in Z$).*

Let $\mathcal{T}^1$ and $\mathcal{T}^{-1}$ be topologies on Y. We say that the *bitopology* $(\mathcal{T}^{-1}, \mathcal{T}^1)$ is *biregular* (regular in [13]) iff each $\mathcal{T}^i$-neighbourhood of a point contains a $\mathcal{T}^{-i}$-closed $\mathcal{T}^i$-neighbourhood of the given point.

COROLLARY 3.2. *[8]. If the conditions in 3.1 are fullfilled, there is a finest extension compatible with $(\mathcal{U}, \mathfrak{s}^-, \mathfrak{s})$; for this $\mathcal{W}$, the bitopology $(\mathcal{T}^-, \mathcal{T})$ is the finest biregular bitopology such that $\mathcal{U}^{-tp} = \mathcal{T}^- \mid X$, $\mathcal{U}^{tp} = \mathcal{T} \mid X$ and the trace of the $\mathcal{T}^-$-($\mathcal{T}$-)neighbourhood filter of $p \in Z$ is equal to $\mathfrak{s}^-(p)$ $(\mathfrak{s}(p))$.*

The bitopology $(\mathcal{T}^-, \mathcal{T})$ described here is said to be the *fine biregular extension of $(\mathcal{U}^{-tp}, \mathcal{U}^{tp})$ associated with $(\mathfrak{s}^-, \mathfrak{s})$*.

In general, the is no coarsest extension in this case ([7]).

4. The case $(\mathcal{U}, \mathcal{T})$

We look now for a $\mathcal{W}$ satisfying $\mathcal{W} \mid X = \mathcal{U}$, $\mathcal{W}^{tp} = \mathcal{T}$ for a given topology $\mathcal{T}$ on Y.

THEOREM 4.1. *[10]. There is an extension compatible with $(\mathcal{U}, \mathcal{T})$ iff $\mathcal{U}^{tp} = \mathcal{T} \mid X$, the trace on X of each $\mathcal{T}$-neighbourhood filter is $\mathcal{U}$-round and $\mathcal{U} \subset \mathcal{U}(\mathcal{T}) \mid X$ for the fine quasi-uniformity $\mathcal{U}(\mathcal{T})$ of $\mathcal{T}$.*

Unfortunately, there is no useful construction for $\mathcal{U}(\mathcal{T})$; therefore it is interesting to look for necessary and for sufficient conditions.

Let us say that $\varphi : Z \to exp\ X$ is a *Z-selector* if $\varphi(p) \in \mathfrak{s}(p)$ for $p \in Z$ and the trace $\mathfrak{s}(p)$ of the $\mathcal{T}$-neighbourhood filter of p. Denote by Φ the collection of all Z-selectors. The collection Ψ of the X-selectors $\psi : X \to exp\ Z$ is defined similarly with the condition $\psi(x) \in \mathfrak{t}(x)$ for $x \in X$ and the trace $\mathfrak{t}(x)$ on Z of the $\mathcal{T}$-neighbourhood filter of $x \in X$.

PROPOSITION 4.2. *[2]. For the existence of an extension compatible with $(\mathcal{U}, \mathcal{T})$ it is necessary that $\mathcal{U}^{tp} = \mathcal{T} \mid X$, each $\mathfrak{s}(p)$ (with the above meaning of $\mathfrak{s}(p)$) $(p \in Z)$ is $\mathcal{U}$-round and, for each $U \in \mathcal{U}$, there are $\varphi \in \Phi$ and $\psi \in \Psi$ such that $\varphi(\psi(x)) \subset U(x)$ for $x \in X$.*

Here $\varphi(A) = \bigcup\{\varphi(p) : p \in A\}$ whenever $A \subset Z$, and $\varphi(B)$ is similarly defined for $B \subset X$.

Suppose now that $\mathcal{T}$ is a *strict* extension of $\mathcal{U}^{tp}$, i.e. the sets $s(G) = G \cup \{p \in Z : G \in \mathfrak{s}(p)\}$, where G is $\mathcal{U}^{tp}$-open, constitute a base for $\mathcal{T}$. In this case, the above necessary condition can be formulated in another way. For this purpose, let us consider a family $\{\mathfrak{s}(p) : p \in Z\}$ of filters in X and say that it is *uniformly tame* iff, for $U \in \mathcal{U}$ and $x \in X$, there are $\varphi \in \Phi$ and $\mathcal{U}^{tp}$-open sets $G(x)$ such that $x \in G(x)$ and $\varphi(p) \subset U(x)$ whenever $G(x) \in \mathfrak{s}(p)$.

Now J. Gerlits has formulated the following result:

COROLLARY 4.3. *[4]. If the topology $\mathcal{T}$ is a strict extension of $\mathcal{U}^{tp}$, then the existence of an extension compatible with $(\mathcal{U}, \mathcal{T})$ implies that the family $\{\mathfrak{s}(p) : p \in Z\}$ of the trace filters is uniformly tame.*

PROBLEM 4.4: Are the conditions in 4.2 sufficient for the existence of an extension compatible with $(\mathcal{U}, \mathcal{T})$, at least in the case of a strict extension $\mathcal{T}$?

In the present case, the usual situation holds:

COROLLARY 4.5. *[4]. If there is a extension compatible with $(\mathcal{U}, \mathcal{T})$ then there is a finest one.*

On the other hand, there is, in general, no coarsest extension compatible with $(\mathcal{U}, \mathcal{T})$ ([9]).

Let us now mention some sufficient conditions.

THEOREM 4.6. *[10]. If X is $\mathcal{T}$-closed and $\mathcal{U}^{tp} = \mathcal{T} \mid X$ then there exists an extension compatible with $(\mathcal{U}, \mathcal{T})$.*

THEOREM 4.7. *[10]. In the case X is $\mathcal{T}$-open, an extension compatible with $(\mathcal{U},\mathcal{T})$ exists iff $\mathcal{U}^{tp} = \mathcal{T} \mid X$ and each $\mathfrak{s}(p)$ $(p \in Z)$ is $\mathcal{U}$-round.*

The following statement was proved in [4] for the case of strict extensions and in [10] in the general case:

THEOREM 4.8. *[10]. If $\mathcal{U}^{tp} = \mathcal{T} \mid X$, and each $\mathfrak{s}(p)$ $(p \in Z)$ is $\mathcal{U}$-round and $\mathcal{U}$-stable then there exists an extension compatible with $(\mathcal{U},\mathcal{T})$.*

Here a filter $\mathfrak{r}$ on X is said to be *$\mathcal{U}$-stable* iff $U \in \mathcal{U}$ implies $\bigcap\{U(R) : R \in \mathfrak{r}\} \in \mathfrak{r}$.

5. The case $(\mathcal{U},\mathcal{T}^-,\mathcal{T})$

We are looking for a $\mathcal{W}$ satisfying $\mathcal{W} \mid X = \mathcal{U}$, $\mathcal{W}^{-tp} = \mathcal{T}^-$, $\mathcal{W}^{tp} = \mathcal{T}$ for given topologies $\mathcal{T}^-$, $\mathcal{T}$ on Y.

THEOREM 5.1. *[7]. If there exists an extension compatible with $(\mathcal{U},\mathcal{T}^-,\mathcal{T})$ then $\mathcal{U}^{-tp} = \mathcal{T}^- \mid X$, $\mathcal{U}^{tp} = \mathcal{T} \mid X$, each trace filter $\mathfrak{s}^-(p)$ $(\mathfrak{s}(p))$ of the $\mathcal{T}^-$-($\mathcal{T}$-) neighbourhood filter of $p \in Z$ is $\mathcal{U}^-$-round ($\mathcal{U}$-round) and each pair $(\mathfrak{s}^-(p))$, $(\mathfrak{s}(p))$ is Cauchy.*

THEOREM 5.2. *[7]. If the conditions in 5.1 are fullfilled and the bitopology $(\mathcal{T}^-,\mathcal{T})$ is the fine biregular extension of $(\mathcal{U}^{-tp},\mathcal{U}^{tp})$, then there exists an extension compatible with $(\mathcal{U},\mathcal{T}^-,\mathcal{T})$.*

An obvious necessary condition for the existing of a compatible extension in this case is:

(*) there is a quasi-uniformity $\mathcal{W}'$ on Y such that

$$\mathcal{W}'^{-tp} = \mathcal{T}^-, \quad \mathcal{W}'^{tp} = \mathcal{T},\ \mathcal{W}'^- \mid X = \mathcal{U}^-, \quad \mathcal{W}' \mid X = \mathcal{U}.$$

Now a sufficient condition can be obtained with the help of the following property: a family $\{(\mathfrak{s}^-(p), \mathfrak{s}(p)) : p \in Z\}$ of filters pairs in X is *uniformly weakly concentrated* iff, for $U \in \mathcal{U}$, there is $U' \in \mathcal{U}$ such that $K, L \in \mathfrak{s}(p), K^-, L^- \in \mathfrak{s}^-(p)$ and $K^- \times K \subset U'$, $L^- \times L \subset U'$ imply $K^- \times L \subset U$.

THEOREM 5.3. *[10]. If the conditions 5.1 and (*) are fullfilled and the family* $\{(\mathfrak{s}^-(p), \mathfrak{s}(p)) : p \in Z\}$ *is uniformly weakly concentrated then there exists an extension compatible with* $(\mathcal{U}, \mathcal{T}^-, \mathcal{T})$.

COROLLARY 5.4. *[7]. If there is an extension compatible with* $(\mathcal{U}, \mathcal{T}^-, \mathcal{T})$ *then there is a finest one.*

In general, there is no coarsest compatible extension in this case ([7]).

6. The case $(\mathcal{U}, \mathcal{V}, \mathcal{T})$

Suppose now that, besides the quasi-uniformity $\mathcal{U}$ on X and the topology $\mathcal{T}$ on Y, a quasi-unifomity $\mathcal{V}$ on Z is given and we look for a $\mathcal{W}$ satisfying $\mathcal{W} \mid X = \mathcal{U}$, $\mathcal{W} \mid Z = \mathcal{V}$, $\mathcal{W}^{tp} = \mathcal{T}$. This is a special case of the problem of looking for *simultaneous extensions.*

THEOREM 6.1. *[2]. An extension compatible with* $(\mathcal{U}, \mathcal{V}, \mathcal{T})$ *exists iff* $\mathcal{U}^{tp} = \mathcal{T} \mid X$, $\mathcal{V}^{tp} = \mathcal{T} \mid Z$, $\mathfrak{s}(p)$ *is* $\mathcal{U}$*-round for* $p \in Z$, $\mathfrak{t}(x)$ *is* $\mathcal{V}$*-round for* $x \in X$, *for each* $U \in \mathcal{U}$ *there are* $\varphi \in \Phi$, $\psi \in \Psi$, $V \in \mathcal{V}$ *such that* $\varphi(V(\psi(x))) \subset U(x)$, *and, for each* $V \in \mathcal{V}$, *there are* $\varphi \in \Phi$, $\psi \in \Psi$, $U \in \mathcal{U}$ *such that* $\psi(U(\varphi(p))) \subset V(p)$ *for* $p \in Z$.

COROLLARY 6.2. *[2]. If* X *is a* $\mathcal{T}$*-open or* $\mathcal{T}$*-closed then the conditions involving* Φ *and* Ψ *can be omitted from 6.1.*

COROLLARY 6.3. *[2]. If there is an extension compatible with* $(\mathcal{U}, \mathcal{V}, \mathcal{T})$ *then there is a finest one.*

In general, there is no coarsest extension in this case ([2]).

7. Questions of density

We say that the extension $\mathcal{W}$ is *dense* iff X is $\mathcal{T}$-dense; it is *doubly dense* iff X is both $\mathcal{T}$-dense and $\mathcal{T}^-$-dense; it is *firm* iff X is $\mathcal{T}^*$-dense for $\mathcal{T}^* = sup(\mathcal{T}, \mathcal{T}^-)$.

In any of the cases 2 to 6, there is a dense extension iff there is an extension and $\mathfrak{s}(p)$ is a proper filter for each $p \in Z$.

There is a doubly dense extension compatible with $(\mathcal{U}, \mathfrak{s}^-, \mathfrak{s})$ or $(\mathcal{U}, \mathcal{T}^-, \mathcal{T})$ iff there is an extension and all $\mathfrak{s}^-(p)$ and $\mathfrak{s}(p)$ are proper filters for $p \in Z$. A doubly dense extension compatible with $(\mathcal{U}, \mathfrak{s})$ exists iff there is a compatible extension and each filter $\mathfrak{s}(p)$ is *D-Cauchy* (Cauchy in [11]), i.e. it is a proper filter admitting a proper *cofilter* $\mathfrak{s}^-(p)$ such that $(\mathfrak{s}^-(p), \mathfrak{s}(p))$ is a Cauchy filter pair.

PROBLEM 7.1: Is there a similar statement in the cases $(\mathcal{U}, \mathcal{T})$ or $(\mathcal{U}, \mathcal{V}, \mathcal{T})$?

A pair of filters $(\mathfrak{r}^-, \mathfrak{r})$ is said to be *linked* iff $R^- \in \mathfrak{r}^-$, $R \in \mathfrak{r}$ imply $R^- \cap R \neq \emptyset$.

THEOREM 7.2. *[8]. There exists a firm extension compatible with* $(\mathcal{U}, \mathfrak{s}^-, \mathfrak{s})$ *or* $(\mathcal{U}, \mathcal{T}^-, \mathcal{T})$ *iff there is a compatible extension and each filter pair* $(\mathfrak{s}^-(p), \mathfrak{s}(p))$ *is linked. This extension is unique.*

If $\mathfrak{r}$ is a filter in X, the $\mathcal{U}$-envelope of $\mathfrak{r}$ is the filter composed of all sets $U(R)$ for $U \in \mathcal{U}$, $R \in \mathfrak{r}$. A filter is said to be *firmly D-Cauchy* iff it is the $\mathcal{U}$-envelope of some filter $\mathfrak{r}$ such that $(\mathfrak{r}, \mathfrak{r})$ is a Cauchy filter pair.

THEOREM 7.3. *[1]. There is a firm extension compatible with* $(\mathcal{U}, \mathfrak{s})$ *iff each filter* $\mathfrak{s}(p)$ $(p \in Z)$ *is* $\mathcal{U}$*-round and firmly* $D-Cauchy$*. This extension is unique.*

COROLLARY 7.4. *[1]. There is a firm extension compatible with* $(\mathcal{U}, \mathcal{T})$ *iff there is a firm extension compatible with* $(\mathcal{U}, \mathfrak{s})$ *and* $\mathcal{T}$ *is a strict extension of* $\mathcal{U}^{tp}$*; this extension is unique.*

8. Transitive extensions

A quasi-uniformity is *transitive* iff it admints (as a filter) a base composed of transitive entourages. For a given topology, there is finest transitive quasi-uniformity inducing it, its *fine transitive quasi-uniformity.*

THEOREM 8.1. *[5]. There is a transitive extension compatible with* $(\mathcal{U}, \mathcal{T})$ *iff* $\mathcal{U}$ *is transitive,* $\mathcal{U}^{tp} = \mathcal{T} \mid X$, $\mathfrak{s}(p)$ *is* $\mathcal{U}$*-round for* $p \in Z$*, and* $\mathcal{U} \subset \mathcal{U}'(\mathcal{T}) \mid X$ *for the fine transitive quasi-uniformity* $\mathcal{U}'(\mathcal{T})$ *of* $\mathcal{T}$.

THEOREM 8.2. *[6]. If* $\mathcal{U}$ *is transitive, each* $\mathfrak{s}^-(p)$ *is* $\mathcal{U}^-$*-round, each* $\mathfrak{s}(p)$ *is* $\mathcal{U}$*-round, each pair* $(\mathfrak{s}^-(p), \mathfrak{s}(p))$ *is* $\mathcal{U}$*-Cauchy* $(p \in Z)$ *and*

$(\mathcal{T}^-, \mathcal{T})$ is the fine biregular extension of $(\mathcal{U}^{tp}, \mathcal{U}^{tp})$ then there exists a transitive extension compatible with $(\mathcal{U}, \mathcal{T}^-, \mathcal{T})$.

Let us say that (U, V) is a *regular pair* iff $U \in \mathcal{U}$, $V \in \mathcal{V}$, U and V are transitive entourages, and there are $\varphi \in \Phi$, $\psi \in \Psi$ such that $\varphi(V(\psi(x))) \subset U(x)$, $\psi(U(\varphi(p))) \subset V(p)$ for $x \in X$, $p \in Z$.

THEOREM 8.3. *[16]. There exists a transitive extension compatible with $(\mathcal{U}, \mathcal{V}\mathcal{T})$ iff $\mathcal{U}^{tp} = \mathcal{T} \mid X$, $\mathcal{V}^{tp} = \mathcal{T} \mid Z$, the filter $\mathfrak{t}(x)$ is $\mathcal{V}$-round for $x \in X$, $\mathfrak{s}(p)$ is $\mathcal{U}$-round for $p \in Z$, and the members U in the regulars pairs (U, V) constitute a base of $\mathcal{U}$, those $\mathcal{V}$ constitute a base of V.*

9. Totally bounded extensions

A quasi-uniformity $\mathcal{U}$ is said to be *totally bounded* iff, for $U \in \mathcal{U}$, there is a finite cover of X whose member A satisfy $A \times A \subset U$.

THEOREM 9.1. *[5]. There is a totally bounded extension compatible with $(\mathcal{U}, \mathcal{T})$ iff $\mathcal{U}$ is totally bounded, $\mathcal{U}^{tp} = \mathcal{T} \mid X$, and each $\mathfrak{s}(p)$ is $\mathcal{U}$-round.*

COROLLARY 9.2. *[5]. Under the above hypotheses, there is a finest totally bounded extension compatible with $(\mathcal{U}, \mathcal{T})$.*

10. Finite extensions

If the set Z is finite, one can formulate more precise statements in many cases.

Let us say that a filter $\mathfrak{r}$ in X is *tame* if, for $U \in \mathcal{U}$, there is $R \in \mathfrak{r}$ such that $R \subset U(x)$ whenever $\mathfrak{r} \to x$ for the topology U^{tp}.

The following sufficient condition can be found in [4] for the case when $\mathcal{T}$ is a strict extension of $\mathcal{U}^{tp}$ and in [6] in the general case:

THEOREM 10.1. *If Z is finite, $\mathcal{U}^{tp} = \mathcal{T} \mid X$, each $\mathfrak{s}(p)$ is $\mathcal{U}$-round and $\mathcal{U}$-tame, then there is an extension compatible with $(\mathcal{U}, \mathcal{T})$.*

Let $\tilde{X}_p$ denote the set of all points $x \in X$ such that $p \in \cap\mathfrak{r}(x)$ $(p \in Z)$.

Theorem 10.2. *[14]. If Z is finite, there is an extension compatible with $(\mathcal{U}, \mathcal{T})$ iff $\mathcal{U}^{tp} = \mathcal{T} \mid X$, each $\mathfrak{s}(p)$ is $\mathcal{U}$-round and, for $U \in \mathcal{U}$, there is $S \in \mathfrak{s}(p)$ such that $x \in \tilde{X}_p$ implies $S \subset U(x)$ $(p \in Z)$.*

If filters $\mathfrak{s}^-(p)$ are prescribed for $p \in Z$, we can give a complete description of the extensions compatible with $(\mathcal{U}, \mathfrak{s}^-, \mathcal{T})$.

Theorem 10.3. *[14]. If Z is finite, there exists an extension compatible with $(\mathcal{U}, \mathfrak{s}^-, \mathcal{T})$ iff $\mathcal{U}^{tp} = \mathcal{T} \mid X$ and, for any $p \in Z$, the filter $\mathfrak{s}(p)$ is $\mathcal{U}$-round, $\mathfrak{s}^-(p)$ is $\mathcal{U}^-$-round, $\cap \mathfrak{s}^-(p) = \tilde{X}_p$, the pair $(\mathfrak{s}^-(p), \mathfrak{s}(p))$ is $\mathcal{U}$-Cauchy, finally, for p, $q \in Z$ $\mathfrak{r}(p) \subset \mathfrak{r}(q)$ implies $\mathfrak{s}^-(q) \subset \mathfrak{s}^-(p)$, and $\mathfrak{r}(p) \not\subset \mathfrak{r}(q)$ implies the existence of $S^- \in \mathfrak{s}^-(q)$ and $S \in \mathfrak{s}(p)$ with $S^- \cap S = \emptyset$.*

Corollary 10.4. *[14]. Under the above hypotheses, the extension compatible with $(\mathcal{U}, \mathfrak{s}^-, \mathcal{T})$ is unique.*

Corollary 10.5. *[14]. If Z is finite, the extensions compatible with $(\mathcal{U}, \mathcal{T})$ constitute a distributive lattice with $\cap$ for inf, containing a finest element but possibly without a coarsest element.*

In some cases, however, there is a coarsest compatible extension; let us say that a filter $\mathfrak{r}$ is *strictly tame* for $\mathcal{U}$ iff there are, for a given $U \in \mathcal{U}$, an entourage $U' \in \mathcal{U}$ and a set $R \in \mathfrak{r}$ such that $U'(x) \in \mathfrak{r}$ implies $R \subset U(x)$.

Theorem 10.6. *[6]. If Z is finite, $\mathcal{T}$ is a strict extension of $\mathcal{U}^{tp}$, and each $\mathfrak{s}(p)$ is round and striclty tame for $\mathcal{U}$, then there is a coarsest extension compatible with $(\mathcal{U}, \mathcal{T})$.*

Let us say that the extension $\mathcal{W}$ is *uniformly strict* (strict in [4]) iff $W \in \mathcal{W}$ implies the existence of $W' \in \mathcal{W}$ such that $s(W'(a) \cap X) \subset W(a)$ for $a \in Y$.

Proposition 10.7. *[4]. If there is a uniformly strict extension then each $\mathfrak{s}(p)$ is strictly tame for $\mathcal{U}$.*

Theorem 10.8. *[4]. If Z is finite, $\mathcal{T}$ is a strict extension of $\mathcal{U}^{tp}$ and $\mathfrak{s}(p)$ $(p \in Z)$ is round and strictly tame for $\mathcal{U}$ then there exists a unique uniformly strict extension compatible with $(\mathcal{U}, \mathcal{T})$.*

The quasi-uniformity $\mathcal{U}$ is said to be *uniformly regular* (regular in [4]) iff, for $U \in \mathcal{U}$, there is $U' \in \mathcal{U}$ such that $\overline{U'(x)} \subset U(x)$ for $x \in X$ and the closure relative to $\mathcal{U}^{tp}$. A filter $\mathfrak{r}$ in X is said to be *regularly tame* iff, for $U \in \mathcal{U}$, there are $U' \in \mathcal{U}$ and $R \in \mathfrak{r}$ such that $U'(x) \cap R' \neq \emptyset$ for every $R' \in \mathfrak{r}$ implies $R \subset U(x)$ $(x \in X)$.

PROPOSITION 10.9. *[4]. If there is a uniformly regular extension $\mathcal{W}$ then $\mathcal{U}$ is uniformly regular, the topology $\mathcal{W}^{tp}$ is regular, and each $\mathfrak{s}(p)$ is round and regularly tame for $\mathcal{U}$.*

Conversely:

THEOREM 10.10. *[3]. If Z is finite, $\mathcal{U}$ is uniformly regular, $\mathcal{T}$ is a regular topology, and each $\mathfrak{s}(p)$ is round and regularly tame for $\mathcal{U}$ and $p \in Z$ then there exists a uniformly regular extension compatible with $(\mathcal{U}, \mathcal{T})$.*

THEOREM 10.11. *[7]. If Z is finite, then an extension compatible with $(\mathcal{U}, \mathcal{T}^-, \mathcal{T})$ exists iff $(\mathcal{T}^-, \mathcal{T})$ is a biregular extension of $(\mathcal{U}^{-tp}, \mathcal{U}^{tp})$, each $\mathfrak{s}(p)$ is $\mathcal{U}$-round, $\mathfrak{s}^-(p)$ is $\mathcal{U}^-$-round, and each pair $(\mathfrak{s}^-(p), \mathfrak{s}(p))$ is $\mathcal{U}$-Cauchy.*

The reader can find more useful information concerning older results on quasi-uniform extensions in the paper [6].

REFERENCES

[1] Á. CSÁSZÁR, *On a class of D-Cauchy filters*, Serdica Math. Journal.
[2] Á. CSÁSZÁR, *On a problem of simultaneous quasi–uniform extensions*, Acta Math. Hungar.
[3] Á. CSÁSZÁR, *Regular extensions of quasi–uniformities*, Studia Sci. Math. Hungar **14** (1979), 15–26.
[4] Á. CSÁSZÁR, *Extensions of quasi–uniformities*, Acta Math. Acad. Sci. Hungar **37** (1981), 121–145.
[5] Á. CSÁSZÁR, *On J. Deák's construction for quasi–uniform extensions*, Rend. Ist. Mat. Univ. Trieste, **30 Suppl.** (1999), 87–90.
[6] J. DEÁK, *A survey of compatible extensions (presenting 77 unsolved problems)*, Coll. Math. Soc. J. Bolyai **55** (1989), 127–175.
[7] J. DEÁK, *Extensions of quasi–uniformities for prescribed bitopologies I*, Stud. Sci. Math. Hungar **25** (1990), 45–67.
[8] J. DEÁK, *Extensions of quasi–uniformities for prescribed bitopologies II*, Stud. Sci. Math. Hungar **25** (1990), 69–91.

[9] J. Deák, *Notes on extensions of quasi–uniformities for prescribed topologies*, Stud. Sci. Math. Hungar **25** (1990), 231–234.

[10] J. Deák, *Quasi–uniform extensions for finer topologies*, Stud. Sci. Math. Hungar **25** (1990), 97–105.

[11] D. Doitchinov, *Another class of complete quasi–uniform spaces*, C. R. Acad. Bulgare Sci. **44** (1991), 1991.

[12] P. Fletcher and W.F. Lindgren, *Quasi–uniform spaces*, Marcel Dekker Inc., New York and Basel, 1982.

[13] J.C. Kelly, *Bitopological spaces*, Proc. London Math. Soc. **13** (1963), no. 3, 71–89.

[14] A. Losonczi, *Finite quasi–uniform extensions i*, Acta Math. Hungar.

[15] A. Losonczi, *Finite quasi–uniform extensions ii*, Acta Math. Hungar.

[16] A. Losonczi, *Special simultaneous quasi–uniform extensions*, Acta Math. Hungar.

[17] L. Nachibin, *Sur les espaces uniformes ordonnés*, C. R. Paris **226** (1948), 774–775.

Received November 10, 1997.