Rend. Istit. Mat. Univ. Trieste
Suppl. Vol. XXX, 21–30 (1999)

A Characterization of Non-archimedeanly Quasimetrizable Spaces

F. G. Arenas and M. A. Sánchez-Granero (*)

Summary. - *In this paper we introduce a new structure on topological spaces which allows us to give a characterization of non-archimedeanly quasipseudometrizable spaces.*

1. Introduction

The concept of fractal (see [6]) is one of the most important in mathematics nowadays, due to the great number of applications it has in economics, physics, mathematics, statistics and so on, as one can see in [7] and [8]. One of the most important classes of fractal are the so called (classical) "(strict) self-similar sets" (see [7, 9.2]). These set are defined by means of a finite set of (similarities) contractions in a compact metric space.

Recently there has been many investigations on the topological structures of (strict) self-similar sets (see [2], [3], [4], [13], [10], [11], [12] and [14]) leading to the notion of symbolic self-similar set (as in [11]), a topological characterization of the classical ones.

Looking for a generalization of symbolic self-similar sets outside compact metric spaces, we develop the concept of GF-space (or generalized fractal space) and we find that it is a common framework

(*) Authors' addresses: F. G. Arenas, Area of Geometry and Topology, Faculty of Science, Universidad de Almería, 04071 Almería, Spain, e-mail: farenas@ualm.es
M. A. Sánchez-Granero, Area of Geometry and Topology, Faculty of Science, Universidad de Almería, 04071 Almería, Spain, e-mail: misanche@ualm.es
1991 Mathematical Subject Classification. 54E15, 54E25, 54B35
Keywords: Transitive quasiuniformity, non-archimedean quasimetric, poset, inverse sequence, GF-space
The first-listed author is partially supported by DGES grant PB95-0737

for the study of self-similar sets and non-archimedeanly quasimetrizable spaces. In this paper we introduce GF-spaces and we use them to characterize non-archimdeanly quasimetrizable spaces in several ways (including some relations with inverse limits of partially ordered sets). The relation between GF-spaces and self-similar sets and self-homeomorphic spaces in the sense of [5] can be found in [1].

Now, we recall some definitions and introduce some notations that will be useful in this paper.

Let $\mathbf{\Gamma} = \{,_n : n \in \mathbb{N}\}$ be a countable family of coverings. Recall that $\mathrm{St}(x,,_n) = \bigcup_{x \in A_n, A_n \in \Gamma_n} A_n$; we also define $U_{xn} = \mathrm{St}(x,,_n) \setminus \bigcup_{x \notin A_n, A_n \in \Gamma_n} A_n$. We also denote by $\mathrm{St}(x, \mathbf{\Gamma}) = \{\mathrm{St}(x,,_n) : n \in \mathbb{N}\}$ and $\mathcal{U}_x = \{U_{xn} : n \in \mathbb{N}\}$.

A (base $\mathcal{B}$ of a) quasiuniformity $\mathcal{U}$ on a set X is a (base $\mathcal{B}$ of a) filter $\mathcal{U}$ of binary relations (called entourages) on X such that (a) each element of $\mathcal{U}$ contains the diagonal Δ_X of $X \times X$ and (b) for any $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ satisfying $V \circ V \subseteq U$. A base $\mathcal{B}$ of a quasiuniformity is called transitive if $B \circ B = B$ for all $B \in \mathcal{B}$. The theory of quasiuniform spaces is covered in [9].

If $\mathcal{U}$ is a quasiuniformity on X, then so is $\mathcal{U}^{-1} = \{U^{-1} : U \in \mathcal{U}\}$, where $U^{-1} = \{(y, x) : (x, y) \in U\}$. The generated uniformity on X is denoted by $\mathcal{U}^*$. A base is given by the entourages $U^* = U \cap U^{-1}$. The topology $\tau(\mathcal{U})$ induced by the quasiuniformity $\mathcal{U}$ is that in which the sets $U(x) = \{y \in X : (x, y) \in U\}$, where $U \in \mathcal{U}$, form a neighbourhood base for each $x \in X$. There is also the topology $\tau(\mathcal{U}^{-1})$ induced by the inverse quasiuniformity. In this paper, we consider only spaces where $\tau(\mathcal{U})$ is T_0.

A quasipseudometric on a set X is a nonnegative real-valued function d on $X \times X$ such that for all $x, y, z \in X$:(i) $d(x, x) = 0$, and (ii) $d(x, y) \leq d(x, z) + d(z, y)$. If in addition d satisfies the condition (iii) $d(x, y) = 0$ iff $x = y$, then d is called a quasi-metric. A non-archimedean quasipseudometric is a quasipseudometric that verifies $d(x, y) \leq \max\{d(x, z), d(z, y)\}$ for all $x, y, z \in X$.

Each quasipseudometric d on X generates a quasiuniformity $\mathcal{U}_d$ on X which has as a base the family of sets of the form $\{(x, y) \in X \times X : d(x, y) < 2^{-n}\}$, $n \in \mathbb{N}$. Then the topology $\tau(\mathcal{U}_d)$ induced by $\mathcal{U}_d$, will be denoted simply by $\tau(d)$.

A space (X, τ) is said to be (non-archimedeanly) quasipseudometrizable if there is a (non-archimedean) quasipseudometric d on X such that $\tau = \tau(d)$.

A relation $\leq$ on a set G is called a partial order on G if it is a transitive antisymmetric reflexive relation on G. If $\leq$ is a partial order on a set G, then $(G, \leq)$ is called a partially ordered set.

$(G, \leq, \tau)$ will be called a poset (partially ordered set) or T_0-Alexandroff space if $(G, \leq)$ is a partially ordered set and τ is that in which the sets $[g, \rightarrow[= \{h \in G : g \leq h\}$ form a neighborhood base for each $g \in G$ (we say that the topology τ is induced by $\leq$). Note that then $\overline{\{g\}} =]\leftarrow, g]$ for all $g \in G$.

Let us remark that a map $f : G \rightarrow H$ between two posets G and H is continuous if and only if it is order preserving, i.e. $g_1 \leq g_2$ implies $f(g_1) \leq f(g_2)$.

2. Non-archimedean quasipseudometrization and inverse limits

To each countable transitive base of a quasiuniformity, one can associate a partition as follows.

PROPOSITION 2.1. *Let X be a countable transitive quasiuniform space, that is, a topological space that has a countable transitive base $\mathcal{B} = \{U_n : n \in \mathbb{N}\}$ of quasiuniformity over X. Suppose that $U_{n+1} \subseteq U_n$ $\forall n \in \mathbb{N}$. Then for each natural number n, $\mathcal{B}^* = \{U_n^*(x) : x \in X\}$ is a partition of X.*

Proof. It is clear that the union of those subsets is X, so it is enough to see that they are disjoint or they are the same.

In order to see that, suppose there is z in $U_n^*(x) \cap U_n^*(y)$. Then $z \in U_n(x)$, $x \in U_n(z)$, and $z \in U_n(y)$, $y \in U_n(z)$. The transitivity of the quasiuniformity and $x \in U_n(z)$, $z \in U_n(y)$ implies that $x \in U_n(y)$ and $y \in U_n(z)$, $z \in U_n(x)$ implies $y \in U_n(x)$, that is $U_n(x) = U_n(y)$. By transitivity again, $U_n^{-1}(x) = U_n^{-1}(y)$. Therefore $U_n(x) \cap U_n(x)^{-1} = U_n(y) \cap U_n(y)^{-1}$ □

In the condition of the Proposition 2.1, we call G_n the quotient space induced by the partition, and we define in G_n the following order relation $U_n^*(x) \leq_n U_n^*(y)$ if $y \in U_n(x)$.

It is easy to see that this is an order relation by definition of the order and the transitivity of the quasiuniformity base. We will consider G_n as the poset with the order relation $\leq_n$, and the induced topology.

Let ρ_n be the quotient map from X onto G_n which carries x in X to $U_n^*(x)$ in G_n. Let see that ρ_n is continuous.

Let O be a basic open set in G_n, then there is $x \in X$ such that $O = \{g_n \in G_n : \rho_n(x) \leq_n g_n\}$. Hence $\rho_n^{-1}(O) = \{y \in X : \rho_n(x) \leq_n \rho_n(y)\} = \{y \in X : y \in U_n(x)\} = U_n(x)$. Therefore $\rho_n^{-1}(O)$ is open, and so ρ_n is continuous.

We also consider the map $\phi_n : \mathrm{G}_n \longrightarrow \mathrm{G}_{n-1}$ defined by $\phi_n(\rho_n(x)) = \rho_{n-1}(x)$. If $\rho_n(x) \leq_n \rho_n(y)$ then $y \in U_n(x) \subseteq U_{n-1}(x)$, what means $\rho_{n-1}(x) \leq_{n-1} \rho_{n-1}(y)$ and by definition of ϕ_n, we have $\phi_n(\rho_n(x)) \leq_n \phi_n(\rho_n(y))$. Therefore ϕ_n is continuous.

Let ρ be the map from X to $\varprojlim G_n$ which carries x in X to $(\rho_n(x))_n$ in $\varprojlim G_n$. Note that ρ is well defined and continuous (by definition of ϕ_n and the continuity of ρ_n and ϕ_n for all n). The inverse limit $\varprojlim G_n$ will be noted hereafter as $\varprojlim(X, \mathcal{B})$ and will be called the inverse limit associated to the countable transitive quasiuniform space $(X, \mathcal{B})$. Note that we do not claim that the inverse limit does not depend of the selected base for a given quasiuniform space.

Note that in [15] it is developped a procedure to associate an inverse system of quasi-ordered spaces to some kinds of families of (locally finite) closed coverings in a similar way.

Proposition 2.2. *Let $(X, \mathcal{B})$ be a countable transitive quasiuniform space. Then $\rho : X \to \varprojlim(X, \mathcal{B})$ is an embedding.*

Proof. Let see that ρ is injective.

Suppose there are $x \neq y$ in X such that $\rho(x) = \rho(y)$. Then $\rho_n(x) = \rho_n(y)$ for all n, that is, $U_n(x) = U_n(y)$ for all n.

Since X is T_0, there is a neighborhood U of (for instance) x, such that $y \notin U$. But then there is a natural number n, with $U_n(x) \subseteq U$, and then $y \notin U_n(x)$. The contradiction shows that ρ is injective.

Now, $y \in U_n(x)$ if and only if $\rho_n(x) \leq_n \rho_n(y)$ if and only if $\rho(y) \in \{g \in \rho(X) : \rho_n(x) \leq_n g_n\}$, then $\rho(U_n(x)) = \{g \in \rho(X) : \rho_n(x) \leq_n g_n\}$ and hence open in $\rho(X)$. $\square$

Now we have a characterization of non-archimedeanly quasipseudometrizable spaces on terms of inverse limits of posets.

THEOREM 2.3. *Let X be a non-archimedeanly quasipseudometrizable space. Then X can be embedded into an inverse limit of a sequence of posets.*

Proof. If X is a non-archimedeanly quasimetrizable space, then the quasiuniformity associated with the metric verifies the conditions of Proposition 2.1. Then by Proposition 2.2 we get that X can be embedded into a inverse limit of a sequence of posets. $\square$

3. GF-spaces

Now, we introduce GF-spaces, the main concept of the paper.

DEFINITION 3.1. *Let X be a topological space. A pre-fractal structure over X is a family of coverings $\mathbf{\Gamma} = \{,_n : n \in \mathbb{N}\}$ such that $\mathcal{U}_x$ is an open neighbourhood base of x for all $x \in X$.*

Furthermore, if $,_n$ is a closed covering and for all n, $,_{n+1}$ is a refinement of $,_n$, such that for all $x \in A_n$, with $A_n \in ,_n$, there is $A_{n+1} \in ,_{n+1} : x \in A_{n+1} \subseteq A_n$, we will say that $\mathbf{\Gamma}$ is a fractal structure over X.

If $\mathbf{\Gamma}$ is a (pre-) fractal structure over X, we will say that $(X, \mathbf{\Gamma})$ is a generalized (pre-) fractal space or simply a (pre-) GF-space. If there is no doubt about $\mathbf{\Gamma}$, then we will say that X is a (pre-) GF-space.

Call $U_n = \{(x,y) \in X \times X : y \in U_{xn}\}$, $U_{xn}^{-1} = U_n^{-1}(x)$ and $\mathcal{U}_x^{-1} = \{U_{xn}^{-1} : n \in \mathbb{N}\}$.

PROPOSITION 3.2. *Let X be a pre-GF-space. Then $U_{xn}^{-1} = \cap_{x \in A_n} A_n$.*

Proof. $y \in U_{xn}^{-1}$ if and only if $x \in U_{yn}$. Now, $x \in A_n$ if and only if $y \in A_n$ (since $x \in U_{yn} = X \setminus \bigcup_{y \notin A_n} A_n$) $\square$

Now we study how fractal structure is induced to subspaces and products.

PROPOSITION 3.3. *Let $(X, \mathbf{\Gamma})$ be a (pre-) GF-space and A a subspace of X. Then $(A, \mathbf{\Gamma}_A)$ is a (pre-)GF-space, with $\mathbf{\Gamma}_A = \{,'_n : n \in \mathbb{N}\}$ and $,'_n = \{A_n \cap A : A_n \in ,_n\}$.*

Proof. For $x \in A$ we have that $U'_{xn} = A \setminus \bigcup_{x \notin A_n \cap A}(A_n \cap A) = A \cap (X \setminus (A \cap (\bigcup_{x \notin A_n} A_n))) = A \cap ((X \setminus A) \cup U_{xn}) = A \cap U_{xn}$

Hence U'_{xn} is an open neighbourhood base of x for all $x \in A$ and therefore $\mathbf{\Gamma}_A$ is a prefractal structure on A.

Suppose that $\mathbf{\Gamma}$ is a fractal structure.

It is clear that $,'_{n+1}$ is a refinement of $,'_n$ and that $,'_n$ is a closed covering for A.

If $x \in A \cap A_n$, then $x \in A_n$, so there exists $A_{n+1} \in ,_{n+1}$ such that $x \in A_{n+1} \subseteq A_n$, and then $x \in A_{n+1} \cap A \subseteq A_n \cap A$. Therefore $\mathbf{\Gamma}_A$ is a fractal structure on A. $\square$

Proposition 3.4. *Let $(X_i, \mathbf{\Gamma}^i)$ be a countable family of (pre-) GF-spaces. Then $(\Pi_{i \in \mathbb{N}} X_i, \Pi_{i \in \mathbb{N}} \mathbf{\Gamma}^i)$ is a (pre-) GF-space, with $\Pi_{i \in \mathbb{N}} \mathbf{\Gamma}^i = \{,_n : n \in \mathbb{N}\}$ and $,_n = \{\cap_{i \leq n} p_i^{-1}(A_n^i) : A_n^i \in ,^i_n\}$ (where p_i is the projection from the product space to X_i).*

Proof. Let us see that $U_{xn} = U^1_{x_1 n} \times \ldots \times U^n_{x_n n} \times X_{n+1} \times \ldots$.

Let $y \in U_{xn}$, then $x \in U^{-1}_{yn} = \bigcap_{y \in A_n} A_n$. Let $i \leq n$ and let A_n^i be such that $y_i \in A_n^i$, if we see that $x_i \in A_n^i$ then $x_i \in \bigcap_{y_i \in A_n^i} A_n^i = (U_{y_i n})^{-1}$, and hence $y_i \in U^i_{x_i n}$.

For $i \neq j \leq n$ let A_n^j be such that $y_j \in A_n^j$. Let $A_n = \bigcap_{k \leq n} p_k^{-1}$ (A_n^k). Then it is clear that $y \in A_n$; hence $x \in \bigcap_{y \in B_n} B_n \subseteq A_n$, and then $x_i \in A_n^i$. Therefore $x_i \in \bigcap_{y_i \in B_n^i} B_n^i = (U^i_{y_i n})^{-1}$ and then $y_i \in U_{x_i n}$. This proves one of the inclusions, and the reverse one is analogous. Therefore $\Pi_{i \in \mathbb{N}} \mathbf{\Gamma}^i$ is a pre-fractal structure over X.

Suppose, now, that each $\mathbf{\Gamma}^i$ is a fractal structure over X_i. It is clear that $,_n$ is a closed covering of $\Pi_{i \in \mathbb{N}} X_i$

Let $x \in A_n$ with $A_n = \bigcap_{i \leq n} p_i^{-1}(A_n^i)$ and $A_n^i \in ,^i_n$. Then $x_i \in A_n^i$ for all $i \leq n$, and since $,^i_n$ is a fractal structure then there exist A^i_{n+1} such that $x_i \in A^i_{n+1} \subseteq A^i_n$ for $i \leq n$. Let $A^{n+1}_{n+1} \in ,^i_{n+1}$ be such that $x_{n+1} \in A^{n+1}_{n+1}$ and let $A_{n+1} = \bigcap_{i \leq n+1} p_i^{-1}(A^i_{n+1})$. Then it is clear that $x \in A_{n+1} \subseteq A_n$. Therefore $\Pi_{i \in \mathbb{N}} \mathbf{\Gamma}^i$ is a fractal structure over X. $\square$

We associate a countable transitive quasiuniformity base to each pre-fractal structure as follows.

Proposition 3.5. *Let X be a pre-GF-space. Then $\mathcal{B}(\mathbf{\Gamma}) = \{U_n : n \in \mathbb{N}\}$ is a transitive quasiuniformity base over X, and $(X, \mathcal{B}(\mathbf{\Gamma}))$ is called the countable transitive quasiuniform space associated to $(X, \mathbf{\Gamma})$.*

Proof. We only have to prove that for $x, y \in X$, $y \in U_{xn}$ implies $U_{yn} \subseteq U_{xn}$. Since y belongs to U_{xn} if and only if $x \in U_{yn}^{-1} = \cap_{y \in A_n} A_n$ (by proposition 3.2), then $U_{yn} = \mathrm{St}(y, , _n) \setminus \bigcup_{y \notin A_n} A_n \subseteq U_{xn}$ □

Proposition 3.6. *Let $(X, \mathbf{\Gamma})$ be a GF-space. Then there exists a non-archimedean quasipseudometric d over X (noted by $d_{\mathbf{\Gamma}}$ and called the canonical quasipseudometric associated to $\mathbf{\Gamma}$), such that $U_{xn} = \{y \in X : d(x, y) < 2^{-n}\}$*

Proof. Let $d(x, y)$ be defined by $2^{-(n+1)}$ if $y \in U_{xn} \setminus U_{x(n+1)}$, by 1 if $y \notin U_{x1}$ and by 0 if $y \in U_{xn}$ for all n. Let see that d is a non-archimedean quasipseudometric. Let $x, y, z \in X$.

Case 1. $d(x, y) = 1$.

Suppose that $d(x, z), d(z, y) \leq 2^{-2}$. Then by definition of d, $z \in U_{x1}$ and $y \in U_{z1}$, but then by transitivity of the quasiuniformity $y \in U_{x1}$, which contradicts that $d(x, y) = 1$.

Case 2. $d(x, y) = 2^{-(n+1)}$ for some natural n.

Suppose that $d(x, z), d(z, y) \leq 2^{-(n+2)}$. Then by definition of d, $z \in U_{x(n+1)}$ and $y \in U_{z(n+1)}$, but then by transitivity of the quasiuniformity $y \in U_{x(n+1)}$, which contradicts that $d(x, y) = 2^{-(n+1)}$.

Case 3. $d(x, y) = 0$. This is clear.

In either of the three cases we have that $d(x, y) \leq \max\{d(x, z), d(z, y)\}$, and hence d is a non-archimedean quasimetric. By construction of d it is clear that $U_{xn} = \{y \in X : d(x, y) < 2^{-n}\}$. □

If we have a fractal structure instead of a pre-fractal one, the neighbourhoods form a decreasing sequence.

Lemma 3.7. *Let X be a GF-space, and let $m \leq n$ be natural numbers, and $x \in X$. Then $U_{xm} \subseteq U_{xn}$.*

Proof. Let $y \in U_{xm}$. If $y \in A_n$, then by induction on the first property on GF-spaces it is easy to prove that there exists $A_m \in , _m$ such that $y \in A_m \subseteq A_n$. Since $x \in U_{ym}^{-1} = \cap_{y \in B_m} B_m$ and $y \in A_m$, then $x \in A_m \subseteq A_n$, that is, $x \in A_n$. Then we have proved $x \in \cap_{y \in A_n} A_n = U_{yn}^{-1}$. Therefore $y \in U_{xn}$. □

Proposition 3.8. *Let G be a poset. Then there exists a fractal structure over G.*

Proof. Let $,_n = \{\overline{\{g\}} : g \in G\}$ for all natural n.

1. $,_{n+1}$ is a refinement of $,_n$ and given $g \in A_n$, with $A_n \in ,_n$, there exists $A_{n+1} \in ,_{n+1}$ $(A_{n+1} = A_n)$ such that $g \in A_{n+1} \subseteq A_n$.

   This is clear since $,_{n+1} = ,_n$ for all n.

2. U_{gn} is an open neighbourhood base of g for all $g \in G$.

   We only have to prove that $U_{gn} = [g, \to[$, that is, $h \in U_{gn}$ if and only if $g \leq h$ for all n.

   Let $h \in U_{gn}$, that is $g \in U_{hn}^{-1} = \bigcap_{h \leq k} \overline{\{k\}}$. Then, since $h \leq h$ we have that $g \in \overline{\{h\}}$, and hence $g \leq h$.

   On the other hand, let $h \in G$ be such that $g \leq h$. If $k \in G$ is such that $h \leq k$, then $g \leq h \leq k$ and $g \leq k$. Hence $g \in \bigcap_{h \leq k} \overline{\{k\}} = U_{hn}^{-1}$ and then $h \in U_{gn}$

□

Finally, we characterize GF-spaces as non-archimedeanly quasi-metrizable spaces.

Theorem 3.9. *Let X be a topological space. The following statements are equivalent:*

1. *There is (at least) a fractal structure over X.*
2. *There is (at least) a pre-fractal structure over X.*
3. *X is non-archimedeanly quasipseudometrizable.*
4. *X can be embedded into the inverse limit of a sequence of posets.*
5. *X can be embedded into a countable product of posets.*

Proof. (1) implies (2) and (4) implies (5) are obvious.

(2) implies (3) By Proposition 3.5, X admits a countable transitive quasiuniform base, and then by Theorem 7.1 of [9] it is a non-arcchimedeanly quasiuniform space.

(3) implies (4) Theorem 2.3

(5) implies (1) By Proposition 3.8, X is homeomorphic to a subspace of a countable product of GF-spaces, and then by Propositions 3.3 and 3.4 it is a GF-space. □

References

[1] F. G. Arenas and M. A. Sánchez-Granero, *GF-spaces and self similar sets*, preprint.

[2] C. Bandt and K. Keller, *Self-Similar Sets 2: a Simple Approach to the Topological Structure of Fractals*, Math. Nachr. **154** (1991), 27–39.

[3] C. Bandt and T. Retta, *Self-similar Sets as Inverse Limits of Finite Topological Spaces*, Topology, measures, and fractals (Warnemunde, 1991) (Berlin), Akademie-Verlag, 1992.

[4] C. Bandt and T. Retta, *Topological spaces admitting a unique fractal structure*, Fund. Math. **141** (1992), 257–268.

[5] W. J. Charatonik and A. Dilks, *On self-homeomorphic spaces*, Top. and its Appl. **55** (1994), 215–238.

[6] G. A. Edgar, *Measure, Topology, and Fractal Geometry*, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 1990.

[7] K. Falconer, *Fractal geometry*, Mathematical Foundations and Applications, John Wiley & Sons, 1990.

[8] J. Feder, *Fractals*, Plenum Press, New York, 1988.

[9] P. Fletcher and W.F. Lindgren, *Quasi-Uniform Spaces*, Lecture Notes Pure Appl. Math., vol. 77, Marcel Dekker, New York, 1982.

[10] M. Hata, *On some Properties of Set-dynamical Systems*, Proc. Japan Acad., Ser. A **61** (1985), 99–102.

[11] M. Hata, *On the structure of Self-Similar Sets*, Japan J. Appl. Math. **2** (1985), 381–414.

[12] M. Hata, *Topological aspects of self-similar sets and singular functions*, Fractal Geometry and Analysis, Kluwer Academic Publishers, Dordrecht, 1991, pp. 255–276.

[13] M. Hata, *Fractals-On Self-similar Sets*, Sugaku Expositions **6** (June 1993), no. 1.

[14] A. Kameyama, *Self-Similar Sets from the Topological Point of View*, Japan J. Indust. Appl. Math. **10** (1993), 85–95.

[15] A. R. Pears, *On quasi-order spaces, normality, and paracompactness*, Proc. London Math. Soc. **23** (1971), no. 3, 428–444.

Received July 12, 1997.