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Invariants on Compact Spaces

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Summary. - *From the dynamical point of view, topological invariants are those characteristics on compact topological spaces which are kept by topological conjugacy. In this paper we consider three of them of special interest: topological entropy, sequence topological entropy and rotational entropy. We survey some of their properties and obtain some useful formulas to their computation over the space of continuous maps on the compact space.*

1. Introduction

Let X be a compact topological space and $f : X \to X$ a continuous map.When we iterate this map over all the points on X we obtain the discrete dynamical system denoted by the pair (X, f). In this setting we are interested in knowing the behavior of all the orbits generated by f (we will call it the *dynamics of* f), that is, all the sequences $(f^n(x))_{n=0}^{\infty}$where $f^n = f^{n-1} \circ f$ for every $n \geq 1$ and $f^0 = Identity$. In particular we want to consider how these orbits mix together. To some extent this can be measured by a topological invariant on X called the *topological entropy* which will be denoted by h.

By a topological invariant we mean some characteristic defined on the set of continuous maps of the topological space into itsef that are not changed by topological conjugacy. We say that two maps on X, f and g are topologically conjugated if there exists an

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homeomorphism d on X such that $f \circ d = d \circ g$. In this case it is held that $h(f) = h(g)$ (see [1]).

The *topological entropy* is an anologous to another invariant, the *metric entropy*, introduced by Kolmogorov and Sinai in the setting of probabilistic measure spaces, [10] and [13]. This invariant measures how complicated is a transformation on the space from the theoretical point of view of the invariant measures. But this metric entropy can not distinguish how is the dynamics concentrated on sets of measure zero. On the contrary, the topological entropy can be "disturbed" by the parts of the dynamics concentrated on very small sets. Surprisingly there is a relationship between these two notions (one purely metric and topological the other) through the set of f-invariant measures which can be defined on X.

Not all the zero topological entropy maps are of the same type. It would be interesting to get another tool to be able to distinguish them, since for this type of maps, topological entropy is a rough measure. To get it, T.N. Goodman [8] introduced as an extention of the notion of topological entropy another topological invariant,the *sequence topological entropy*, where not all but the iterates given by a sequence of numbers are considered.

Recently, W. Geller and M. Misiurewicz [7], in the setting of torus transformations, have introduced another invariant on this type of transformations, *the rotational entropy,* which allows us to take an idea of how complicated the dynamics is if we consider the rotational behavior of the maps.

All the invariants hold some formulas which allow to evaluate them for some of the maps and compare them in most cases. But not all the invariants behave in the same way, that is, not all the invariants hold similar formulas and we find differences.

The aim of this paper is to review some of these formulas and improve others for topological entropy and give some news for the others invariants. This work can be also done in an analogous way in the setting of metric invariants, but we will concentrate here in the topological aspects.

We organize the paper in the following way. First we introduce the notions of topological and sequence topological entropy (shortly t.e. and s.t.e respectively). Secondly we consider some known and others new formulas for them. Finally, we will consider the rotational entropy. In the inside we will present some open problems.

2. Topological and Sequence Topological Entropy

The notion of topological entropy was introduced in 1972 by Adler, Konheim and McAndrew [1].Let X be a compact topological space and α an open covering. The *entropy* of the covering α will be

$$H(\alpha) = log\ N(\alpha)$$

where $N(\alpha)$ means the minimum number of open sets in any finite subcover on X. Given two covers α and β, $\alpha \vee \beta$ will denote the set $\{A \cap B : A \in \alpha, B \in \beta\}$. If f is a continuous map on X ,

$$f^{-1}\alpha = \{f^{-1}(S) : S \in \alpha\}$$

The entropy of f referred to α is given by

$$h(f,\alpha) = \lim_{n\to\infty} \frac{1}{n} H(\alpha \vee f^{-1}\alpha \vee f^{-2}\alpha \vee \ldots \vee f^{-n+1}\alpha)$$

this limit always exists and finally the *topological entropy* of f is given by

$$h(f) = \sup_{\alpha} h(f,\alpha)$$

where this supremum is calculated over all open finite covers on X.

When we consider only the open covers given by the preimages $f^{-a_1}\alpha, f^{-a_2}\alpha, \ldots, f^{-a_n}\alpha$ where $A = (a_i)_{i=1}^{\infty}$ is a sequence of positive integers we have the notion of *sequence topological entropy* given by

$$h_A(f) = \sup_{\alpha} h_A(f,\alpha)$$

where

$$h_A(f,\alpha) = \limsup_{n\to\infty} \frac{1}{n} H(\bigvee_{i=1}^{n} f^{-a_i}\alpha)$$

This notion has been used in the setting of dynamical systems of the type (I,f) where $I = [0,1]$ and $f \in C(I,I)$ to distinguish between *chaotic* and *non chaotic* dynamical systems in the Li-Yorke sense (see [6]).

Now we need to introduce some others notions of topological dynamics. We will denote by *Per(f)* the set of periodic points of f and by $\Omega(f)$ the set of *nonwandering points* of f.A point $x \in X$ is a *nonwandering point* if given any neighborhood of x, $U(x)$, there is a positive integer m such that $U \cap f^m(U) = \emptyset$. Given $x \in X$, the set of limit points of the sequence $(f^n(x))_{n=0}^{\infty}$ is the $\omega-$*limit set* of f, denoted by $\omega_f(x)$ and $\omega(f) = \bigcup_{x \in X} \omega_f(x)$ will be the omega-limit set of f. If $x \in \omega_f(x)$ then x is a *recurrent* point of f. Let *Rec(f)* be the set of recurrent points of f. The closure $\overline{\mathrm{Rec}(f)} = C(f)$ is the *center of* f.

For topological entropy the following formulas are held:

(i)
$$h(f^k) = k\ h(f) \qquad \textit{for every positive integer } k.$$

An easy proof can be found in [4].

(ii) The Bowen's formula

$$h(f) = h(f \mid \Omega(f))$$

which can be found in [5].

When X is a compact metric space, the Bowen´s formula can be improved since $C(f) = \overline{\mathrm{Rec}(f)} \subseteq \Omega(f)$.

Theorem 2.1. *Let X be a compact metric space and $f \in C(X, X)$. Then is*

$$h(f) = h(f \mid \overline{\mathrm{Rec}(f)})$$

Proof. Let $M(X, f)$ be the set of invariant measures on X (it is supposed to have a $\sigma - Borel$ algebra defined on X),that is, if $\mu \in M(X, f)$ then is $\mu(A) = \mu(f^{-1}(A))$ for every borelian set A.It can be seen in [14] that for every invariant measure $\mu \in M(X, f)$ is $\mu(\Omega(f)) = 1$. But can be also seen that in fact is $\mu(\mathrm{Rec}(f)) = 1$. To see this,let $(U_n)_{n=1}^{\infty}$ be a base on the topology of X.Then we have,

$$X \setminus \mathrm{Rec}(f) = \bigcup_{n=1}^{\infty} (U_n \cap \bigcap_{k=1}^{\infty} f^{-k}(X \setminus U_n))$$

and for every n and $\mu \in M(X,f)$ we have,

$$\mu(U_n \cap \bigcap_{k=1}^{\infty} f^{-k}(X \setminus U_n)) = 0$$

using the recurrence theorem of Poincaré (see [14]).

Therefore, if $\mu \in M(X,f)$ then is $\mu(\mathrm{Rec}(f)) = 1$ and consequently the metric entropy of f associated to μ will be

$$h_\mu(f) = h_\mu(f \mid \mathrm{Rec}(f))$$

and finally by the *variational principle* (see for example [14]) we have:

$$h(f) = \sup_{\mu \in M(X,f)} h\mu(f) = \sup_{\mu \in M(X,f)} h_\mu(f \mid \overline{\mathrm{Rec}(f)}) =$$
$$h(f \mid \overline{\mathrm{Rec}(f)}) = h(f \mid C(f)).$$

□

3. On the Commutativity of t.e. and s.t.e.

When considering dynamical systems, one interesting problem is to study what happens when we compose the maps defining them, namely, if (X,f) and (X,g) are two dynamical systems, try to understand the behavior of the systems $(X, f \circ g)$ and $(X, g \circ f)$. In this setting we are considering what happens with the two invariants we have considered, the t.e. and s.t.e.

In [11] S. Kolyada and L.Snoha have proved that t.e. holds the formula $h(f \circ g) = h(g \circ f)$. Their proof is based in a type of entropy adapted to sequences of maps.Now jointly with J. Cánovas we are able to give a direct proof of it.

THEOREM 3.1. *Let f,g: X→ X be continuous maps from the compact topological space X into itself. Then*

$$h(f \circ g) = h(g \circ f)$$

Proof. Let α be a finite open cover of X. First it is easy to prove that $h(f,\alpha) = h(f, f^{-1}\alpha)$.

Then

$$N(\bigvee_{i=1}^{\infty}(f\circ g)^{-i}(\alpha)) = N(g^{-1}(\bigvee_{i=1}^{\infty}(g\circ f)^{-i+1}(f^{-1}\alpha))) \leq$$

$$\leq N((\bigvee_{i=1}^{\infty}(g\circ f)^{-i+1}(f^{-1}\alpha)))$$

therefore

$$h(f\circ g,\alpha) \leq h(g\circ f, f^{-1}\alpha) \leq h(g\circ f),$$

and finally $h(f\circ g) \leq h(g\circ f)$. Interchanging f and g , we have the reverse inequality and we end the proof. □

The problem for s.t.e. is more complicated since in general is not true that $h_A(f,\alpha) = h_A(f,f^{-1}\alpha)$ for any sequence of positive integers A.It can only be proved that $h_A(f,\alpha) \geq h_A(f,f^{-1}\alpha) = h_{A+1}(f,\alpha)$ where if $A = (a_i)_{i=1}^{\infty}$then $A+1 = (a_i+1)_{i=1}^{\infty}$. If the map is surjective, then the commutativity can be proved.

Proposition 3.2. *Let f,g: X→ X be continuous maps fron the compact topological space X into itself such that f∘g and g∘f be surjective. Then for any sequence* A=$(a_i)_{i=1}^{\infty}$*we have* $h_A(f\circ g) = h_A(g\circ f)$.

Proof. Since the map $f\circ g$ is surjective, we have that for any cover of X,α,

$$h_A(f\circ g,(f\circ g)^{-1}\alpha) = h_A(f\circ g,\alpha)$$

Then following the same line of reasoning of Theorem 3.1 we have

$$h_A(f\circ g,\alpha) \leq h_A(g\circ f, f^{-1}\alpha) \leq h_A(g\circ f).$$

Therefore $h_A(f\circ g) \leq h_A(g\circ f)$. Interchanging f and g we have the result. □

A key point to prove that $h_A(f) = h_{A+1}(f)$ would be the validity of the formula $h_A(f) = h(f \mid \bigcap_{n\geq 0} f^n(X))$, but this is at the moment an open question.

If the conditions of the following Goodman' theorem are held then we obtain also the commutativity of the s.t.e.

THEOREM 3.3. *Let f: $X \to X$ be a continuous map and suppose X has a finite covering dimension. We consider a sequence of integers A. Then:*

$$\mathrm{h}_A(f) = \begin{cases} 0 & \text{if } K(A) = 0 \\ K(A)h_\mu(f) & \text{if } 0 < h_\mu(f) < \infty \\ 0 & \text{if } 0 < K(A) < \infty,\ h_\mu(f) = 0 \\ \infty & \text{if } 0 < K(A) \leq \infty,\ h_\mu(f) = \infty \end{cases}$$

where

$$K(A) = \lim_{k\to\infty} (\limsup_{n\to\infty} \frac{S_A(n,k)}{n})$$

and

$$S_A(n,k) = Card \bigcup_{i=1}^{\infty} \{-k + a_i, \dots , k + a_i\}$$

COROLLARY 3.4. *Let A and $f \circ g : X \to X$ satisfy the conditions given in Theorem 4. Then $h_A(f \circ g) = h_A(g \circ f)$.*

As an illustration of the differences of behavior between t.e.and s.t.e. with respect to commutativity, we will consider the case of antitriangulatr maps. We will say that $F : X \times X \to X \times X$ is an antitriangular map if $F(x, y) = (f(y), g(x))$ where $f$and g are continuous maps from X into itsel.

PROPOSITION 3.5. *Let $F{:}X{\times}X \to X \times X$ an antitriangular map given by $F(x,y)=(f(y),g(x))$ for all $(x,y) \in X \times X$. Then $h(F) = h(f \circ g) = h(g \circ f)$.*

Proof. If we calculate $F^2(x, y)$ we can use the formula $h(f_1 \times f_2) = h(f_1)+h(f_2)$ (see [5]), where $f_i : X_i \to X_i$, $i = 1, 2$ are two continuous maps and obtain, $F^2(x, y) = (f \circ g(x), g \circ f(y))$ and $h(F^2) = h(f \circ g) + h(g \circ f) = 2h(f \circ g) = 2h(g \circ f)$ by the commutativity. By other hand $h(F^2) = 2h(F)$ and therefore $h(F) = h(f \circ g) = h(g \circ f)$. □

But the above formula does not hold for s.t.e. To prove it, consider the following example.

EXAMPLE 3.6. *Let $T_{a,b}$ the transformation on the torus $\mathbb{T}^2$into itself, given by $T_{a,b}(x, y) = (ax, by)$ where $a, b \in \mathbb{S}^1$.It is not dificult to see that given the sequence $A = (2^i)_{i=1}^{\infty}$, is $h(T_{a,b}) = \log 2$.*

Now we consider the map $F_{a,b} : \mathbb{T}^2 \times \mathbb{T}^2 \to \mathbb{T}^2 \times \mathbb{T}^2$ given by $F_{a,b}(x,y) = (T_{a,b}(y), T_{a,b}(x))$. Then we have $h_A(F^2_{a,b}) = 2\ h_A(T^2_{a,b})$ using a formula given by Goodman [8]. Then using the previous proposition we have $h_A(F_{a,b}) = h_A(F^2_{a,b}) = 2\ h_A(T^2_{a,b}) = 2\ h_A(T_{a,b}) = \log 4$.

4. The Formula $h_A(f^k) = k\ h_A(f)$ is not true for s.t.e.

In this section we are giving an example proving that in general $h_A(f^k) \neq k\ h_A(f)$
This example is not easy, so we will give only a sketch of the construction.To this end we are introducing the *Morse sequences.*

A sequence $B = (b_0, b_1, \dots, b_{k-1})$ of zeros and ones is called a *block*.The *length* of the block B is denoted by $\mid B \mid = k$.We denote by $B[i,j] = (b_i, \dots, b_j), B[i,i] = B[i]$,and $\widetilde{B} = (\widetilde{b}_0, \dots, \widetilde{b}_{k-1})$ where $\widetilde{b}_k = 1 - b_k$ for $k = 0, \dots, k-1$.If $C = (c_0, \dots, c_{m-1})$ is another block, then we define

$$B \times C = B^{(c_0)} B^{(c_1)} \dots B^{(c_{m-1})}$$

where $B^{(0)} = B, B^{(1)} = \widetilde{B}$.

Assume that $\mid B \mid \leq \mid C \mid$ .Then $fr(B,C)$ denotes the frequency of B in C, That is

$$fr(B,C) = card\{0 \leq j \leq \mid C \mid - \mid B \mid : C[j, j + \mid B \mid - 1] = B\}$$

Let $b^0, b^1, \dots$ be finite blocks with length at least two starting with zero, and let

$$x = b^0 \times b^1 \times \dots$$

Next, we consider

$$r_i^* = \min\left\{\frac{1}{\lambda_i} fr(0, b^i), \frac{1}{\lambda_i} fr(1, b^i)\right\} \quad i \geq 0, \lambda_i = \mid b^i \mid$$

Definition 4.1. *The sequence x defined as above is called a generalized Morse sequence (or in brief a Morse sequence) if*

(1) infinitely many of the b's are different from 0 ... 0,

(2) infinitely many of the b's are different from 0101 ... 01,

(3) $\sum_{i=0}^{\infty} r_i^* = \infty$

More details about this sequences can be seen in [10].

Now consider the case $k = 2$ and the sequence

$$x = 01 \times 00 \times 01 \times 00 \times 00 \times 01 \times 00 \times 00 \times 00 \times \dots$$

We can give for x the two representations:

(1)
$$x = (01 \times 00) \times (01 \times 00) \times \dots$$

(2)
$$x = (01 \times 00 \times 01) \times (00 \times 01) \times (00 \times 01) \times \dots$$

Assume that $(n_i)$and (m_i) are the corresponding sequences of the products of lengths of the successive blocks in (1) and (2) respectively.

Using the notion of *n_i-entropy of a Morse sequence x,* $h_{(n_i)}(x)$ and some results by M.Lemanczyk in [12] we can see that $h_{(n_i)}(x) = \log \frac{1}{2}(1 + \sqrt{5})$ and $h_{(m_i)}(x) = \log 2$. As a consequence we can construct an homeomorphism f preserving a measure on X, and such that $h_{(n_i)}(f^2) \neq 2\, h_{(n_i)}(f)$. For more details see [12].

5. The Rotational Entropy

Given the dynamical system (X, f) and an observable map $\phi : X \to \mathbb{R}^d$ we are looking at its ergodic averages at different points. When these averages converge at a given point, the limit is the *rotation vector* at this point and the set of all rotation vectors is the rotation set.

When the observable map ϕ is the displacement fuction, that is, a map measuring the vector by which the point is displaced in the universal covering space, we obtain the classical notion of *rotation sets.*

Given $x \in X$, we define the rotation vector at x as

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{\infty} \phi(f^i x)$$

if this limit exists.If μ denotes an *ergodic measure* on X, using the Birkoff's ergodic theorem (see for example [14]), the above limit exists for μ-almost point x and is equal to $\int_X \phi \, d\mu$. We will say that μ is an *ergodic invariant measure μ-directional* if the rotation set of f restricted to the support of μ consists of only one vector called the *direction* of μ.

The *directional entropy* of f in the direction $v \in \mathbb{R}^d$ (or shortly *v-entropy* of f) is defined by (using the variational principle):

$$h_v(f) = \sup\{h_\mu(f) : \mu \textit{ is ergodic with direction } v\}$$

We are asking if it is true that $h_v(f \circ g) = h_v(g \circ f)$.

When the ergodic measure μ has not a direction we say that it is *lost, (*we follow the terminology and definitions introduced by W.Geller and M.Misiurewicz in [7]). In this case we define the *lost entropy* of f by:

$$h_l(f) = \sup\{h_\mu(f) : \mu \textit{ is ergodic and lost}\}$$

We ask again if it is true that $h_l(f \circ g) = h_l(g \circ f)$.

Let $\mathcal{E}(X, f \circ g)$ be the set of all ergodic invariant measures on $X, \mu \in \mathcal{E}(X, f \circ g)$ and its image $\widetilde{g}\mu \in \mathcal{E}(X, g \circ f)$ (see [14] for definitions), first we need to state the relationship between the supports of the two measures.

Lemma 5.1. *It is held that $f^{-1}(supp\ \mu) \supseteq supp\ \widetilde{g}\mu$.*

Proof. Let $E = \text{supp}\ \mu$.Then E is closed and $\mu(E) = 1$.Now consider the closed set $f^{-1}(E) \in \beta(X)$.Then we have:

$$\widetilde{g}\mu(f^{-1}(E)) = \mu(g^{-1}(f^{-1}(E))) = \mu((f \circ g)^{-1}(E)) = \mu(E) = 1$$

and from this it follows supp $\widetilde{g}\mu \subseteq f^{-1}(E)$. $\square$

In what follows we will denote by $\mathcal{E}(X, f \circ g, v)$ the set of ergodic measures of $f \circ g$ having as direction v.

LEMMA 5.2. *Let $\mu \in \mathcal{E}(X, f \circ g, v)$ and let $y \in$supp $\widetilde{g}\mu$. Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{\infty} \phi\left((g \circ f)^i(y)\right) = \int_X \phi \, d\widetilde{g}\mu$$

Proof. It is necessary to prove that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{\infty} \phi\left((f \circ g)^i(x)\right) = \lim_{n\to\infty} \sum_{i=1}^{\infty} \phi\left((f \circ g)^i(x)\right)$$

For this we consider

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{\infty} \phi\left((f \circ g)^i(x)\right) = \lim_{n\to\infty} \left[\frac{1}{n} \sum_{i=1}^{\infty} \phi\left((f \circ g)^i(x)\right) + \frac{1}{n}\phi(x)\right],$$

and since $\lim\limits_{n\to\infty} \frac{1}{n}\phi(x) = 0$ we have the proof.
Using Lemma 5.1, if $y \in$ supp $\widetilde{g}\mu$,then $f(y) = x \in$supp μ. We apply now the last formula to obtain:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{\infty} \phi\left((g \circ f)^i(y)\right) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{\infty} \phi\left((g \circ f)^i(y)\right) =$$

$$= \lim_{n\to\infty} \frac{1}{n-1} \sum_{i=0}^{n-2} \phi \circ g\left((f \circ g)^i(x)\right) = \int_X \phi \circ g \, d\mu = \int_X \phi \, d(\widetilde{g}\mu).$$

□

LEMMA 5.3. *There exists a one to one map from $\mathcal{E}(X, f \circ g, v)$ into $\mathcal{E}(X, g \circ f, u)$, where $v = \int \phi \, d\mu$ and $u = \int \phi \, d(\widetilde{g}\mu$.*

Proof. If $\mu \in \mathcal{E}(X, f \circ g, v)$, then it is ergodic and has as rotation vector v.It can be proved (see [3]) that there exists a one to one map between the sets $\mathcal{E}(X, f \circ g)$ and $\mathcal{E}(X, g \circ f)$. Therefore $\widetilde{g}\mu$ is ergodic, and by Lemma 5.2, u has as rotation vector, that is $\widetilde{g}\mu \in \mathcal{E}(X, g \circ f, u)$.

In this case the map $\widetilde{g}\ |_{\mathcal{E}(X,f\circ g,v)}\ (\mu) = \widetilde{g}\mu$ defined for all $\mu \in \mathcal{E}(X, f \circ g, v)$ is well defined and using the mentioned result it is easy to see that the map is one to one. □

Now the main theorem is immediate:

THEOREM 5.4. *Let* $v=\int_X \phi\, d\mu$ *and* $u=\int_X \phi\, d\widetilde{g}\mu$. *Then the rotational entropy with respect to the vectors* u *and* v *satisfies the formula:*

$$h_v(f \circ g) = h_u(g \circ f)$$

Proof. It is an immediate consequence of Lemma 5.3 and the fact that if $\mu \in M(X, f \circ g)$ then $h_\mu(f \circ g) = h_{\widetilde{g}\mu}(g \circ f)$ (see [3]). □

For the lost measures we obtain analogous results. If we denote by $\mathcal{E}(X, f \circ g, l)$ the set of the lost ergodic measures of $f \circ g$, we have

LEMMA 5.5. $\mathcal{E}(X, f \circ g, l) = \widetilde{g}\mathcal{E}(X, g \circ f, l)$

Proof. Let $\mu \in \mathcal{E}(X, f \circ g, l)$ and suppose that $\widetilde{g}\mu \notin \mathcal{E}(X, g \circ f, l)$.Then $\widetilde{g}\mu$ has a rotation vector and by Lemma 5.3, $\widetilde{f} \circ \widetilde{g}\mu$ has a rotation set which is a contradiction. □

We finish with the following result:

THEOREM 5.6. *For the lost entropy it is held:*

$$h_l(f \circ g) = h_l(g \circ f)$$

Proof. It is enough to apply Lemma 5.3 and the result mentioned in the proof of Theorem 5.4. □

6. Remarks

In the cases of t.e. and s.t.e. we have given some examples which are concerned with certain topological spaces where we have studied the two topological invariants. Now we can state the following question: What is the role played by the topology of the spaces?. In other words, given a compact topological space, can we construct continuous functions on it having a prescribed topological entropy?. The same question for the sequence topological entropy.

In the setting of $X = [0, 1]$ the problem is solved (see [2]). Given any positive number α it is possible to construct a continuous map on it having α as a topological entropy.The construction can be also made for zero and infinite topological entropy. If we wish a C^1-map the problem can be also be solved except in the case of infinite entropy for which is impossible.

But unfortunately in the general case we can not give an answer to this problem. We guess that if the topological space contains a type Cantor set then the answer could be affirmative and the construction can be made.

References

[1] R.L. Adler, A.G. Konheim, and M.H. McAndrew, *Topological entropy*, Trans. Amer. Math. Soc. (1965), 309–319.

[2] F. Balibrea and J. Cánovas, *On the construction of maps with a prescribed entropy*, Annales Mathematicae Silesianae, to appear.

[3] F. Balibrea, J. Cánovas, and V. Jiménez, *Some results on entropy and sequence entropy*, International Journal of Bifurcation and Chaos, to appear.

[4] L. Block and A. Coppel, *Dynamics in one dimension*, Lecture Notes in Mathematics, no. 1513, Springer Verlag, 1992.

[5] R. Bowen, *Entropy for group endomorphisms and homogeneous spaces*, Trans. Amer. Math. Soc. **153**, 401–414.

[6] N. Franzová and J. Smital, *Positive sequence topological entropy characterizes chaotic maps*, Proc. Amer. Math. Soc. **112** (1991), 1083–1086.

[7] W. Geller and M. Misiurewicz, *Rotation and entropy*, 1997.

[8] T.N.T. Goodman, *Topological sequence entropy*, Proc. London. Math. Soc. **3** (1974), no. 29, 331–350.

[9] M. Keane, *Generalized Morse sequences*, Z.Wahrsch.Verw.Gebiete **10** (1968), 335–353.

[10] N. Kolmogorov, *New metric invariants of transitive dynamical systems and automorphisms of Lebesgue spaces*, Dokl. Akad. Nauk. SSSR **119** (1958), 861–864.

[11] S. Kolyada and L. Snoha, *Topological entropy of nonautonomous dynamical systems*, Random and Comp. Dynamics **4** (1996), 205–233.

[12] M. Lemanczyk, *The sequence entropy for morse shifts and some counterexamples*, Studia Mathematica **82** (1985), 221–245.

[13] Y. Sinai, *On the notion of entropy of a dynamical system*, Dokl. Acad. Nauk. SSSR **124** (1958), 768–771.

[14] P. Walters, *An introduction to ergodic theory*, Graduate Texts in Mathematics (1982), no. 79.

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