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Determination of Convex Bodies from $\pm\infty$-chord Functions

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Summary. - *We generalize the concept of i-chord function to the cases $i = +\infty$ and $i = -\infty$, and we extend two results concerning the determination of convex bodies from $i-$chord functions to those new values of i.*

1. Introduction

If $K \subset I\!R^n$ is a convex body, i.e. a convex compact set with non empty interior, its radial function at the origin O is defined ([3], chapter 0) by

$$\rho_K(u) := \max\{c : c \in R, c \cdot u \in K\},$$

for all unit vectors $u \in S^{n-1}$ such that the line through 0 and parallel to u intersects K.

The $i-$chord function $\rho_{i,K}$ for $i \in R$, has been defined in [2] and [4] as follows. We define $\rho_{i,K}(u) = 0$, if the line through the origin parallel to $u \in S^{n-1}$ does not intersect K. Otherwise, if $i \neq 0$, we let

$$\rho_{0,K}(u) := \begin{cases} |\rho_K(u)|^i + |\rho_K(-u)|^i & \text{if } 0 \in K \\ ||\rho_K(u)|^i - |\rho_K(-u)|^i| & \text{if } 0 \notin K. \end{cases}$$

For $i = 0$ we let

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$$\rho_{0,K}(u) := \begin{cases} \rho_K(u) \cdot \rho_K(-u) & \text{if } 0 \in K \\ \exp|\log|\rho_K(u)/\rho_K(-u)|| & \text{if } 0 \notin K. \end{cases}$$

The latter definition is motivated by the relation

$$\rho_{0,K}(u) = \lim_{i \to 0} \left(\frac{1}{2} \rho_{i,K}(u) \right)^{\frac{2}{i}}$$

if $0 \in K$ and

$$\rho_{0,K}(u) = \lim_{i \to 0} \exp\left(\frac{\rho_{i,K}(u)}{|i|} \right)$$

if $0 \notin K$.

In the literature ([1], [2], [4] and [3] Chapters 2, 5, 6, 7) there are several results on the determination of convex (and even star) bodies from i-chord functions.

If K is a convex body, let us denote by ∂K its boundary and by $\text{int} K$ its interior.

In this paper we will generalize the concept of i-chord function to the cases $i = +\infty$ and $i = -\infty$. The extension will be obtained by a limiting process, as the 0-chord function.

The following well known properties suggest the appropriate way how to do it.

If a and b are positive real numbers, then

$$\lim_{i \to +\infty} |a^i + b^i|^{\frac{1}{i}} = \max\{a, b\} \tag{1}$$

$$\lim_{i \to +\infty} |a^i - b^i|^{\frac{1}{i}} = \begin{cases} 0 & \text{if } a = b \\ \max\{a, b\} & \text{if } a \neq b \end{cases} \tag{2}$$

$$\lim_{i \to -\infty} |a^i + b^i|^{\frac{1}{i}} = \min\{a, b\} \tag{3}$$

$$\lim_{i \to -\infty} |a^i - b^i|^{\frac{1}{i}} = \begin{cases} 0 & \text{if } a = b \\ \min\{a, b\} & \text{if } a \neq b. \end{cases} \tag{4}$$

Let us now define $+\infty$- and $-\infty$-chord functions. As before, these chord functions are defined to be zero if the line through the origin parallel to $u \in S^{n-1}$ does not intersect K. Otherwise, we let

$$\rho_{+\infty,K} := \max\{|\rho_K(u)|, |\rho_K(-u)|\}$$

$$\rho_{-\infty,K} := \min\{|\rho_K(u)|, |\rho_K(-u)|\}.$$

Note that the definitions differ slightly from the limits in (1), (2), (3) and (4). This happens when the line through the origin parallel to u supports K in a single point and the limits are zero. Our definition has the advantage that it makes the two chord functions continuous on their support.

The rest of this paper is devoted to the extension of two results concerning the determination of convex bodies from i-chord functions to the case when i belongs to the extended real line $\overline{I\!R} := I\!R \cup \{-\infty\} \cup \{+\infty\}$.

2. Determination of convex bodies from $\pm\infty$-chord functions at two distinct points

In this section we analyse the question of whether the $+\infty$- or $-\infty$-chord functions at two distinct points p_1, p_2 determine a convex body.

First of all, let us show by an example that if the line p_1p_2 does not intersect K, then in most cases K is not determined by two $+\infty$-chord (or $-\infty$-) chord functions.

EXAMPLE 2.1. *Let K be a plane convex body and suppose that p_1 and p_2 are two distinct points such that the line p_1p_2 does not intersect K. To simplify our considerations, let us assume that K is strictly convex.*

Let us denote by τ_k, $k = 1, 2$, the arc of ∂K, whose radial representation, with the origin at p_k, $k = 1, 2$, is

$$\rho_k(u) = \rho_{+\infty,K}(u)$$

for all $u \in S^1$ such that $t \cdot u \in K$ for some $t > 0$. Let us denote by $s_{k,2}$, $k = 1, 2$, the halflines issuing from p_k, supporting K and *separating K from p_h, with $h \neq k$. We shall denote by $s_{k,1}$, $k = 1, 2$, the other two supporting halflines issuing from p_k. It is clear that if we are given the $+\infty$-chord functions of K at p_1 and p_2, then $\tau_1 \cup \tau_2$ is determined. Note that, since K is strictly convex, $\tau_1 \cup \tau_2 = \partial K$ if and only if*

$$s_{k,2} \cap s_{h,2} \cap \partial K \neq \emptyset, \quad h \neq k$$

or

$$s_{k,1} \cap s_{h,1} \cap \partial K \neq \emptyset, \quad h \neq k$$

or both. If K has a shape of a drop dripping from the intersection of two appropriate supporting lines, K is uniquely determined. If not, there exist infinitely many convex bodies K', strictly larger than K, such that $\tau_1 \cup \tau_2 \subset \partial K'$ and having the same $+\infty$-chord functions as K at p_1 and p_2.

Similar considerations when $p_1p_2 \cap K = \emptyset$ can be made for $-\infty$-chord functions. In general it can be observed that $-\infty$-chord functions give a poorer information on K than $+\infty$-chord functions. It is not difficult to see that a strictly convex plane body K is not uniquely determined by its $-\infty$-chord functions at p_1 and p_2, if the quadrangle containing it, determined by the supporting halflines through p_1 and p_2, is unbounded. On the other hand, if that quadrangle is bounded, K is uniquely determined if and only if it is a drop contained in that quadrangle, dripping from the intersection point farer from the line p_1p_2.

REMARK 2.2. *There is a remarkable similarity between the conclusions in the previous example and the main result of [5]. In that paper the author proves that K is uniquely determined by its 1-chord functions at p_1 and p_2, when $p_1p_2 \cap K = \emptyset$ and ∂K contains a point of intersection of two distinct supporting halflines $s_{h,k}$.*

The negative result given by Example 2.1 is not surprising in view of the analogous difficulties for i-chord functions with $i \in \mathbb{R}$ when $p_1p_2 \cap K = \emptyset$. One of the most important open questions in this area is in fact whether the main result of [6], where at least three points are considered, is the best possibile, or if two points are always sufficient.

Let us now give a positive result.

THEOREM 2.3. *If $K \subset \mathbb{R}^n$ is convex body, then there exist infinitely many pairs of points p_1 and p_2, interior to K, such that the $+\infty$-chord functions at p_1 and p_2 determine K uniquely.*

Proof. Let G_1 and G_2 be two distinct parallel hyperplanes intersecting $\mathrm{int}K$. Let E_k and F_k, $k = 1, 2$ be the closed half-spaces determined by G_k and containing, and not containing respectively, G_h, $h = 1, 2$, $h \neq k$. Let C_k be the cap $K \cap F_k$. Let $t_k \in C_k$ be a point of maximal distance from G_k, $k = 1, 2$. If p_k is interior to C_k and close enough to t_k, then for any chord c of K through p_k intersecting $K \cap G_k$, p_k is closer to the endpoint of c which belongs to C_k than to the other endpoint.

If H is a convex body with the same $+\infty$-chord functions at p_1 and p_2, then H either contains $K \cap E_j$, $j = 1, 2$ (and in this case $H \cap E_j = K \cap E_j$), or H contains the reflection of $K \cap E_j$ with respect to p_j and $H \cap E_j = \emptyset$ and $H \subset F_j$.

Obviously it is impossible that $H \subset F_1$ and $H \subset F_2$ at the same time. On the other hand, if $H \cap E_j = K \cap E_j$, H cannot contain at the same time the reflection of $K \cap E_h$ with respect to p_h, $h \neq j$, $h = 1, 2$, which is also contained in E_j. Therefore $H = K$. □

The previous theorem has a negative counterpart. We first need two lemmas.

LEMMA 2.4. *If q is an exposed point of a convex body K, then a neighborhood base for q in the relative topology of ∂K is given by sets of the type $V := \partial K \cap E$, where E is an open halfspace.*

Proof. Let U be a neighborhood of q in the relative topology of ∂K. Let H_0 be a hyperplane supporting K at q and let us denote by H_n the hyperplane parallel to H_0 at distance $\frac{1}{n}$ intersecting $\mathrm{int}K$. Let us denote by E_n the open halfspace determined by H_n and containing q. The intersection between ∂K and the closure $\overline{E}_n$ of E_n is compact. Since q is exposed, $\cap_n \overline{E}_n = \{q\}$. It follows that $C_n := (\overline{E}_n \cap \partial K) \setminus U$ is a decreasing sequence of compact sets with empty intersection. Therefore $\overline{E}_n \cap \partial K \subset U$ for some n and we may then take $V := E_n \cap \partial K$. □

Let now $K \subset \mathbb{R}^n$ be a convex body and suppose $p \in \text{int} K$. Let us define the following mapping from ∂K to ∂K. If $x \in \partial K$, the chord determined by the line px intersects ∂K in a point $y \neq x$. We put $p(x) := y$.

LEMMA 2.5. *The mapping $p : \partial K \to \partial K$ is continuous.*

Proof. We may assume that p is the origin. The mapping p can be written as a composition of three continuous mappings. The first associates to $x \in \partial K$ the point $u := \frac{x}{||x||} \in S^{n-1}$. The second sends u in $-u$. Finally the third maps $-u$ in $y := -\rho_K(-u)u$. $\square$

THEOREM 2.6. *If K is a convex body, then there exist infinitely many pairs of distinct points p_1 and p_2 interior to K such that infinitely many convex bodies have the same $+\infty$-chord functions in p_1 and p_2 as K.*

Proof. Let $q \in \partial K$ be an exposed point. Consider two distinct chords c_1, c_2 of K with one endpoint in q and let us take $p_i \in c_i$, $i = 1, 2$, such that p_i is closer to q than to the other endpoint of c_i. It follows from Lemma 2.5 that the sets $U_i := \{x : |x - p_i| < |p_i - p_i(x)|\}$ are open in the relative topology of ∂K. Since $q \in U_i$, $i = 1, 2$, $U = U_1 \cap U_2$ is a neighborhood of q. By Lemma 2.4, there exists a hyperplane H such that U contains the intersection of ∂K with the open halfspace E determined by H and containing q.

It is clear now that if we consider any convex body K' contained in K and containing $K \setminus E$, then K and K' have the same $+\infty$-chord functions at p_1 and p_2. $\square$

For $-\infty$-chord functions there is a strong negative result. First we need a lemma.

LEMMA 2.7. *If $K \subset \mathbb{R}^n$ is convex body which is strictly convex and if $K_i \subset K$, $1 \leq i \leq 3$, are three convex bodies such that* $\text{Conv}(K_i \cup K_j) = K$ *whenever $i \neq j$, $1 \leq i, j \leq 3$, then $K_i = K$ for some i, $1 \leq i \leq 3$.*

Proof. Assume $K_1 \neq K$, $K_2 \neq K$ and $\text{Conv}(K_1 \cup K_2) = K$. It is in fact $K_1 \cup K_2 = K$, because if it were not so, then for some $x \in \partial K \setminus (K_1 \cup K_2)$ there would exist $y \in K_1$, $z \in K_2$ and $t \in]0, 1[$ such that

$x = ty+(1-t)z$, contradicting the strict convexity of K. Therefore, if we denote by $E_i := K_i \cap \partial K$, we have that $E_1 \cup E_2 = \partial K$. Similarly, if $\mathrm{Conv}(K_1 \cup K_3) = K$, we have $E_1 \cup E_3 = \partial K$, and therefore $E_3 \supset E_2 \setminus E_1$. The same argument shows that $\mathrm{Conv}(K_2 \cup K_3) = K$ implies that $E_3 \supset E_1 \setminus E_2$ and therefore $E \supset E_1 \cup E_2 = \partial K$, so $K_3 = K$. $\square$

REMARK 2.8. *The Lemma is in general not true for bodies which are not stricly convex, as the following example shows. Let us consider three points* p_1, p_2, p_3 *in* $\mathbb{R}^n$, *such that* $||p_i - p_j|| = 3$ *for* $i \neq j$, $1 \leq i, j \leq 3$ *and let us denote by* B_i *the ball centered at* p_i *with radius* 1. *Let* $K_i := \mathrm{Conv}(B_j \cup B_k)$, *with* $\{i, j, k\} = \{1, 2, 3\}$. *Let moreover* $K := \mathrm{Conv}(K_1 \cup K_2 \cup K_3)$. *Then* $\mathrm{Conv}(K_i \cup K_j) = K$ *for all* i, j, $i \neq j$, $1 \leq i, j \leq 3$. *On the other hand,* $K \neq K_i$ *for each* i, $1 \leq i \leq 3$.

THEOREM 2.9. *If* $K \subset \mathbb{R}^n$ *is strictly convex, then given any three distinct points* p_1, p_2, p_3 *belonging to* $\mathrm{int}K$, *we can choose two of them so that the* $-\infty$*-chord functions at the two chosen points do not determine* K *uniquely.*

Proof. Consider the three convex bodies K_i, $1 \leq i \leq 3$, whose radial function at p_i is $\rho_{K,-\infty}$. Obviously $K_i \subset K$ for all i. Note that the bodies K_i are centrally symmetric with center at p_i. If $\mathrm{Conv}(K_i \cup K_j) \neq K$ for some $i \neq j$, we are done, since K and $\mathrm{Conv}(K_i \cup K_j)$ are distinct and have the same $-\infty$-chord functions at p_i and p_j respectively. If on the other hand $\mathrm{Conv}(K_i \cup K_j) = K$ for any choice of $i \neq j$, $1 \leq i, j \leq 3$, then it follows from the previous lemma that $K_i = K$ for at least one i. In our case this may happen only for one index i, $1 \leq i \leq 3$, since if $K = K_i = K_j$, central symmetry would imply that $p_i = p_j$. So suppose $K_1 = K$ and take $x \in \partial K \setminus K_2$. We shall show that K is not determined by its $-\infty$-chord functions at p_1 and p_2. Let A_0 be a hyperplane separating x from $p_1 \cup K_2$. Clearly, the set U of all $u \in S^{n-1}$ such that the halfline $p_1 + tu$, $t \geq 0$, does not intersect $A_0 \cap K$, has the property that for each $u \in S^{n-1}$, either u or $-u$ belongs to U. Therefore, if we consider any convex body $H \supset K$ such that $\rho_K(u) = \rho_H(u)$ for all $u \in U$, K and H will have the same $-\infty$-chord function at p_1. The convex bodies H and K will also have the same $-\infty$-chord function at p_2, since $H \setminus K$ does not intersect K_2. To prove the assertion it will be enough to prove that there exists a point $y_0 \notin K$ such that K and $H = \mathrm{Conv}(K \cup \{y_0\})$have the same radial function at p_1 for all $u \in U$. Consider the halfline l issuing from p_1 and containing x.

Given $y \in l \setminus K$, denote by L_y the set of all points $z \in \partial K$ such that the segment $[y, z]$ does not intersect int$K$. L_y is compact and $L_{y_1} \subset L_{y_2}$ if y_1 is between y_2 and x.

Let $z \in \partial K$ be such that A_0 separates z from x. Then, if y is sufficiently close to x, $z \notin L_y$ by strict convexity of K. Moreover, let $C \subset \partial K$ be the component of $K \setminus A_0$ containing x. Then the intersection of compact sets $\cap(L_y \setminus C)$, the intersection being taken for $y \in l - K$, is empty, therefore there exists y_0 such that $L_{y_0} \setminus C = \emptyset$. It follows that $H := \mathrm{Conv}(K \cup \{y_0\})$ has the desired properties. □

THEOREM 2.10. *There exist a convex body $K \subset \mathbb{R}^n$, and two points p_1 and p_2 interior to K, such that K is uniquely determined by its $-\infty$-chord functions in p_1 and p_2.*

Proof. Let B_1 and B_2 be balls with radius 1 centered at $p_1 := v$ and $p_2 := -v$ respectively, with $||v|| \geq 1$, and let $K := \mathrm{Conv}(B_1 \cup B_2)$. We shall prove that K is uniquely determined by the $-\infty$-chord functions at p_1 and p_2. Let $\rho^{(k)}_{K,-\infty}$ be the $-\infty$-chord function of K at p_k, $k = 1, 2$. It is $B_1 \subset K$, otherwise $\rho^{(1)}_{K,-\infty}$ would be smaller, and analogously $B_2 \subset K$. Then $B_1 \cup B_2 \subset K$, and therefore

$$K \supset \mathrm{Conv}(B_1 \cup B_2). \tag{5}$$

Let E_j, $j = 1, 2$, be the closed halfspace not containing B_k with $k \neq j$ determined by the hyperplane H_j through p_j, orthogonal to the line p_1p_2. Let $u \in S^{n-1}$ be such that $p_1 + u \in E_1$. Then (5) implies that $\rho^{(1)}_K(-u) > 1$ and hence $\rho^{(1)}_K(u) = 1$. Analogously we see that $\rho^{(2)}_K(u) = 1$ for any $u \in S^{n-1}$ such that $p_2 + u \in E_2$. It follows then that ∂K contains the two semispheres centered at p_k with radius 1 and contained in E_i. Therefore the only supporting hyperplanes at the points of $K \cap H_j$, $j = 1, 2$, also support K at the points of $K \cap H_k$, with $k = 1, 2$, $k \neq j$. Therefore K coincides with the convex hull of $B_1 \cup B_2$. □

THEOREM 2.11. *There exists a convex body K which is not determined by its $-\infty$-chord functions in any pair of distinct points of* int$K$.

Proof. Let $K \subset \mathbb{R}^n$ be a ball centered at the origin and radius 1 and let $p_1, p_2 \in$int$K$. Consider a hyperplane H through the origin such that both points belong to the same closed halfspace E determined by H. Denote by $u_0 \in S^{n-1}$ the vector orthogonal to H, such that $u \notin E$. It is easy to see now that $K' :=$Conv$(K \cup \{2u\})$ has the same $-\infty$-chord functions of p_1 and p_2 as K. ☐

Conjecture. If K is a convex body and $p_1 \in$ int$K$, then there exist infinitely many points $p_2 \in$ int$K$ such that infinitely many convex bodies have $+\infty$-chord functions in p_1 and p_2 equal to those of K.

3. Determination of convex bodies from distinct i-chord functions at an interior point

In this section we shall consider an extension of Theorem 5.2 from [4] to i-chord functions with $i \in \overline{\mathbb{R}}$. In that paper the Authors considered determination of star bodies and this more general setting brings with it several technical difficulties concerning the domain of the radial function and its support (cfr. Lemma 2.2, Corollary 3.2 and the example following Corollary 3.3). To avoid those difficulties we shall only consider i-chord functions at a point interior to a convex body K. In this way it will be clearer to the reader what is the novelty of our results.

Contrary to the conclusions we got in Section 2, we will see that in this case the result is fully preserved if we take i in the extended real line $\overline{\mathbb{R}}$.

THEOREM 3.1. *Let $K_1, K_2 \subset \mathbb{R}^n$ be convex bodies containing the origin in their interiors. Let $i, j \in \overline{\mathbb{R}}$. Then $\rho_{i,K_1} = \rho_{i,K_2}$ and $\rho_{j,K_1} = \rho_{j,K_2}$ if and only if*

$$\{\rho_{K_1}(u), \rho_{K_1}(-u)\} = \{\rho_{K_2}(u), \rho_{K_2}(-u)\} \tag{6}$$

for all $u \in S^{n-1}$ and hence, if and only if $\rho_{t,K_1} = \rho_{t,K_2}$ for all $t \in \overline{\mathbb{R}}$.

Proof. If $i, j \in \mathbb{R}$, the conclusion follows from Theorem 5.2 of [4].

If $i = -\infty$ and $j = +\infty$, we have that

$$\min(\rho_{K_1}(u), \rho_{K_1}(-u)) = \min(\rho_{K_2}(u), \rho_{K_2}(-u))$$

and

$$\max(\rho_{K_1}(u), \rho_{K_1}(-u)) = \max(\rho_{K_2}(u), \rho_{K_2}(-u))$$

and therefore (6) follows immediately.

Suppose now $i = -\infty,\ j \in \mathbb{R} \setminus \{0\}$. Then

$$\min(\rho_{K_1}(u), \rho_{K_1}(-u)) = \min(\rho_{K_2}(u), \rho_{K_2}(-u))$$

and

$$\rho^j_{K_1}(u) + \rho^j_{K_1}(-u) = \rho^j_{K_2}(u) + \rho^j_{K_2}(-u). \tag{7}$$

The last equality can be written as

$$\begin{aligned}\min(\rho_{K_1}(u), \rho_{K_1}(-u))^j + \max(\rho_{K_1}(u), \rho_{K_1}(-u))^j =\\ = \min(\rho_{K_2}(u), \rho_{K_2}(-u))^j + \max(\rho_{K_2}(u), \rho_{K_2}(-u))^j.\end{aligned}$$

Therefore

$$\begin{aligned}\max(\rho_{K_2}(u), \rho_{K_2}(-u)) = \min(\rho_{K_1}(u), \rho_{K_1}(-u))^j +\\ + (\max(\rho_{K_1}(u), \rho_{K_1}(-u))^j - \min(\rho_{K_2}(u), \rho_{K_2}(-u))^j)^{\frac{1}{j}} =\\ \max(\rho_{K_1}(u), \rho_{K_1}(-u))\end{aligned}$$

so we are again in the previous case. If $j = 0$, then (7) is substituted by

$$\rho_{K_1}(u) \cdot \rho_{K_1}(-u) = \rho_{K_2}(u) \cdot \rho_{K_2}(-u). \tag{8}$$

As previously, from (8) we deduce that

$$\begin{aligned}\max(\rho_{K_2}(u), \rho_{K_2}(-u)) =\\ = \frac{\min(\rho_{K_1}(u), \rho_{K_1}(-u)) \cdot \max(\rho_{K_1}(u), \rho_{K_1}(-u))}{\min(\rho_{K_2}(u), \rho_{K_2}(-u))} =\\ = \max(\rho_{K_1}(u) \cdot \rho_{K_1}(-u)),\end{aligned}$$

and now the conclusion follows as before.

The case when $i \in \mathbb{R}$ and $j = +\infty$ can be treated in a similar way.

The converse is obvious, since from (6) it follows that $\rho_{t,K_1} = \rho_{t,K_2}$ for all $t \in \overline{\mathbb{R}}$. $\square$

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