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Estimates and existence theorems for a class of nonlinear degenerate elliptic equations

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Summary. - *Let* $\{a_{ij}(x,\eta)\}$ *be a matrix of bounded Carathéodory functions such that* $a_{ij}(x,\eta)\xi_j\xi_i \geq b(|\eta|)\,\nu(x)|\xi|^2 \quad \forall\,\xi \in \mathbb{R}^n$, *where* $b : [0,+\infty[\to \mathbb{R}$ *is a positive bounded continuous function and* $\nu \in L^1$, $\dfrac{1}{\nu} \in L^t$ *with* $t > 1$. *A priori estimates for solutions of the homogeneous Dirichlet problem related to the equation* $-(a_{ij}(x,u)u_{x_j})_{x_i} = f$ *are proved under various summability assumptions on* f. *As a consequence, existence theorems are obtained.*

1. Introduction

Let Ω be an open bounded subset of $\mathbb{R}^n$, $n \geq 2$. Let $\{a_{ij}(x,\eta)\}$ be a matrix of Carathéodory functions (i.e., $\forall\, i,j = 1,2\ldots n$, $a_{ij}(x,\eta)$ is measurable with respect to x for every $\eta \in \mathbb{R}$ and continuous with respect to η for a.e. $x \in \Omega$).

We consider the following Dirichlet problem

$$(I) \qquad \begin{cases} -(a_{ij}(x,u)u_{x_j})_{x_i} = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

under the assumptions

$$(II) \qquad a_{ij}(x,\eta)\xi_j\xi_i \geq b(|\eta|)\nu(x)\ |\xi|^2 \quad \text{a.e. } x \in \Omega,\ \eta \in \mathbb{R},\ \xi \in \mathbb{R}^n,$$

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where

$$b \, : \, [0,+\infty[\to [0,+\infty[\ \text{ is bounded and continuous,}$$
$$\nu \, : \, \Omega \to [0,+\infty[\ \text{ is such that } \ \nu \in L^1(\Omega) \ \text{ and } \frac{1}{\nu} \in L^t(\Omega), \ t>1.$$

In this paper we will give some a priori estimates for weak solutions of problem (I). All results in this direction will be obtained by symmetrization techniques in the same spirit as [1] and [2].

Moreover, a priori estimates will be used to prove, by methods analogous to those used in [5], the existence of weak solution of problem (I) under further assumptions

(III) $|a_{ij}(x,\eta)| \leq c \, \nu(x)$ for a.e. $x \in \Omega$, $\eta \in \mathbb{R}$, $i,j = 1,2,\ldots,n$,

where c is a constant, and

(IV) $b(|\eta|) = \dfrac{1}{(1+|\eta|)^\alpha}$, $\qquad 0 \leq \alpha \leq 1$.

We assume condition (IV) only for the sake of simplicity. Most of our results are true under more general hypotheses on the function b (see also [2]).

We shall prove the following existence theorems.

THEOREM 1.1. *Let* $\frac{1}{\nu} \in L^t(\Omega)$ *and* $f \in L^r(\Omega)$ *with* $p > \frac{n}{2}$, $\frac{1}{p} = \frac{1}{r} + \frac{1}{t}$, $n \geq 2$. *Under the assumptions (II), (III) and (IV), there exists* $u \in W_0^{1,2}(\nu) \cap L^\infty(\Omega)$ *such that*

$$\int_\Omega a_{ij}(x,u) u_{x_j} v_{x_i} \, dx = \int_\Omega f v \, dx, \qquad \forall \, v \in W_0^{1,2}(\nu) \ ,$$

i.e. u is a weak solution of problem (I) (for the definition of $W_0^{1,s}(\nu)$ *see preliminaries).*

THEOREM 1.2. *Let* $\frac{1}{\nu} \in L^t(\Omega)$ *and* $f \in L^r(\Omega)$ *with* $\frac{2n}{n+2-\alpha(n-2)} < p < \frac{n}{2}$, $\frac{1}{p} = \frac{1}{r} + \frac{1}{t}$, $n > 2$. *Under the assumptions (II), (III) and (IV), there exists* $u \in W_0^{1,2}(\nu) \cap L^q(\Omega)$, $q = \frac{np(1-\alpha)}{n-2p}$, *such that*

$$\int_\Omega a_{ij}(x,u) u_{x_j} v_{x_i} dx = \int_\Omega f v \, dx, \qquad \forall v \in W_0^{1,2}(\nu),$$

i.e. u is a weak solution of problem (I).

THEOREM 1.3. *Let* $\frac{1}{\nu} \in L^t(\Omega)$, $f \in L^r(\Omega)$ *and* $\nu \in L^{\frac{s}{2(s-1)}}(\Omega)$, *with* $t > 1$, $r > 1$, $s > 1$ *and* $\frac{n}{n+1-\alpha(n-1)} < p \leq \frac{2n}{n+2-\alpha(n-2)}$, $\frac{2t}{2t-1} < s < \frac{np(1-\alpha)}{n-p(1+\alpha)}$, $\frac{1}{p} = \frac{1}{r} + \frac{1}{t}$, $n > 2$.
Under the assumptions (II), (III) and (IV) there exists $u \in W_0^{1,s}(\nu^{s/2}) \cap L^q(\Omega)$, $q = \frac{np(1-\alpha)}{n-2p}$, *such that*

$$\int_\Omega a_{ij}(x,u) u_{x_j} \varphi_{x_i} \, dx = \int_\Omega f \varphi \, dx, \qquad \forall \varphi \in C_0^\infty(\Omega),$$

i.e. u is a solution of problem (I) in the sense of distributions.

THEOREM 1.4. *Let* $\frac{1}{\nu} \in L^t(\Omega)$, $f \in L^r(\Omega)$ *and* $\nu \in L^{t'}(\Omega)$ *with* t, t', $r > 1$ *and* $\frac{n}{n+1-\alpha(n-1)} < p \leq \frac{2n}{n+2-\alpha(n-2)}$, $1 + \frac{1}{t} < \frac{t'-1}{t'} \times \frac{np(1-\alpha)}{n-p(1+\alpha)}$, $\frac{1}{p} = \frac{1}{r} + \frac{1}{t}$, $n > 2$. *Under the assumptions (II), (III) and (IV) there exists u such that*

$$\int_\Omega a_{ij}(x,u) u_{x_j} \varphi_{x_i} \, dx = \int_\Omega f \varphi \, dx, \qquad \forall \varphi \in C_0^\infty(\Omega),$$

u belongs to $W_0^{1,s}(\nu) \cap L^q(\Omega)$ *for every* $1 + \frac{1}{t} < s < \frac{t'-1}{t} \times \frac{np(1-\alpha)}{n-p(1+\alpha)}$ *and* $q = \frac{np(1-\alpha)}{n-2p}$.

The scheme of the paper is as follows.

In Section 2 we recall some properties of the rearrangements and some functional spaces which are useful for our a priori estimates.

In Section 3 we prove a priori estimates for weak solutions of problem (I).

Section 4 is devoted to the proof of the existence Theorems 1.1, 1.2, 1.3 and 1.4.

2. Preliminaries

Let T be a measurable subset of $\mathbb{R}^n$ and let u be a real-valued measurable function defined on T. The distribution function μ_u of u is defined by

$$\mu_u(\tau) = |\{x \in T \,:\, |u(x)| > \tau\}|, \qquad \tau \geq 0$$

where $|T'|$ denotes the Lebesgue measure of set $T' \subseteq T$. The decreasing rearrangement of u, denoted by u^*, is the distribution function of μ_u, i.e.

$$\begin{aligned} u^*(\sigma) &= |\{\tau \in [0, +\infty[\,:\, \mu_u(\tau) > \sigma\}| \\ &= \inf\{\tau \in \mathbb{R} \,:\, \mu_u(\tau) \leq \sigma\} \qquad \sigma \in (0, |T|). \end{aligned}$$

The increasing rearrangement u_* of u and the symmetric rearrangement $u^\#$ of u are respectively defined by $u_*(\sigma) = u^*(|T| - \sigma)$, $\sigma \in (0, |T|)$,

$$u^\#(x) = u^*(C_n|x|^n), \qquad x \in T^\# = \{x \in \mathbb{R}^n \,:\, C_n|x|^n < |T|\}$$

where C_n is the measure of the unit ball of $\mathbb{R}^n$.
Recall that u and u^* are equimeasurable, i.e. $\mu_u(\tau) = \mu_{u^*}(\tau)$, $\tau \geq 0$. As it is well known, Hardy - Littlewood inequality holds

$$\begin{aligned} \int_0^{|T|} u^*(\sigma) v_*(\sigma)\, d\sigma \leq \int_T |u(x)v(x)| dx &\leq \int_0^{|T|} u^*(\sigma) v^*(\sigma)\, d\sigma \qquad (1) \\ &= \int_{T^\#} u^\#(x) v^\#(x)\, dx. \end{aligned}$$

In particular

$$\int_{T'} |u(x)|\, dx \leq \int_0^{|T'|} u^*(\sigma)\, d\sigma, \qquad \forall\, T' \subset T. \qquad (2)$$

Furthermore

$$\int_T \psi(|u(x)|)\,dx = \int_0^{|T|} \psi(u^*(\sigma))\,d\sigma \tag{3}$$

for any monotone function ψ (see [12]).
Equality (3) and the definition of u^* imply

$$\|u\|_p \equiv \|u\|_{L^p(\Omega)} = \|u^*\|_{L^p(0,|T|)}, \qquad 1 \le p \le +\infty \ .$$

An exhaustive treatment of theory of rearrangement can be found for example in [4], [10], [12].

Recall that the Lorentz $L(p,q)$ space, $p > 1$ and $q \ge 1$, is the collection of all real-valued measurable functions defined on T such that

$$\|u\|_{p,q} = \left(\int_0^{+\infty} [\sigma^{1/p}\overline{u}(\sigma)]^q \frac{d\sigma}{\sigma}\right)^{1/q} < +\infty, \qquad q < +\infty$$

where $\overline{u}(\sigma) = \dfrac{1}{\sigma}\displaystyle\int_0^\sigma u^*(\tau)\,d\tau$,

$$\|u\|_{p,+\infty} = \sup_{\sigma>0} \sigma^{1/p}\overline{u}(\sigma) \ .$$

As it is well known $L(p,p) = L^p(T)$ (for $p > 1$) $L(p,1) \subset L(p,q) \subset L(p,r) \subset \subset L(p,\infty)$, for $p > 1$, $1 \le q < r \le \infty$.
Let ν be a measurable function, defined on T. As in [1], $L_\nu(p,q)$ will denote the collection of all real valued measurable functions u defined on T and such that

$$\|u\|_{p,q,\nu} = \left(\int_0^{+\infty} \frac{1}{\nu_*(\sigma)} \left[\sigma^{1/p}\overline{u}(\sigma)\right]^q \frac{d\sigma}{\sigma}\right)^{1/p} < +\infty,$$

$1 \le q < \infty$, $1 < p < \infty$.

Finally, $W_0^{1,s}(\nu)$ will denote the closure of $C_0^\infty(T)$ under the norm

$$\|u\|_{W_0^{1,s}(\nu)} = \left(\int_T \nu(x)\,|Du|^s dx\right)^{1/s} ,$$

where $\dfrac{1}{\nu} \in L^t(\Omega)$ and $s \ge 1 + \dfrac{1}{t}$, (see [9] for more details on weighted Sobolev space $W_0^{1,s}(\nu)$).

3. A priori estimates

As in [1, Section 2] we consider a function $\underline{\nu}(\sigma)$ defined on $[0,|\Omega|[$ such that

$$\int_{|u|>\tau} \frac{1}{\nu(x)}\,dx = \int_0^{\mu_u(\tau)} \frac{1}{\underline{\nu}(\sigma)}\,d\sigma, \qquad \text{for a.e.} \quad \tau \in [0,\,|\Omega|[\tag{4}$$

where $\frac{1}{\nu} \in L^t(\Omega)$, $t \geq 1$, and u is a given measurable function defined in Ω.

Lemma 2.2 of [1] ensures the existence of a sequence $\{\nu_m\}$ with the following properties:

$$\nu_m^* = \nu^*, \qquad \text{in } [0,|\Omega|[\ ,$$

and, if $t > 1$,

$$\frac{1}{\nu_m} \to \frac{1}{\underline{\nu}} \ \text{ weakly in } \ L^t([0,|\Omega|[), \tag{5}$$

if $t = 1$,

$$\lim_{m\to\infty} \int_0^{|\Omega|} \frac{1}{\nu_m(\sigma)} g(\sigma)\,d\sigma = \int_0^{|\Omega|} \frac{1}{\underline{\nu}(\sigma)} g(\sigma)\,d\sigma, \quad \forall\, g \in BV([0,|\Omega|[). \tag{6}$$

For $k > 0$ set (see [2])

$$T_k(\eta) = \begin{cases} k \ \mathrm{sign}\,(\eta) & \text{if } |\eta| \geq k, \\ \eta & \text{if } |\eta| < k. \end{cases} \tag{7}$$

We will consider solutions $u \in W_0^{1,1}(\Omega)$ of problem (1) satisfying the following conditions

$$\begin{cases} T_k(u) \in W_0^{1,2}(\nu) \\ \displaystyle\int_\Omega a_{ij}(x,u)\,u_{x_j}\,(T_k(u))_{x_i}\,dx = \int_\Omega f\ T_k(u)\,dx, \qquad \forall\, k > 0. \end{cases} \tag{8}$$

THEOREM 3.1. *Let u be a solution of problem (1) satisfying (8). Then we have*

$$-\frac{d}{d\sigma}\,B(u^*(\sigma)) \leq \frac{1}{n^2 C_n^{2/n} \sigma^{2-2/n} \underline{\nu}(\sigma)} \int_0^\sigma f^*(\tau)\,d\tau, \tag{9}$$

for a.e. $\sigma \in (0, |\Omega|)$, *where* $\underline{\nu}$ *satisfies (4) and* $B(\tau)$ *is defined by*

$$B(\tau) = \int_0^\tau b(\sigma)\, d\sigma, \qquad \forall \tau \in [0, +\infty[. \tag{10}$$

Proof. For τ, $h > 0$ we can use (8) with $k = \tau$ and $k = \tau + h$, obtaining

$$\int_{\tau < |u| \le \tau + h} a_{ij}(x,u) u_{x_j} u_{x_i}\, dx =$$
$$= \int_{\tau < |u| \le \tau + h} f\,(|u| - \tau)\ \mathrm{sign}\, u\ dx + h \int_{|u| > \tau + h} f\ \mathrm{sign}\ u\, dx.$$

Therefore (2) implies

$$\int_{\tau < |u| \le \tau + h} b(|u|)\, \nu(x) |Du|^2\, dx \le h \int_{|u| > \tau} |f|\, dx. \tag{11}$$

On the other hand, Schwartz inequality implies:

$$\int_{\tau < |u| \le \tau + h} b(|u|)|Du| dx \le \tag{12}$$
$$\le \left(\int_{\tau < |u| \le \tau + h} b(|u|) \nu(x) |Du|^2 dx \right)^{1/2} \left(\int_{\tau < |u| \le \tau + h} \frac{b(|u|)}{\nu(x)}\, dx \right)^{1/2}.$$

From (11) and (12), we get

$$\left(\frac{1}{h} \int_{\tau < |u| \le \tau + h} b(|u|)\, |Du| dx \right)^2$$
$$\le \left(\frac{1}{h} \int_{\tau < |u| \le \tau + h} \frac{b(|u|)}{\nu(x)}\, dx \right) \left(\int_{|u| > \tau} |f|\, dx \right). \tag{13}$$

Passing to the limit as h goes to zero in (12) and observing that b is continuous we have, for a.e. $\tau > 0$,

$$b(\tau) \left(-\frac{d}{d\tau} \int_{|u| > \tau} |Du|\, dx \right) \le \left(-\frac{d}{d\tau} \int_{|u| > \tau} \frac{1}{\nu(x)}\, dx \right) \int_{|u| > \tau} |f|\, dx. \tag{14}$$

It is well known (see, e.g., (43) of [10]) that a consequence of isoperimetric inequality is the following one

$$nC_n^{1/n}\mu_u(\tau)^{1-1/n} \le -\frac{d}{d\tau}\int_{|u|>\tau}|Du|\,dx.$$

Together with (14) it gives

$$b(\tau) \le \frac{-\mu_u'(\tau)}{n^2 C_n^{2/n}\mu_u(\tau)^{2-2/n}}\ \frac{1}{\underline{\nu}(\mu_u(\tau))}\int_0^\tau f^*(\sigma)\,d\sigma.$$

Replacing $\mu_u(t)$ by s, and using the properties of rearrangements, we obtain (9).

COROLLARY 3.2. *If u is a solution of (1) satisfying (8) and if v is solution of the following problem*

$$\begin{cases} -\displaystyle\sum_{i=1}^n \partial_{x_i}(\underline{\nu}(C_n|x|^n)\partial_{x_i}v(x)) = f^{\#} & \text{in } \Omega^{\#} \\ v=0 & \text{on } \partial\Omega^{\#}, \end{cases} \tag{15}$$

then

$$B(u^*(\sigma)) \le v^*(\sigma), \qquad \sigma\in(0,|\Omega|).$$

Proof. It is sufficient to integrate between s and $|\Omega|$ both sides of (9) and to observe that (see [1])

$$v(x) = \frac{1}{n^2C_n^{2/n}}\int_{C_n|x|^n}^{|\Omega|}\frac{\sigma^{-2+2/n}}{\underline{\nu}(\sigma)}\,d\sigma\int_0^\sigma f^*(\tau)\,d\tau$$

is solution of (15). □

COROLLARY 3.3. *Let u be a solution of (1) satisfying (8). Assume $\lim\limits_{\tau\to+\infty} B(\tau) = +\infty$. If B^{-1} denotes the inverse function of B, the following inequalities hold true:*

a) if $f\in L_\nu\left(\frac{n}{2},1\right)$, then $\|u\|_\infty \le B^{-1}\left(\frac{1}{n^2C_n^{2/n}}\|f\|_{n/2,1,\nu}\right)$;

b) if $f \in L^r(\Omega)$, $\dfrac{1}{\nu} \in L^t(\Omega)$ *and* $\dfrac{1}{t} + \dfrac{1}{r} < \dfrac{2}{n}$, *then* $\|u\|_\infty \le B^{-1}(A\|f\|_r)$,

where $A = \dfrac{1}{n^2 C_n^{2/n}} \left(\displaystyle\int_0^{|\Omega|} \left(\int_{\sigma'}^{|\Omega|} \frac{\sigma^{2/n-2}}{\nu_*(\sigma)} \, d\sigma \right)^{r'} d\sigma' \right)^{1/r'}$ *and* $\dfrac{1}{r} + \dfrac{1}{r'} = 1.$

Proof. Integrating both sides of (8) and by (6) we have, for $\eta \in (0, |\Omega|)$

$$\begin{aligned} B(u^*(\eta)) &\le \frac{1}{n^2 C_n^{2/n}} \int_\eta^{|\Omega|} \frac{1}{\underline{\nu}(\sigma)} \sigma^{-2+2/n} \, d\sigma \int_0^\sigma f^*(\tau) \, d\tau \\ &= \lim_{m \to +\infty} \frac{1}{n^2 C_n^{2/n}} \int_\eta^{|\Omega|} \frac{1}{\nu_m(\sigma)} \sigma^{-2+2/n} \, d\sigma \int_0^\sigma f^*(\tau) \, d\tau. \quad (16) \end{aligned}$$

Taking into account the fact that $\sigma^{-2+2/n} \displaystyle\int_0^\sigma f^*(\tau) \, d\tau$ is decreasing and that $\left(\dfrac{1}{\nu_m}\right)^* = \dfrac{1}{\nu_*}$ $\forall m \in N$, we have

$$\begin{aligned} \int_\eta^{|\Omega|} \frac{1}{\nu_m(\sigma)} \sigma^{-2+2/n} d\sigma \int_0^\sigma f^*(\tau) d\tau \le \\ \le \int_0^{|\Omega|} \frac{\sigma^{2/n}}{\nu_*(\sigma)} \overline{f}(\sigma) \frac{d\sigma}{\sigma} = \|f\|_{n/2,\,1,\,\nu}. \end{aligned}$$

Together with (16) this implies

$$B(u^*(0)) \le \frac{1}{n^2 C_n^{2/n}} \|f\|_{n/2,\,1,\,\nu}.$$

Observing that B is increasing and invertible, we obtain (a). Part (b) follows immediately using the arguments of Theorem 3.2 of [1].

Corollary 3.4. *Under the assumptions of Corollary 3.3, we have*

$$\|B(|u|)\|_{q,k} \le \frac{1}{n^2 C_n^{2/n}} \frac{q^2}{q-1} \|f\|_{s,k,\nu}, \quad (17)$$

with $1 < s < \frac{n}{2}$, $\frac{1}{q} = \frac{1}{s} - \frac{2}{n}$, $k > 1$.
If, moreover, $f \in L^r(\Omega)$, $\frac{1}{\nu} \in L^t(\Omega)$, *with* $\frac{2}{n} < \frac{1}{r} + \frac{1}{t} < 1 + \frac{2}{n}$ *then*

$$\|B(|u|)\|_q \leq D\,\|f\|_r, \tag{18}$$

where $\frac{1}{q} = \frac{1}{r} + \frac{1}{t} - \frac{2}{n}$ *and* D *is a constant.*

Proof. If one observes that $B(|u|)^*(\sigma) \leq B(u^*(\sigma)) \quad \forall\,\sigma \in [0, |\Omega|[$ (see, e.g. [12]), the Theorem can be proved as Theorems 3.3 and 3.4 in [1]. □

For the sake of simplicity, from now on, we will suppose

$$b(\tau) = \frac{1}{(1+\tau)^\alpha}, \qquad 0 \leq \alpha \leq 1. \tag{19}$$

We observe explicitely that, under hypothesis (19), if $f \in L^r(\Omega)$ and $\frac{1}{\nu} \in L^t(\Omega)$, $\frac{1}{p} = \frac{1}{t} + \frac{1}{r}$, Corollaries 3.3 and 3.4 imply that:

$$\text{if } \ p > \frac{n}{2}, \quad n \geq 2, \quad 0 \leq \alpha \leq 1, \text{ then } u \in L^\infty(\Omega) \tag{20}$$

$$\text{if } \ \frac{n}{n+1-\alpha(n-1)} < p < \frac{n}{2}, \ 0 \leq \alpha < 1, \ n > 2, \tag{21}$$

then $u \in L^{\frac{np(1-\alpha)}{n-2p}}(\Omega)$.

THEOREM 3.5. *Let* u *be a solution of (1) satisfying (8). If (19) holds,* $f \in L^r(\Omega)$, $\frac{1}{\nu} \in L^t(\Omega)$ *and* $p > \frac{2n}{n+2-\alpha(n-2)}$, $\frac{1}{p} = \frac{1}{r} + \frac{1}{t}$, *then*

a) $p > \frac{n}{2}$, $n \geq 2 \quad 0 \leq \alpha \leq 1 \quad \Longrightarrow \quad u \in W_0^{1,2}(\nu) \cap L^\infty(\Omega)$

b) $\frac{2n}{n+2-\alpha(n-2)} < p < \frac{n}{2}$, $n > 2$, $0 \leq \alpha < 1 \Longrightarrow$
$u \in W_0^{1,2}(\nu) \cap L^q(\Omega)$,

where $q = \frac{np(1-\alpha)}{n-2p}$.

Proof. As in Theorem 3.1 one can still obtain (11).
Dividing by h both sides of (11) and passing to limit as h goes to zero, we have

$$-\frac{d}{d\tau}\int_{|u|>\tau}\nu(x)\,|Du|^2\,dx \le (1+\tau)^\alpha\int_0^{\mu_u(\tau)} f^*(\sigma)\,d\sigma.$$

Then integrating between 0 and $+\infty$ we get

$$\int_\Omega \nu(x)\,|Du|^2\,dx \le \int_0^{+\infty}(1+\tau)^\alpha d\tau\int_0^{\mu_u(\tau)} f^*(\sigma)\,d\sigma.$$

Hence, by the same calculation as in [2] (see Theorem 3.1), we can write

$$\int_\Omega \nu(x)\,|Du|^2 dx \tag{22}$$
$$\le \|f\|_r\left(2^\alpha|\Omega|^{1-1/r}+\left(\frac{\|u\|_q^q}{q}\right)^{1/r'}\left(\int_1^{+\infty}\frac{(1+\tau)^{\alpha r}}{\tau^{(q-1)(r-1)}}d\tau\right)^{1/r}\right),$$

for any $q>0$ and $\frac{1}{r'}+\frac{1}{r}=1$.

On the other hand, if $p>\frac{n}{2}$, by (20) we have $u\in L^\infty(\Omega)$. Then, for sufficiently large q, (22) gives $u\in W_0^{1,2}(\nu)$, and the part a) is proved.
For part b) we have to consider only the case $0\le\alpha<1$. We can use $q=\frac{np(1-\alpha)}{n-2p}$ into (22). The summability of the function $\frac{(1+\tau)^{\alpha r}}{\tau^{(q-1)(r-1)}}$ appearing in the integral on the right hand side of (22), can be proved by observing that, from hypotheses, we have: $r>p$, $q-1-\alpha>0$ and

$$\begin{aligned}(q-1)(r-1)-\alpha r-1 &= (q-1-\alpha)r-(q-1)-1\\ &> (q-1-\alpha)p-(q-1)\alpha=(q-1)(p-1)-\alpha p-1>0.\end{aligned}$$

From this, from (21) and (22), part b) follows. □

THEOREM 3.6. *Under the assumption of Theorem 3.5, with* $r > 1$, $t > 1$, $\frac{n}{(n+1)-\alpha(n-1)} < p \le \frac{2n}{(n+2)-\alpha(n-2)}$, $0 \le \alpha < 1$ *and* $n > 2$, *we have*

$$\left(\int_\Omega \nu(x)^{s/2} |Du|^s dx\right)^{1/s} < +\infty, \quad \text{for any } 0 < s < \frac{np(1-\alpha)}{n-p(1+\alpha)}.$$

Proof. Once again we start from (11). Hölder inequality implies

$$\begin{aligned}
&\frac{1}{h}\int_{\tau<|u|\le\tau+h} b(|u|)\nu(x)^{1/2} |Du|^s \, dx \\
&\le \frac{1}{h}\left(\int_{\tau<|u|\le\tau+h} b(|u|)\nu(x) |Du|^2 dx\right)^{1/2} \left(\int_{\tau<|u|\le\tau+h} b(|u|)dx\right)^{1-s/2} \\
&\le \left(\frac{1}{h}\int_{\tau<|u|\le\tau+h} b(|u|)\right)^{1-s/2} \left(\int_{|u|>\tau} |f| \, dx\right)^{1/2}.
\end{aligned}$$

Passing to the limit as h goes to zero we have

$$\begin{aligned}
&-\frac{d}{d\tau}\int_{|u|>\tau} (\nu(x))^{s/2} |Du|^s \, dx \\
&\qquad \le \left(-\mu_u'(\tau)\right)^{1-s/2} \left((1+\tau)^\alpha \int_0^{\mu_u(\tau)} f^*(\sigma)d\sigma\right)^{s/2}.
\end{aligned}$$

At this point one can use the argument of the proof of Theorem 3.2 in [2]. □

THEOREM 3.7. *Let u be a solution of problem (1) satisfying (8). If* $b(\tau) = \frac{1}{(1+\tau)^\alpha}$, $0 \le \alpha < 1$, $f \in L^r(\Omega)$, $\frac{1}{\nu} \in L^t(\Omega)$, $\nu \in L^{t'}(\Omega)$, *with* r, t, $t' > 1$, $\frac{n}{n+1-\alpha(n-1)} < p \le \frac{2n}{n+2-\alpha(n-r)}$ *and* $n > 2$, $\frac{1}{p} = \frac{1}{r} + \frac{1}{t}$, *then*

$$\int_\Omega \nu(x)|Du|^s \, dx < +\infty, \quad \text{for any} \quad 0 < s < \frac{t'-1}{t'}\frac{np(1-\alpha)}{n-p(1+\alpha)}.$$

Proof. Using Schwartz - Hölder inequality, by (11) we have $\forall\, s \in]0,2[$

$$\frac{1}{h}\int_{\tau<|u|\le\tau+h} b(|u|)\nu(x)\,|Du|^s\,dx$$
$$\le \left(\frac{1}{h}\int_{\tau<|u|\le\tau+h} b(|u|)\nu(x)\,dx\right)^{1-s/2}\left(\int_{|u|>\tau}|f|dx\right)^{s/2}.$$

By a passage to the limit as h goes to zero in the above inequality, we obtain

$$-\frac{d}{d\tau}\int_{|u|>\tau}\nu(x)|Du|^s\,dx \tag{23}$$
$$\le \left(-\mu_u'(\tau)\,\tilde{\nu}(\mu_u(\tau))\right)^{1-s/2}\left((1+\tau)^\alpha\int_0^{\mu_u(\tau)} f^*(\sigma')\,d\sigma'\right)^{s/2}$$
$$\le \|f\|_r^{s/2}\left(-\mu_u'(\tau)\tilde{\nu}(\mu_u(\tau))\right)^{1-s/2}\left((1+\tau)^\alpha(\mu_u(\tau))^{1-1/r}\right)^{s/2}$$

where, as in the definition of $\frac{1}{\underline{\nu}}$, $\tilde{\nu}$ is defined on $[0,|\Omega|[$ and it is such that

$$\int_{|u|>\tau}\nu(x)dx = \int_0^{\mu_u(\tau)}\tilde{\nu}(\sigma')\,d\sigma', \quad \text{for a.e. } \tau\in[0,|\Omega|[,$$

(for the construction of $\tilde{\nu}$ see e.g. [1]).
Lemma 2.2 of [1] ensures the existence of a sequence $\{\nu_m\}$ with $\nu_m^* = \nu^*$ and such that

$$\nu_m \longrightarrow \tilde{\nu}, \quad \text{weakly in } L^{t'}([0,|\Omega|[), t'>1. \tag{24}$$

From (23), integrating between 0 and $+\infty$, using Schwartz - Hölder inequality and taking into account of (24), we obtain

$$\int_\Omega \nu(x)\,|Du|^s\,dx \tag{25}$$
$$\le \|f\|_r^{s/2}\left(\int_0^{+\infty}(1+\tau)^{\frac{t'-1}{t'}}\,\tilde{\nu}(\mu_u(\tau))(-\mu'(\tau))d\tau\right)^{1-2/s}\times$$
$$\times\left(\int_0^{+\infty}(1+\tau)^{\alpha+\frac{t'-1}{t'}q(1-2/s)}\,\mu_u(\tau)^{1-1/r}\,d\tau\right)^{s/2}$$

$$\leq \|f\|_r^{s/2}\left(\|\nu\|_{t'}\left(\int_0^{+\infty}(1+\tau)^q(-\mu_u'(\tau))d\tau\right)^{1-1/t'}\right)^{1-2/s}\times$$
$$\times\left(\int_0^{+\infty}(1+\tau)^{\alpha+\frac{t'-1}{t'}q(1-2/s)}\,\mu_u(\tau)^{1-1/r}d\tau\right)^{s/2}$$
$$\leq \|f\|_r^{s/2}\,\|\nu\|_{t'}^{1-2/s}\,\|1+|u|\|_q^{q(1-1/t')(1-2/s)}\times$$
$$\times\left(\int_0^{+\infty}(1+\tau)^{\alpha+\frac{t'-1}{t'}q(1-2/s)}\mu_u(\tau)^{1-1/r}d\tau\right)^{s/2}.$$

The integral in the right hand side of (25) can be estimated as in Theorem 3.2 from [2]:

$$\int_0^{+\infty}(1+\tau)^{\alpha+\frac{t'-1}{t'}q(1-2/s)}\mu_u(\tau)^{1-1/r}\,d\tau \tag{26}$$
$$\leq 2^{\alpha+\frac{t'-1}{t'}q(1-2/s)}|\Omega|^{1-1/r}+$$
$$+\left(\frac{\|u\|_q^q}{q}\right)^{1/r'}\left(\int_1^{+\infty}\frac{(1+\tau)^{\alpha r+q\,\frac{t'-1}{t'}r(1-2/s)}}{\tau^{(q-1)(r-1)}}\,d\tau\right)^{1/r}.$$

Now we observe that if $q=\dfrac{np(1-\alpha)}{n-2p}$, $0\leq\alpha<1$, then

$$(q-1)(r-1)-\alpha r-q\frac{t'-1}{t'}r\left(1-\frac{2}{s}\right)-1$$
$$\geq(q-1)(p-1)-\alpha p-q\frac{t'-1}{t'}p\left(1-\frac{2}{s}\right)-1>0, \tag{27}$$

for any $s<\dfrac{t'-1}{t'}\,\dfrac{np(1-\alpha)}{n-2p(1+\alpha)}$.

Put $q=\dfrac{np(1-\alpha)}{n-2p}$ into (25) and (26). From (21), (25), (26) and (27) we have the assertion. □

4. Existence Theorems

Proof of Theorem 1.1.
As in [5], the existence of u will be obtained by approximation. To

this aim, let us define the following sequence of problems

$$\begin{cases} A_h(u) = -\dfrac{\partial}{\partial_{x_i}}\left(a_{ij}\,(x,\,T_h(u_h))\,\dfrac{\partial}{\partial_{x_j}}\,u_h\right) = f & \text{in } \ \Omega \\ u = 0 & \text{on } \ \partial\Omega \end{cases} \tag{28}$$

Observe first that, if $f \in L^r(\Omega)$ and $\dfrac{1}{\nu} \in L^t(\Omega)$ with $p \geq \dfrac{2n}{n+2}$, $\dfrac{1}{p} = \dfrac{1}{r} + \dfrac{1}{t}$, $n \geq 2$, then $f \in W^{-1,2}\left(\dfrac{1}{\nu}\right)$ ($W^{-1,2}\left(\dfrac{1}{\nu}\right)$ denotes the dual space of $W_0^{1,2}(\nu)$).

On the other hand, (II) and (III) imply

$$a_{ij}(x,\,T_h(\eta))\xi_j\xi_i \geq \frac{1}{(1+h)^\alpha}\nu(x)\,|\xi|^2 > 0 \tag{29}$$

$$\text{for a.e. } \ x \in \Omega,\ \eta \in \mathbb{R},\ |\xi| \neq 0\,, h \in N\,,$$

$$|a_{ij}(x,\,T_h(\eta))| \leq c\nu(x) \qquad \forall\, i,j = 1,\,2,\,\ldots\,n\,,\quad h \in N.$$

Inequalities (29) enable to deduce that, $\forall\, h \in N$, A_h is an operator of the calculus of variations type (in the sense of Definition (2.2) from [8], see Section 2.5, p. 180-182) from $W_0^{1,2}(\nu)$ into its dual $W^{-1,2}\left(\dfrac{1}{\nu}\right)$. Then there exists $u_h \in W_0^{1,2}(\nu)$ such that

$$\int_\Omega a_{ij}(x,\,T_h(u_h))\,\frac{\partial u_h}{\partial_{x_j}}\,\frac{\partial v}{\partial_{x_i}}\,dx = \int_\Omega f\,v\,dx, \qquad \forall\, v \in W_0^{1,2}(\nu) \tag{30}$$

Observing that (28) is analogous to problem (I), from (20) we have $u_h \in L^\infty(\Omega)$ and by (b) of Corollary 3.3 the norm u_h in L^∞ is bounded by a constant independent of h. Therefore, for h large, $T_h(u_h) = u_h$ which, together with (30), proves our assertion.

Proof of Theorem 1.2.

The assertion follows by the same method as in Theorem 1.1. In fact, we can analogously show that there exists a weak solution $u_h \in W_0^{1,2}(\nu)$ of (28), i.e. u_h satisfies (30). By (21) and by Theorem 3.5 (see in particular (18) and (22)), we have

$$\|u_h\|_q \leq C, \qquad \forall\, h \in N \ \text{ and } \ q = \frac{np(1-\alpha)}{n-2p}\,,$$

$$\|u_h\|_{W_0^{1,2}(\nu)} \leq C', \qquad \forall\, h \in N,$$

the constants C and C' being independent of h.
By Hölder inequality we get

$$\|Du_h\|_s \le \|u_h\|_{W_0^{1,2}(\nu)} \left\|\frac{1}{\nu}\right\|_t^{1/2}, \quad t = \frac{s}{2-s}, \quad 1 < s < 2.$$

Then there exists a subsequence of $\{u_h\}$, still denoted $\{u_h\}$, and $u \in W_0^{1,2}(\nu) \cap L^q(\Omega)$ such that $u_h \to u$ almost everywhere in Ω, as a consequence of the Rellich Theorem, $\nu(x)^{1/2} Du_h \to \nu(x)^{1/2} Du$ weakly in $L^2(\Omega; \mathbb{R}^n)$.
Moreover, by Theorem 2.1 from [7], we have

$$\frac{1}{\nu(x)^{1/2}} a_{ij}(x, T_h(u_h)) \to \frac{1}{\nu(x)^{1/2}} a_{ij}(x, u), \qquad \text{strongly in } L^2(\Omega).$$

Replacing v by $\varphi \in C_0^\infty(\Omega)$ into (30) and then passing to the limit as h goes to infinity, we get

$$\int_\Omega a_{ij}(x, u) u_{x_j} \varphi_{x_i} \, dx = \int_\Omega f \varphi \, dx, \qquad \forall \varphi \in C_0^\infty(\Omega)$$

From this, since C_0^∞ is dense in $W_0^{1,2}(\nu)$, $f \in W_0^{-1,2}\left(\frac{1}{\nu}\right)$ and $u \in W_0^{1,2}(\nu)$, we obtain our assertion.

Proof of Theorem 1.3.
In the same way as in [5] we consider $\forall\, h \in N$ the following problems

$$\begin{cases} -\dfrac{\partial}{\partial_{x_i}} \left((a_{ij}(x, T_h(u_h)) \dfrac{\partial}{\partial_{x_j}} u_h \right) = f_h & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{31}$$

where $f_h \in L^m(\Omega)$, $\dfrac{1}{m} = \min\left\{\dfrac{n+2}{2n} - \dfrac{1}{t}, \dfrac{1}{r}\right\}$, $f_h \to f$ strongly in L^r and

$$\|f_h\|_r \le \|f\|_r, \qquad \forall\, h \in N. \tag{32}$$

In the case $m = r$ obviously $f_h = f$, $\forall\, h \in N$.
Get $\dfrac{1}{p'} = \dfrac{1}{m} + \dfrac{1}{t}$, $p' \ge \dfrac{2n}{n+1}$. Problem (31) is of the same kind of (28). Then there exists $u_h \in W_0^{1,2}(\nu)$ such that

$$\int_\Omega a_{ij}(x, T_h(u_h)) \frac{\partial u_h}{\partial_{x_j}} \frac{\partial v}{\partial_{x_i}} dx = \int_\Omega f_h v \, dx, \quad \forall\, v \in W_0^{1,2}(\nu) \tag{33}$$

By (21), (32) and by Theorem 3.6 we have

$$\|u_h\|_q \leq C, \quad \forall\, h \in N, \quad q = \frac{np(1-\alpha)}{n-2p} \tag{34}$$

$$\|u_h\|_{W_0^{1,s}(\nu^{s/2})} \leq C', \quad \forall\, h \in N, \quad \frac{2t}{2t-1} < s < \frac{np(1-\alpha)}{n-p(1+\alpha)}, \tag{35}$$

the constants C, C' being independent of h.
By Hölder inequality we get

$$\begin{aligned} \|Du_h\|_\tau &\leq \left(\int_\Omega \nu^{s/2}|Du_h|^s dx\right)^{1/s} \left(\int_\Omega \left(\frac{1}{\nu}\right)^{s\tau/2(s-\tau)} dx\right)^{\frac{s-\tau}{s\tau}} \\ &= \|u_h\|^{1/s}_{W_0^{1,s}(\nu^{s/2})} \left\|\frac{1}{\nu}\right\|_t^{1/2}, \end{aligned} \tag{36}$$

$t = \dfrac{s\tau}{2(s-\tau)} > \dfrac{s}{2(s-1)}$, $1 < \tau < s$.
From (34), (35), (36) we deduce that there exists a subsequence of $\{u_h\}$, still denoted $\{u_h\}$, and $u \in W_0^{1,s}(\nu^{s/2}) \cap L^q(\Omega)$ such that $u_h \to u$ almost everywhere in Ω, as a consequence of the Rellich Theorem, $\nu(x)^{1/2} Du_h \longrightarrow \nu(x)^{1/2}\, Du$ weakly in $L^s(\Omega)$. Moreover, by Theorem 2.1 from [7], we have $\dfrac{1}{\nu(x)^{1/2}} a_{ij}(x,\, T_{h_l}(u_h)) \longrightarrow \dfrac{1}{\nu(x)^{1/2}}\, a_{ij}(x, u)$ strongly in $L^{\frac{s}{s-1}}(\Omega)$.
From this, replacing v by $\varphi \in C_0^\infty(\Omega)$ into (33), and passing to the limit when h goes to infinity, we have

$$\int_\Omega a_{ij}(x,\, u)\, u_{x_j}\, \varphi_{x_i}\; dx = \int_\Omega f\;\, \varphi\; dx, \qquad \forall\, \varphi \in C_0^\infty(\Omega),$$

and the proof is complete. □

Using (21) and Theorem 3.7, by argument analogous to that in Theorem 1.3, we obtain Theorem 1.4.

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