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Weierstrass Points, Inflection Points and Ramification Points of Curves

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Summary. - *Let C be an integral curve of the smooth projective surface S and $P \in C$. Let $\pi : X \to C$ be the normalization and $Q \in X$ with $\pi(Q) = P$. We are interested in the case in which Q is a Weierstrass point of X. We compute the semigroup $N(Q,X)$ of non–gaps of Q when S is a Hirzebruch surface $F_e, P \in C_{reg}$ and P is a total ramification point of the restriction to C of a ruling $F_e \to P^1$. We study also families of pairs (X,Q) such that the first two integers of $N(Q,X)$ are k and d. To do that we study families of pairs (P,C) with C plane curve , $deg(C) = d$, C has multiplicity $d-k$ at P, C is unibranch at P and a line through P has intersection multiplicity d with C at P.*

1. Introduction

We work over an algebraically closed field $\mathbf{K}$ with $char(\mathbf{K}) = 0$. Let C be an integral Gorenstein projective curve with $g := p_a(C) \geq 2$. Fix $P \in C_{reg}$. Since C is smooth at P, for every integer t the sheaf $\omega_C(-tP)$ is a line bundle. Hence, exactly as in the case of a smooth curve we may define the numerical semigroup of non–gaps of C at P, say $\mathbf{N}(P,C) \subset \mathbf{N}$, such that $card(\mathbf{N}\backslash\mathbf{N}) = g$ ([15], [14]). P is not a Weierstrass point of C if and only if $\mathbf{N}\backslash\mathbf{N}(P,C) = \{1,\dots,g\}$. We just recall that in general $\mathbf{N}(P,C)$ is non a semigroup (for any conceivable definition of $\mathbf{N}(P,C)$) without the assumption $P \in C_{reg}$ ([14]). Reading [18], it seems obvious that the last assertion

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of [18] ,Th. 1, p. 545, is not quite true as stated. Hence it seems natural to try to obtain a recipe for the construction of many triples $(g, k, d), k < d - 2k$, of a pair (X, Q), where X is a smooth curve of genus g and $Q \in X$ is a Weierstrass point with k and d as first two positive integers of $\mathbf{N}(Q, X)$. Even more: we want to construct nice families of such pairs. For any such pair (Q, X) the complete linear system $|dQ|$ is a base point free and induces a morphism $f : X \to \mathbf{P}^2$. Assume f birational and set $C := f(X)$, $P := f(Q)$. The reduction of the tangent cone of C at P is given by a unique line, L, which has intersection multiplicity d with C at P. Furthermore, C is unibranch at P and it has multiplicity $d-k$ at P because every line $D \neq L$ with $P \in D$ has intersection multiplicity $d - k$. See Theorem 2.5 which clarifies the case in which C is smooth outside P, except for a small number of nodes which are in general $\mathbf{P}^2$. According to [17], Remark 13. 12, our approach should be the classical approach considered in [4], p. 59, and in [13], p. 547. For the case in which g is the first non gap and $g + x$, $1 \leq x \geq g - 1$, is the only gap $> g$, see [17], Th. 14. 7. For the case in which then map $f : X \to C \subset \mathbf{P}^2$ is not birational, see Remark 2.3. In section 3 we compute $\mathbf{N}(P, C)$ when C is an integral curve contained in a Hirzebruch surface, $P \in C_{reg}$, and P is a total ramification point of the restriction of a ruling of F_e to C (see Theorems 3.1 and 3.5). We solve the same problem for the curve X which is a partial normalization of C at a small number of nodes and cusps which are general points of F_e (Theorem 3.3).

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2. Plane curves

In this section we consider the case of plane curves. Fix an integer $d \geq 4$ and an integer y with $0 \leq y \leq d$. At the beginning of this section (Lemma 2.1 and Proposition 2.2) we consider the set–up of [9], i. e. we consider a pair (C, P) such that C is an integral plane curve of degree d, $P \in C_{reg}$ and such that the tangent line $T_P C$ of C at P has intersection multiplicity y with C at P. With the terminology of [9], P is called a $(y - 2)$–inflection point of P. Set $V(y, d) := \{x \in \mathbf{N} : x = ay + b$ for some integers $a,\ b$ with $0 \leq a \leq$

$d-3$ and $0 \leq b \leq d-3-a\}$ and $N(y,d) := \{t \in \mathbf{N} : t-1 \notin V(y,d)\}$. With the definition of the semigroup of non–gaps for smooth points of integral Gorenstein curves outlined in the introduction, the proofs of [9], Lemma 2.1 and Prop. 2.2, work verbatim and give the following lemma.

LEMMA 2.1. *Let $u : X \to C$ be a partial normalization of an integral plane curve C and $P \in C_{reg}$ such that the tangent line $T_P C$ of C at P has intersection multiplicity y with C at P. Then $N(y,d) \subseteq N(P,C)$.*

PROPOSITION 2.2. *Fix integers d, y with $d \geq 4$ and $d-2 \leq y \leq d$. Let C be an integral plane curve and $P \in C_{reg}$ such that the tangent line $T_P C$ of C at P has intersection multiplicity y with C at P. Then $N(y,d) = N(P,C)$.*

Now we consider the problem described in the introduction. We want to construct for several triples (g,k,d) good families of pairs (X,Q) with X smooth genus g curve and Q Weierstrass point of X whose first non–gaps are k and d. We will try to find a plane model (C,P) for (X,Q) such that $C \setminus \{P\}$ has only nodes as singularities. We cannot apply [6], Th. 0. 2, because in our situation the two numerical assumptions of [6] Th. 0. 2, are never simultaneously satisfied, except in the trivial case $k = d-1$, i. e. the case in which C is smooth at Q; for this case, see appendix of [9] in which more is proved. However, we may easily adapt the proofs and ideas contained in [6] to cover our situation.

REMARK 2.3. *As recalled in the introduction the set–up considered in this section covers all pairs (X,Q) with Q Weierstrass point on X whose first non–gaps are k and d with $k < d < 2k$ and such that the complete linear system $|dQ|$ on X induces a birational morphism $f : X \to \mathbf{P}^2$. Now we will show how to reduce to this case the construction of all the possible examples for which f is not birational. Start with a pair (X',Q') with X' of genus $g' \geq 3$ and P' such that the first non–gaps are k' and d' with $k' < d' < 2k'$. Take an integer $t \geq 2$ and a ramified degree t convering $\pi : X \to X'$ which is totally ramified over P' and with X smooth of genus g; set $k := tk'$, $d := td'$ and $Q := \pi^{-1}(Q')_{red}$. Using Castelnuovo–Severi inequality (see e. g. [1], Ch. 3) we see that for many values of t, g, g', d' and*

k', the integers k and d must be the first non–gaps of P; for instance if $t = 2$ is true if $d \leq g - 2g'$. Viceversa, start with the integers g, k, d and assume thath $f : X \to \mathbf{P}^2$ is not birational. Let g' be the genus of the normalization of $f(X)$. Since $h^0(f(X), \mathbf{O}_f(X)(1)) = 3$ and $k < d < 2k$, it is easy to check that $g' \geq 3$.

We recall the ideas introduced in [10] to study equisingular deformations of plane curves. We need this reference for our problem because we need deformations of a unibranch point which preserve the condition "unibranch" and the multiplicity. We fix integers k and d with $3 \leq k < d$; in the application to gap–sequences it is sufficient to consider the case $d < 2k$. For simplicity we work over the field of complex numbers $\mathbf{C}$. We fix $P \in \mathbf{P}^2$ and the germ at P (in the analytic category) of a curve T with a unibranch singularity at P with multiplicity $d-k$. Call L the reduced tangent cone of T at P seen as a line in $\mathbf{P}^2$; we assume that L has intersection multiplicity d with L at P. The construction of all such examples is obvious in terms of Puiséux expansions. The paper [10] contains the definition of a zero–dimensional subscheme Z' of $\mathbf{P}^2$ (called the generalized singularity scheme) such that the condition $h^1(\mathbf{P}^2, \mathbf{I}_{Z'}(d)) = 0$ gives the existence of a plane curve U of degree d with $P \in U$ and such that U is topologically equivalent to T at P. For any $Q \in \mathbf{P}^2$ and integer $x > 0$, let xQ the fat point of order x in $\mathbf{P}^2$ supported by Q, i. e. the $(x-1)$-th infinitesimal neighborhood of Q in $\mathbf{P}^2$. Hence $dP|L$ is the effective divisor of degree d on L given by the multiple of order d of P. Taking $Z'' := Z' \cup (dP|L) \cup (d-k)Q$ instead of Z' if $h^1(\mathbf{P}^2, \mathbf{I}_{Z''}(d)) = 0$ we may obtain an irreducible plane curve V of degree d topologically equivalent to T near P, V and L having intersection multiplicity d at P and such that V has multiplicity at least $d-$ at P. If Z'' does not contain $(d-k+1)P$, then we may even find such V with multiplicity $d-k$ at P. Take the germ T at P of any integral unibranch curve at P whose ideal sheaf I_T at P contains $\mathbf{I}_{Z''}$ and let $\mathbf{c}$ be its conductor; since T is Gorenstein at P, the partial normalization T'' of T at P has $dim_{\mathbf{K}}(\mathbf{O}_{T''}/\mathbf{O}_T) = dim_{\mathbf{K}}(\mathbf{O}_{T,P}/\mathbf{c})$; we are interested mainly in the case in which $dim_{\mathbf{K}}(\mathbf{O}_{T,P}/\mathbf{c})$ is as small as possible (compatibility with the order data). Let Z be the minimal zero–dimensional subscheme of $\mathbf{P}^2$ containing both Z' and the scheme $T/\mathbf{c}$; since every deformation of T preserving $\mathbf{c}$ preserves the geometric genus, if $h^1(\mathbf{P}^2, \mathbf{I}_Z(d)) = 0$ we may obtain an irreducible plane curve V of degree d topologically equivalent to T near P, V and L having intersection multiplicity d at P, such that V has multiplicity at least $d-k$ at P and such that the arithmetic genus of the partial normalization at P is fixed and is it given by $(d-1)(d-2)/2 - dim_{\mathbf{K}}(\mathbf{O}_{T,P}/\mathbf{c})$. We will call any such Z suitable for the pair (k,d).

DEFINITION 2.4. *Let Z be a zero-dimensional subscheme of $\mathbf{P}^2$ and $L \subset \mathbf{P}^2$ a line. Set $Z\{1\} := Z$, $Z(1) := Z \cap L$ and call $Z\{2\}$ the residual scheme $Res_L(Z)$ of Z with respect to L. Define inductively $Z(i)$ and $Z\{i+1\}$ using the formulas $Z(i) := Z\{i\} \cap L$ and $Z\{i+1\} := Res_L(Z\{i\})$. Set $z(i) := h^0(Z(i), \mathbf{O}_{Z(i)})$. Notice that $z\{i+1\} + z(i) = z\{i\}$ and $z(i+1) \leq z(i)$ for every $i \geq 1$. We will call the non–increasing sequence $\{z(1), z(2), \dots\}$ (resp. $\{z\{1\}, z\{2\}, \dots\}$) the first (resp. second) associated sequence of Z with respect to L.*

THEOREM 2.5. *Fix integers d, k, x with $d > k \geq 3$. Fix $P \in \mathbf{P}^2$ and a zero–dimensional scheme Z suitable for the pair (k,d), say with respect to the line L. Let $z := h^0(Z, \mathbf{O}_Z)$ be the length of Z and let $\{z(1), z(2), \dots\}$ be the first associated sequence of Z with respect to L. Assume $z(1) = d$, $z(i) \leq max\{d-i-1, 0\}$ for every $i \geq 2$ and $0 \leq 3x \leq d(d+3)/2 - 3 - z$. Then for x general points $Q_1, \dots, Q_x$ of $\mathbf{P}^2$ there exists an integral degree d curve $C \subset \mathbf{P}^2$ with $P \in C$, C unibranch at P, L intersecting C with multiplicity d, $Z \subset C$, $Q_i \in C$ for every i, Q_i with ordinary nodes at each Q_i and C smooth outside $\{P, Q_1, \dots, Q_x\}$. Furthermore, fixing Z and varying $Q_1, \dots, Q_x$ in a Zariski open subset of symmetric product $S^x(\mathbf{P}^2)$ we obtain an irreducible family $M(Z,x)$ of dimensiond $d(d+3)/2$ of plane curves with geometric genus $(d-1)(d-2)/2 - \delta - x$, where δ is the arithmetic genus of the singularity (C,P).*

Proof. Step 1) Here we will show that the union of Z and the first infinitesimal neighborhoods $2Q_i$, $1 \leq i \leq x$, of x general points Q_i of $\mathbf{P}^2$ imposes $z + 3x$ independent conditions to the linear system of degree d plane curves containing them. Since $z(1) = d$ is the maximal length of a subscheme of Z contained in a line through P, we have $Z \subseteq dP$. Hence the case $x = 0$ is obvious. Hence we will assume $x > 0$. Take a general $Q \in L$ and let $\mathbf{t}$ the length 2 zero–dimensional subscheme of L with $\mathbf{t}_{red} = \{Q\}$; the scheme $\mathbf{t}$ is the second simple residue of Q with respect to L in the sense of [2], Definition 2. 2. Take $x-1$ general points Q_i, $1 \le i \le x-1$, of $\mathbf{P}^2$ and let W be the union of $Z\{2\}$, the schemes $2Q_i$, $1 \le i \le x-1$, and $\mathbf{t}$. By [2], Lemma 2. 3 it is sufficient to prove the vanishing of $h^1(\mathbf{P}^2, \mathbf{I}_W(d-1))$. Notice that $length(W \cap L) = z(2) + 2 \le d-2$. Now we continue making again an application of Horace Lemma with respect to the line L. We specialize $[(d-z(2)-2)/2]$ of the points Q_i, say Q_i, $1 \le i \le [(d-z(2)-2)/2]$, to general points Q_i', $1 \le i \le [(d-z(2)-2)/2]$ of L; if $d-z(2)-2$ is even we call W' the union of $Z\{2\}$, $2Q_i'$, $1 \le i \le [(d-z(2)-2)/2]$ and $2Q_j$, $[(d-z(2)-2)/2] < j \le x$ with Q_j' general in $\mathbf{P}^2$. Set $W'' := Res_L(W')$. Since $length(W' \cap L) = d$, we have $h^1(\mathbf{P}^2, \mathbf{I}_{W'}(d-1) = h^1(\mathbf{P}^2, \mathbf{I}_{W''}(d-2)$ and hence we may continue the construction. If $d-z(2)-2$ is odd instead of a general point Q_α, $\alpha := [(d-z(2)-2)/2]+1$, of $\mathbf{P}^2$ we specialize it to a general $Q' \in L$ and call $\mathbf{t}'$ the length 2 zero–dimensional subscheme of L with $\mathbf{t}'_{red} = \{Q'\}$; set $W(1) := (W'' \setminus 2Q_{x-1}) \cup \mathbf{t}'$. By [2], Lemma 2. 3, to prove the vanishing of $h^1(\mathbf{P}^2, \mathbf{I}_{W(}d-1))$ it is sufficient to prove the vanishing of $h^1(\mathbf{P}^2, \mathbf{I}_{W(1)}(d-2))$. Then we continue to reduce the vanishing we need to the vanishing of some group $H^1(\mathbf{P}^2, \mathbf{I}_{A(}t))$ for some integer $t < d$ and some zero–dimensional scheme A whose connected component, B, supported by P is $Z\{d-t+1\}$. Hence at each step we add at most one double point, $\mathbf{t}''$, supported by a general point of L and this length 2 zero–dimensional scheme gives no contribution when we take the residual scheme with respect to L. Since $z(d-t) \le t+1$,we are sure that at every step we may add on L a scheme,$\mathbf{t}'$ of length 2 when $t+2-z(d-t-1)$ is odd, and still have the vanishing of the cohomology group H^1 for $A \cap L$. If we finish the x points before arriving to the case $t = 0$, we have won. Since $z + 3e \le h^0(\mathbf{P}^2, \mathbf{O}_{\mathbf{P}^2}(d)) - (h^0(\mathbf{P}^2, \mathbf{O}_{\mathbf{P}^2}(1)))$, we are sure to finish the points Q_i.

Step 2) The generalized associated scheme of an ordinary double point at Q is just ideal sheaf of $2Q$ ([10], Ex. 2 after Def. 2. 3). Hence by [10] we obtain that the family of plane curves with $x+1$ prescribed singularities (one at P with Z as associated scheme plus x nodes in general position) is smooth of the expected dimension and its general member has only the expected singularities. $\square$

REMARK 2.6. *Since in characteristic zero every smooth curve of genus at least* 2 *has only finitely many Weierstrass points, it is obvious that the "parametrization" of smooth curves with a Weierstrass point with first non–gaps* k *and* d *given by Theorem 2.5 is, up to an element of* $Aut(\mathbf{P}^2)$, *finite – to – one.*

EXAMPLE 2.7. *Here we will compute the associated scheme in the particular case in which* d *and* k *are coprime. In this case the germ at* $(0,0)$ *of the curve* $t \to (t^{d-k}, t^d)$ *is the germ of a unibranch curve which has all the properties we want. We drop the condition* $d < 2k$. *We apply the standard blowing–up procedure to this curve whose local equation near* $(0,0)$ *is* $y^{d-k} = x^d$. *Set* $x' = x$, $y' = yx$. *In the first infinitesimal neighborhood we obtain an equation* ${y'}^{d-k} = {x'}^k$. *Since* k *and* $d-k$ *are coprime, we continue in the same way and find that the tree of the resolution is a Dynkin diagram of type* A_W *for some* w. *With the notations of [19], p. 15, the characteristic of this branch are the integers* $(d-k, d)$ *and* $g = 1$. *This example is important because any nearby germ with the same characteristic exponents is formally equivalent to it and in particular it has the same* δ*-invariant, i.e. the conductor has the same colength.*

EXAMPLE 2.8. *Now we will construct in some cases plane curves with two associated schemes of type* (k,d), *but with* $2k < d$. *More generally, we fix two integers* k, k' *with* $2 \leq k \leq k' \leq d-k$, *take two distinct points* P, P', *of* $\mathbf{P}^2$, *lines* L, L', *on* $\mathbf{P}^2$ *with* $P \in L$, $P' \in L'$, $P \notin L'$, $P' \notin L$ *and look at integral plane curves,* C, *of degree* d *passing through* Q *(resp.*Q'*) with multiplicity* $d-k$ *(resp.* $d-k'$*), unibranch at* P *and* P' *and with* L *(resp.* L'*) with intersection multiplicity* d *with* C *at* P *(resp.* P'*). Let* Z *(resp.* Z'*) be the associated scheme for the data* (k,d,P,L) *(resp.* (k',d,P',L')*). We want to prove (under suitable assumptions on* Z *and* Z'*) that* $h^1(\mathbf{P}^2, \mathbf{I}_{Z\cup Z'}(d)) = 0$. *Set* $W := Res_L(Z)$ *and* $W' := Res_{L'}(Z')$. *We have* $length(W) = length(Z) - d$ *and* $length(W') = length(W') - d$. *We apply first Horace Lemma with respect to* L *(loosing one condition) and then Horace Lemma with respect to* L' *(without loosing anything because* L' *is in the base locus of* $H^0(\mathbf{P}^2, \mathbf{I}_{W\cup W'}(d-1))$*). Hence to prove the vanishing of* $h^1(\mathbf{P}^2, \mathbf{I}_{Z\cup Z'}(d)) = 0$ *it is sufficient to prove the vanishing of* $h^1(\mathbf{P}^2, \mathbf{I}_{W\cup W'}(d-2))$. *Let* $\{z(1), z(2), \dots\}$ *(resp.* $\{z'(1), z'(2), \dots\}$*) be the first associated sequence of* Z *(resp.*

$Z')$. The proof of Theorem 2.5 works verbatim exploiting alternatively L and L' if for every integer $i \geq 2$ we have $z(i) \geq d-2i$ and $z'(i) \geq d-2i-1$. However, if for some integers k, k' we want to construct such integral curves with a few nodes, say at general point $Q_1, \dots, Q_y$ with y very small, it is better to use the following trick to check the vanishing of $h^1(\mathbf{P}^2), \mathbf{I}_{W \cup W' \cup 2Q_1 \cup \dots \cup 2Q_y}(d-2))$. We apply Horace Lemma $[(d-2)/2]$ times using a smooth conic passing through P, P' and perhaps some of the points Q_i. Call D a general conic through P and P' and define the schemes $W(i)$ (resp. $W'(i)$), $i \geq 1$, and the first associated D–sequence to W (resp. W') exploiting D instead of the line L (resp. L') and denote it with $\{w(1), \dots\}$ (resp. $\{w'(1), \dots\}$). Hence $w(1) = w'(1) = d-2$. Suppose that after $i-1$ applications of Horace Lemma using a smooth conic through P and P' one need to check the vanishing $h^1(\mathbf{P}^2, \mathbf{I}_{W(i-1) \cup W'(i-1) \cup 2Q_1 \cup \dots \cup 2Q_W}(d-2i))$ for some non–negative integers w, z. By the generality of the points $Q_1, \dots, Q_y$ it is sufficient to have $h^1(\mathbf{P}^2, \mathbf{I}_{W(i-1) \cup W'(i-1) \cup 2Q_1 \cup \dots \cup 2Q_W}(d-2i)) = 0$ and $length(W(i-1)) + length(W'(i-1)) + 3w + z \leq (d-2i+2)(d-2i+1)/2$. Fix an integer m with $0 \leq m \leq min\{w, 5\}$ and take a general smooth conic A through P, P' and m of the points $Q_1, \dots, Q_W$. To apply another time Horace Lemma exploiting A it is sufficient to have $2m + w(i-1) + w'(i-1) \leq 2(d-2i)$.

3. Curves on Hirzebruch surfaces

In this section we fix an integer $e \geq 0$ and set $S := F_e$, where $F_e = \mathbf{P}(\mathbf{O}_{P^1} \oplus \mathbf{O}_{P^1})$ is a Hirzeburch surface. Let $\pi : S \to \mathbf{P}^1$ be the associated ruling. We take as a basis of $Pic(S) \cong \mathbf{Z}^2$ a fiber, f, of π and a section h of π with minimal self–intersection. Hence we have $h^2 = -e$, $h \cdot l = 1$ and $f^2 = 0$. For the cohomological properties of the line bundles on S, see [11], pp. 379–381. We will use both the additive and multiplicative notation for line bundles, divisors and linear systems on S. We just observe that (as in [11], pp. 379–381) everything follows from the projection formula $\pi_*(\mathbf{O}_S(ah+bf)) \cong \oplus_{0 \leq t \leq a} \mathbf{O}_{P^1}(b-te)$ for all integers a, b with $a \geq 0$. We have $\omega_S = -2h + (e-2)f$. We are interested in integral curves $C \in |kh + xf|$, some $k \geq 2$, $x \geq ke$, $x > 0$, such that there are some points, say

$P_1, \dots, P_z \in C_{reg}$, $z \geq 1$, such that the fiber of f at each point P_i has intersection multiplicity k at P_i, i.e. such that every point P_i is a total ramification point of π. By the adjunction formula we have $p_a(C) = kx - ek(k-1)/2 - x - k + 1$. Since the case $k = 2$ corresponds to hyperelliptic curves, we will assume $k \geq 3$.

THEOREM 3.1. *Fix integers k, x with $k \geq 3$, $x > 0$, $x \geq ke$, an integral curve $C \in |kh + xf|$ and $P \in C_{reg}$ which is a total ramification point of π. If $e > 0$ assume that P is not contained in the section h of π with minimal self–intersection. If $e = 0$ we have $h^0(C, \mathbf{O}_C(akP)) = a + 1$ if $0 \leq 0 \leq x$, $h^0(C, \mathbf{O}_C(tP)) = [t/k] + 1$ if $bk < t < (b+1)k$ for some b with $b = 1 \leq x$, while $h^0(C, \mathbf{O}_C(tP)) = t + 1 - g$ (i.e. $h^1(C, \mathbf{O}_C(tP)) = 0$) if $t > xk$, i.e. the semigroup $\mathbf{N}(P, C)$ is the minimal one compatible with its first positive value, k, and the constraint card $(\mathbf{N} \setminus \mathbf{N}(P, C)) = g$. Assume $e > 0$; we have $h^1(C, \mathbf{O}_C(\alpha kP)) = 0$, i.e. $h^0(C, \mathbf{O}_C(\alpha kP)) = \alpha k - kx + ek(k-1)/2 + x + k$ if and only if $\alpha \geq x - e - 1$; fix an integer $\beta \leq x - e - 2$; let u be the minimal non–negative integer with $e(k - 2 - u) \leq x - e - 2 - \beta$; we have $h^0(C, \mathbf{O}_C(\beta kP)) = \beta - kx + ek(k-1)/2 + x + k + \sum_{0 \leq i \leq k-2-u} (x - e - 1 - \beta - ie)$; the non–gaps of P in the interval $k\beta < t \leq k\beta + k$ are at last $h^0(C, \mathbf{O}_C((\beta + 1)kP)) - h^0(C, \mathbf{O}_C(\beta kP))$ integers in this interval.*

Proof. It is sufficient to compute the value of $h^0(C, \omega_C(-tP))$ for every integer $t > 0$. For every integer $t > 0$, call $D(t)$ the zero–dimensional subscheme of S corresponding to the positive Cartier divisor tP on C. Since $h^0(S, K_S) = h^1(S, K_S) = 0$, the restriction map $H^0(S, \mathbf{O}_S((k-2)h + (x - e - 2)f)) \to H^0(C, \omega_C)$ is bijective. Hence it is sufficient to compute $h^0(S, \mathbf{O}_S((k-2)h + (x + e - 2)f) \oplus \mathbf{I}_{D(t)})$ for every integer $t > 0$. Set $w := [t/k]$ and $z := t - kw$. Hence $0 \leq z \leq k$ and $h^0(S\mathbf{O}_S((k-2)h + (x+e-2)f) \oplus \mathbf{I}_{D(t)}) = h^0(S, \mathbf{O}_S((k-2)h + (x + e - 2 - w)f) \oplus \mathbf{I}_{D(z)})$. By the cohomology exact sequence of the restriction of K_S to α fibers, $\alpha \geq 0$ we obtain $h^1(S, K_S(-\alpha f)) = 0$ for every positive integer α. Since $h^0(S, K_S(-\alpha f)) = 0$ for every positive integer α, by Serre duality on C we obtain $h^1(C, \mathbf{O}(\alpha tP)) = 0$ if and only if $\alpha \geq x - e - 1$ and $h^1(C, \mathbf{O}_C(\beta tP)) = h^0(S, \mathbf{O}_S((k-2)h + (x - e - 2 - \beta)f))$ for every positive integer β; let u be the multiplicity of h as base component of $|(k-2)h + (x - e - 2 - \beta)|$, i.e. let u be the minimal non–negative integer with $e(k-2-u) \leq x - e - 2 - \beta$;

by the projection formula we obtain $h^0(S, \mathbf{O}_S((k-2)h+(x-e-2-\beta)f)) = h^0(S, \mathbf{O}_S((k-2-u)h+(x-e-2-\beta)f)) = \sum_{0\le i\le k-2-u}(x-e-1-\beta-ie)$. Hence we know $h^0(C, \mathbf{O}_C(wkP))$ and $h^0(C, \mathbf{O}C((w+1)kP))$ and we have $1 \le h^0(C, \mathbf{O}_C((w+1)kP)) - h^0(C, \mathbf{O}_C(wkP)) \le k-1$, until $\mathbf{O}_C((w+1)kP)$ is a special line bundle, i.e. for $w \le x-e-2$. Furthermore since $\mathbf{O}_C((w+1)kP)$ is spanned, we know that $h^0(C, \mathbf{O}_C((wk+k-1)kP)) = h^0(C, \mathbf{O}_C((w+1)kP)) - 1$. For every integer q we have $h^0(C, \mathbf{O}_C(qP)) \le h^0(C, \mathbf{O}_C((q+1)P)) \le h^0(C, \mathbf{O}_C(qP)) + 1$. To prove the theorem it is sufficient to check that the $h^0(C, \mathbf{O}_C((w+1)kP)) - h^0(C, \mathbf{O}_C(wkP))$ jumps for the values of $h^0(C, \mathbf{O}_C(qP))$ in the interval $wk \le q < (w+1)k$ occur for $q = wk$ and the last $h^0(C, \mathbf{O}_C((w+1)kP)) - h^0(C, \mathbf{O}_C(wkP)) - 1$ values of q. First we assume $e > 0$. Call $m \ge 0$ the multiplicity of h as base component of $|(k-2)h+(x-e-2-w-1)f|$, i.e. set $m := 0$ if $x-e-2-w \ge ek$ and set $m := k-2-[(x-e-2-w-1)/e]$, otherwise. By Serre duality on C it is sufficient to note that the restriction of $|(k-2-m)h+(x-e-2-w-1)|$ to any fiber of π (even to $\pi^{-1}(\pi(P))$) is a complete linear system of degree $(k-2-m)$, that $h^0(C, \mathbf{O}_C((w+1)kP)) - h^0(C, \mathbf{O}_C(wkP)) = k-1-m$ and that for every integer z with $0 \le z \le t$ the zero–dimensional scheme $D(z)$ is contained in the fiber $\pi^{-1}(\pi(P))$. The case $e = 0$ is similar and easier. □

REMARK 3.2. *The proof of Theorem 3.1 works verbatim if C is any integral curve such that $P \in C_{reg}$ and such that the fiber of $\pi|C$ is just P with multiplicity k. Of course, here we use the notation of gap sequence at a smooth point of any Gorestein curve. The power of this tool will be shown by the proof of Theorem 3.3 we assume $Sing(C) = \{Q_1, \dots, Q_y\}$, i.e. even in the case in which we are studing Weierstrass points on a smooth curve (the normalization of C).*

Now we consider the case of the normalization of nodal or cuspidal curves. The case $e = 0$ was considered (at least for the nodes) in [5] and [7]. Again, the aim is to show that when we have very few nodes and cusps and/or their position is sufficiently general, then the situation is the best possible one with the numerical constraints we have and in particular the Weierstrass point has the a priori possible minimal weight. Since it comes for free, we will consider a more gen-

eral statement allowing the partial normalization of a very singular curve.

THEOREM 3.3. *Fix integers k, x, y with $k \geq 3$, $x > 0$, $x \geq ke$ and $y \geq 0$. Fix y general points $Q_1, \dots, Q_y$ of $S := F_e$. Assume the existence of an integral curve C with $C \in |kh + xf|$ such that each Q_i is an ordinary node or an ordinary cusp of C and such that there is $P \in C_{reg}$ which is a total ramification point of π. If $e > 0$ assume that P is not contained in the section h of π with minimal self–intersection. Let X be the partial normalization of C at $Q_1, \dots, Q_y$. Set $g := kx - ek(k-1)/2 - k - x + 1 - y$. Let $C' \in |kh + xf|$ be a smooth curve with $P \in C$ and such that P is a total ramification point for $\pi|C'$. Then for every integer $t > 0$ we have $h^0(C, \omega_C(-tP)) = max\{h^0(C', \omega_{C'}(-tP)) - y, 0\}$.*

Proof. The adjoint linear system to X is given by the curves in $|(x-2)k+(x+e+2)f|$ passing through the points $Q_1, \dots, Q_y$. Since these points are in general points of S, they impose the maximal possible number of conditions (i.e. $min\{y, dim(V)\}$) to any complete linear system V on S. Hence the proof of Theorem 3.1 and the claim that the same proof works for a singular curve (Remark 3.2) give the result.

For the existence of a curve C as in the statement of Theorem 3.3 and with $Sing(C) = \{Q_1, \dots, Q_y$, see Remark 3.4. □

REMARK 3.4. *Modifying the proof of [3], Prop. 3.7 and Prop. 4.1, Theorem 3.3 may be applied to several cases in which C has an ordinary node at each point $Q_1, \dots, Q_y$ as unique singularities and hence in which X is smooth; the only difference in each cohomological computation comes from the condition " $\pi|C$ with total ramification at P"; for some case for $e = 0$, see [5] or [7], Proof of Theorem 0.1; for similar cases in the plane, see [9] and [8]; anyway if y is very small with respect to $x - ke$, the existence of such curve C is an easy exercise. Now we will consider the case in which in the statement of Theorem 3.1 we have $e > 0$ and $P \in h$.*

THEOREM 3.5. *Fix integers e, k, x with $e > 0$, $k \geq 3$, $x > 0$, $x \geq ke$, an integral curve $C \in |kh + xf|$, $p \in h$ and $P \in C_{reg}$ which is a total ramification point of π. We have $h^1(C, \mathbf{O}_C(\alpha kP)) = 0$,* *i.e.* $h^0(C, \mathbf{O}_C(\alpha kP)) = \alpha k - kx + ek(k-1)/2 + x + k$ *if and only if* $\alpha \geq x - e - 1$; *fix an integer* $\beta \leq x - e - 2$; *let* u *be the minimal non–negative integer with* $e(k-2-u) \leq x - e - 2 - \beta$; *we have* $h^0(C, \mathbf{O}_C(\beta kP)) = \beta - kx + ek(k-1)/2 + x + k + \sum_{0 \leq i \leq k-2-u}(x - e - 1 - \beta - ie)$; *the non–gaps in the interval* $k\beta < t \leq k\beta + k$ *are in the first* $h^0(C, \mathbf{O}_C((\beta+1)kP)) - h^0(C, \mathbf{O}_C(\beta kP)) - 1$ *values of* t *and the integer* $k\beta + k$.

Proof. We use the notations of the proof of 3.1. To prove 3.5 it is sufficient to check that the $h^0(C, \mathbf{O}_C((w+1)kP)) - h^1(C, \mathbf{O}_C(wkP))$ jumps for the values of $h^0(C, \mathbf{O}_C(qP))$ in the interval $wk \leq t < (w+1)k$ occurs for the value $t = wk$ (which is true because $\mathbf{O}_C(wkP)$ is spanned) and for the next $h^0(C, \mathbf{O}_C((w+1)kP)) - h^1(C, \mathbf{O}_C(wkP)) - 1$ values of t. The latter assertion is true because uh contains $D(u)$. $\square$

THEOREM 3.6. *Fix integers* k, x *and* z *with* $z \geq 1$, $k \geq 2$, $x > 0$, $x \geq ke$. *Let* $C \in |kh + xf|$ *be an integral projective curve and* $P_1, \ldots P_Z \in C_{reg}$ *total ramification points of* π. *Thus* $g := p_a(C) = kx + ke - k - x + 1$. *Call* $M(P_1, \ldots, P_z, k, x)$ *the subset of* $|kh + xf|$ *parametrizing the curves,* D, *containing the effective divisor* $\sum_{1 \leq i \leq z} kP_i$ *of* C, *i.e. containing each* P_i *and such that* $\pi|D$ *has total ramification at each* P_i. *Call* $M'(P_1, \ldots, P_Z, k, x)$ *the subset of* $M(P_1, \ldots, P_z, k, x)$ *formed by the curves,* D, *with* $P_i \in D_{reg}$ *for every* i. *Let* N_C *be the normal bundle of* C *in* S. *Hence* $N_C = \mathbf{O}_C(kh + xf)$ *and* $deg(N_C) = C^2 = 2g - 2 - K_S \cdot C = 2g - 2 + 2k + 2x + ke$. *Hence we have the following remark.*

REMARK 3.7. *If* $zk < 2k + 2x + ke$ *we have* $h^1(C, N_C(-\sum_{1 \leq i \leq z} kP_i)) = 0$.

REMARK 3.8. *By Remark 3.7 and [17], Th. 1.5 if* $zk < 2k + 2x + ke$ *the scheme* $M'(P_1, \ldots, P_Z, k, x)$ *is smooth at* P *of dimension* $h^0(C, N_C(-\sum_{1 \leq i \leq z} kP_i)) = g - 1 + 2x + 2k + ke - zk = kx + 2ke + k(e - z - 1)$. *Furthermore, for every* w *with* $1 \leq w \leq z$ *the subscheme of* $Hilb(S)$ *parametrizing the curves near* C *containing the divisors* kP_i *for all integers* i *with* $1 \leq i \leq w$ *is smooth and of the expected dimension. Hence we obtain the following result.*

PROPOSITION 3.9. *Fix integers k, x and z and w with $z \geq 1$, $k \geq 2$, $x \geq ke$ and $zk < 2k + 2x + ke$. Let $C \in |kh + xf|$ be an integral projective curve and $P_1, \dots, P_z \in C_{reg}$ total ramification points of π. For every integer w with $1 \leq w \leq z$ there exists a generically smooth irreducible open subset $M'(w)$ of $M'(P_1, \dots, P_w, k, x)$ such that $dim(M'(w)) = kx = 2ke + k(e - w - 1)$, $M'(w + 1)$ is contained in the closure of $M'(w)$ if $w < z$ and $C \in M'(z)$.*

Now we some integers e, k, x, and z we will construct the data $(C, P_1, \dots, P_Z)$ we were looking for.

PROPOSITION 3.10. *Fix integers e, k, x, z, with $e \geq 0$, $k \geq 3$, $x > 0$, $x \geq ke + z$. Fix z distinct points $Q_1, \dots, Q_z$ of $\mathbf{P}^1$ and $P_i \in S$ with $\pi(P_i) = Q_i$ for every i. Then there exists a smooth curve $C \in |kh + xf|$ such that every integer i $P_i \in C$ and P_i is a total ramification point of $\pi|C$.*

Proof. Call Z the zero–dimensional subscheme of S contained in $\bigcup_{1 \leq i \leq z} \pi^{-1}(Q_Z)$, with $D_{red} = \{P_1, \dots, P_Z\}$ and such that each connected component of Z has length k. By assumption the linea system $|kx + (x - z)f|$ is a base point free. Hence the sheaf $\mathbf{I}_Z\mathbf{O}_S(kh + bf)$ is spanned outside $\bigcup_{1 \leq i \leq z} \pi^{-1}(Q_Z)$. Since it is easy to check that no line of $\bigcup_{1 \leq i \leq z} \pi^{-1}(Q_Z)$ is in the base locus of $\mathbf{I}_{Z \otimes}\mathbf{O}_S(kh + bf)$, by Bertini theorem a general curve $C \in |kh + bf|$ containing D is smooth outside $\{P_1, \dots, P_Z\}$. Taking reducible curves union of $\bigcup_{1 \leq i \leq z} \pi^{-1}(Q_Z)$ and of a curve $C' \in |kh + (x - z)f|$ with $P_i \notin C'$ for every i, we obtain the smoothness of a general curve $C \in |kh + xf|$ containing D. $\square$

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