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On the Construction of Compatible Data for Hyperbolic-Parabolic Initial-Boundary Value Problems

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Summary. - *We study various questions related to compatibility conditions for a class of hyperbolic-parabolic initial-boundary value problems.*

1. Introduction

1.1. In our paper [10] we considered global existence and comparison results for solutions of the quasilinear dissipative hyperbolic initial value problem

$$\begin{cases} \varepsilon u_{tt} + u_t - \sum_{i,j=1}^{n} a_{ij}(\nabla u)\partial_i\partial_j u = f(x,t), \\ u(x,0) = u_0(x), \qquad u_t(0) = u_1(x), \end{cases} \tag{1}$$

for small $\varepsilon > 0$, and those of the formal limit parabolic initial value problem corresponding to $\varepsilon = 0$, i.e.

$$\begin{cases} v_t - \sum_{i,j=1}^{n} a_{ij}(\nabla v)\partial_i\partial_j v = g(x,t), \\ v(x,0) = v_0(x). \end{cases} \tag{2}$$

We considered smooth solutions in the sense of Sobolev-Kato (as described in Sections 3 and 4 below), corresponding to data $\{f, u_0, u_1\}$

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(respectively $\{g, v_0\}$), of *arbitrary* size: in this case, blow up of solutions in finite time may be expected for both problems, but in [10] we were able to show that, roughly speaking, global solvability of problem (1) and that of (2) are equivalent, in the sense that problem (2) is solvable on an arbitrary time interval $[0, T]$ for arbitrary data if and only if the same is true for problem (1) and ε is sufficiently small. (Local in time solvability of both problems (1) and (2) for arbitrary data follows from Kato, [4]; if the data are "sufficiently small", a direct application of Matsumura's technique of [7] shows that both problems (1) and (2) are globally solvable).

1.2. We recall that, in [10], the "if" part is proven by controlling a singular convergence process as $\varepsilon \to 0$, while the "only if" part is proven by introducing the change of variable $y = u - v$, and applying to the problem satisfied by y the aforementioned global existence result of Matsumura. To carry out this procedure, the first step in both cases consists in the construction of a suitable set of data for the type of problem we assume we can solve globally, starting with the given data of the problem we wish to solve globally; for the pure initial value problems (1) and (2), this essentially amounts to the choice of a suitable approximation of the data that are given.

It is of course natural to ask to what extent we can follow an analogous procedure when we consider problems (1) and (2) in a bounded domain $\Omega \subset \mathbb{R}^n$, with appropriate boundary conditions. In this case, the question is complicated by the requirement of compatibility conditions on the data at the boundary $\partial\Omega$ for $t = 0$, which are necessary for the solvability of either problem: we have then to deal with both the order and the type of the compatibility conditions. The latter depends of course on the problem we consider (we shall indeed speak of "hyperbolic compatibility conditions", HCC in short, for problem (1), and of "parabolic compatibility conditions", PCC in short, for problem (2)), while the order depends, for either problem, on the regularity desired of the solution. Both these questions arise naturally if we try to follow the approximation procedure of [10]: indeed, if we want to approximate a given set of data (of either problem) by a sequence of smoother data, we must ensure that these satisfy the higher order compatibility conditions of the proper type. In addition, if we want to construct a suitable set of data for one type of problem, starting with the given data of the other problem, we must also be able to do so respecting the corresponding type of compatibility conditions.

1.3. These are precisely the questions we investigate in this paper, in preparation for the extension of our equivalency result of [10] to initial-boundary value problems, which we shall pursue in a following paper. In this respect, we remark that the fact that problems (1) and (2) are quasilinear does not play any essential role; indeed, in the sequel we shall only deal for simplicity with the linear problems, and will indicate the necessary modifications for the quasilinear case when appropriate. Finally, again for simplicity, we shall only consider homogeneous Dirichlet boundary conditions at the boundary; we believe that other types of boundary conditions can be treated in a similar way.

This paper is organized as follows: in Section 2 we introduce notations and prepare several technical results needed in the sequel; in Section 3 we consider the hyperbolic problem and, given data $\{f, u_0, u_1\}$ satisfying the HCC of a certain order, we show how to construct approximating smooth data $\{f^\delta, u_0^\delta, u_1^\delta\}$ satisfying the HCC of higher order; in Section 4 we do the same for the parabolic problem; finally, in Section 5 we show how to suitably construct one set of data, satisfying the HCC or the PCC, from given data satisfying the other type of conditions, i.e. the PCC or HCC.

2. Preliminaries

2.1. In the sequel, $\Omega \subset \mathbb{R}^n$ is a bounded open domain with smooth boundary $\partial\Omega$. For integer $m \geq 1$ we denote by $||.||_m$ the norm in the Sobolev space $H^m(\Omega)$, and set $H_*^m(\Omega) \doteq H^m(\Omega) \cap H_0^1(\Omega)$. We also set $L^2(\Omega) = H^0(\Omega)$, and denote by $||.||$ its norm; finally, we set $H^\infty(\Omega) \doteq \cap_{m\geq 0} H^m(\Omega)$. The following density result can be proven exactly as in [12], where only the case $m = 1$ was considered:

Proposition 2.1. *For all $m \geq 1$, $H^\infty(\Omega) \cap H_0^1(\Omega)$ is dense in $H_*^m(\Omega)$.*

Given $T > 0$ and $\tau \in (0, T)$, we set $Q_\tau \doteq \Omega \times (\tau, T)$, and will abbreviate $Q = Q_0 = \Omega \times (0, T)$. If $u = u(x, t)$ is defined in Q, we denote space derivatives by $\partial_i u = \partial u/\partial x_i$, and set $\nabla u = \{\partial_1, \dots, \partial_n\}$; time differentiation is denoted by $\partial_t u = \partial u/\partial t$, and we write u_t and u_{tt} instead of $\partial_t u$ and $\partial_t^2 u$.

In the sequel we shall need the following modification of a result by Shibata and Kikuchi, [14]:

PROPOSITION 2.2. *For $0 \le j \le m$, let $u_j \in H^{m-j}(\Omega)$. There exists a function $u \in \cap_{j=0}^m C^j(\mathbb{R}; H^{m-j}(\Omega))$, such that*

$$\text{for } 0 \le j \le m, \qquad (\partial_j u)(\cdot, 0) = u_j \tag{3}$$

$$\text{for } 0 \le j \le m, \qquad \sup_{t\in\mathbb{R}} ||(\partial_t^j u)(\cdot,t)||_{m-j} \le C \sum_{k=0}^m ||u_k||_{m-k}, \tag{4}$$

$$\forall \tau \in (0,T), \qquad \int_\tau^T ||(\partial_t^{m+1} u)(\cdot,t)||^2 dt \le \frac{C}{\tau} \sum_{k=0}^m ||u_k||^2_{m-k}, \tag{5}$$

with $C > 0$ independent of u, T and τ.

Proof. We adapt the argument of [14], Theorem Ap5. After extending the functions u_j to $v_j \in H^{m-j}(\mathbb{R}^n)$, we set

$$\psi(\xi,t) \doteq \sum_{j,l=0}^m (\exp\{(i-1)(l+1)(1+|\xi|^2)^{1/2} t\}) q_{jl} (1+|\xi|^2)^{-j/2} \hat{v}_j(\xi), \tag{6}$$

where $i = \sqrt{-1}$, $\hat{v}$ denotes the Fourier Transform of v, and the $(m+1)^2$ numbers q_{jl} are defined as the solutions of the linear algebraic system

$$\begin{cases} \sum_{l=0}^m ((i-1)(l+1))^k q_{jl} = \delta_{kj} \\ j,k = 0,\dots,m \end{cases} \tag{7}$$

($\delta_{jk} = 1$ if $j = k$, $= 0$ otherwise). It is then easy to verify that $(\partial_t^j \psi)(\cdot,0) = \hat{v}_j$ for $0 \le j \le m$; defining then $v(x,t)$ by $\hat{v}(\xi,t) = \psi(\xi,t)$ and calling $u = u(x,t)$ the restriction of v to Ω, we easily see by Parseval's formula that u has the desired properties. In particular,

(5) follows from the estimates

$$\begin{aligned}
&\int_\tau^T ||(\partial_t^{m+1}\psi)(\cdot,t)||^2\,dt \\
&\quad\le C\sum_{j=0}^m \int_{\mathbb{R}^n}(1+|\xi|^2)^{m+1-j}|\hat v_j(\xi)|^2 \times \\
&\qquad\qquad \times \int_\tau^T \exp\{-2(1+|\xi|^2)^{1/2}t\}\,dt\,d\xi \\
&\quad\le \frac{1}{2}C\sum_{j=0}^m \int_{\mathbb{R}^n}(1+|\xi|^2)^{m-j+1/2}|\hat v_j(\xi)|^2\exp\{-2(1+|\xi|^2)^{1/2}\tau\}\,d\xi,
\end{aligned} \tag{8}$$

and from the inequality

$$\sup_{\xi\in\mathbb{R}^n}(1+|\xi|^2)^{1/2}\exp\{-2(1+|\xi|^2)^{1/2}\tau\}\le\frac{1}{2e\tau}. \tag{9}$$

□

Remark. In a similar way, we can prove that if $v_j \in H^{m-2j}(\Omega)$ for $0\le j\le \left[\frac{m}{2}\right]$, there is a function $v\in\cap_{j=0}^{[m/2]}C^j([0,T];H^{m-2j}(\Omega))$ such that $(\partial_t^j v)(\cdot,0)=v_j$ and

$$\int_\tau^T ||v(\cdot,t)||_{m+1}^2\,dt \le \frac{C}{\tau^{m+1}}\sum_{j=0}^{[m/2]}||v_j||_{m-2j}^2 \quad \forall\,\tau\in(0,T), \tag{10}$$

$$\int_0^T ||(\partial_t^{r+1}v)(\cdot,t)||^2\,dt \le C\sum_{j=0}^{[m/2]}||v_j||_{m-2j}^2 \quad \text{if} \quad m=2r+1, \tag{11}$$

$$\int_0^T ||(\partial_t^{r+1}v)(\cdot,t)||_{H^{-1}(\Omega)}^2\,dt \le C\sum_{j=0}^{[m/2]}||v_j||_{m-2j}^2 \quad \text{if} \quad m=2r, \tag{12}$$

we shall use estimates (3), (4) and (5) in Section 3 on the hyperbolic problem, and estimates (10), (11) and (12) in Section 4 on the parabolic problem.

3. Hyperbolic Compatibility Conditions

3.1. In this section we consider the linear hyperbolic initial boundary value problem

$$\begin{cases} \varepsilon u_{tt} + u_t - \sum_{i,j=1}^n a_{ij}(x,t)\partial_i\partial_j u = f(x,t), \\ u(x,0) = u_0(x), \qquad u_t(0) = u_1(x), \\ u(\cdot,t)_{|\partial\Omega} = 0, \end{cases} \tag{13}$$

with $\varepsilon > 0$, f given in Q, and u_0, u_1 given in Ω. Following Kato, [5], we consider solutions of (13) in the spaces

$$X_m(0,T) \doteq \cap_{j=0}^m C^j([0,T]; H^{m-j}(\Omega))$$

for sufficiently large integer m (at least so that such solutions are also classical ones, by Sobolev's imbedding theorems). We shall assume that the coefficients a_{ij} in (13) are smooth and symmetric (i.e. $a_{ij} = a_{ji}$), and satisfy the uniformly strong ellipticity condition

$$\exists\, \nu > 0 \ \forall\,(x,t) \in \overline{Q},\ \ \forall\, q \in \mathbf{R}^n, \qquad \sum_{i,j=1}^n a_{ij}(x,t)q^i q^j \geq \nu|q|^2. \tag{14}$$

From Kato, [5] (Theorem 12.1), we have

THEOREM 3.1. *Let $s \in \mathbb{N}$, $s \geq \left[\frac{n}{2}\right] + 2$, and assume that*

HA1) $a_{ij} \in C^{s-1}(\overline{Q})$, $f \in X_{s-1}(0,T)$, $\partial_t^s f \in L^2(Q)$, $u_0 \in H_*^{s+1}(\Omega)$, $u_1 \in H_*^s(\Omega)$,

HA2) *the coefficients $\{a_{ij}\}$ and the data $\{f, u_0, u_1\}$ satisfy the HCC of order s described below.*

There exists then a unique $u \in X_{s+1}(0,T)$, solution of (13) *(also in the classical sense).*

3.2. We recall the definition of HCC of order s: following Kato, [5], we introduce the operator

$$L(t)u \doteq \sum_{i,j=1}^n a_{ij}(\cdot,t)\partial_i\partial_j u \tag{15}$$

and, given smooth data $\{f, u_0, u_1\}$ and coefficients $\{a_{ij}\}$, starting with u_0 and u_1 we generate a sequence of functions $\{u_k\}$ on Ω by the recursive definition

$$\varepsilon\, u_{k+2} = (\partial_t^k f)(\cdot, 0) - u_{k+1} + A_k[u_0, \ldots, u_k], \qquad k \geq 0, \tag{16}$$

where

$$A_k[u_0, \ldots, u_k] \doteq \sum_{j=0}^{k} \binom{k}{j} (\partial_t^j L)(0) u_{k-j}; \tag{17}$$

i.e., for instance,

$$\varepsilon\, u_2 = f(\cdot, 0) - u_1 + \sum_{i,j=1}^{n} a_{ij}(\cdot, 0)\partial_i \partial_j u_0, \tag{18}$$

$$\varepsilon\, u_3 = f_t(\cdot, 0) - u_2 + \sum_{i,j=1}^{n} a_{ij}(\cdot, 0)\partial_i \partial_j u_1 + \tag{19}$$

$$+ \sum_{i,j=1}^{n} (\partial_t a_{ij})(\cdot, 0)\partial_i \partial_j u_0,$$

etc. The following result is easily proven by induction on k, recalling that $H^{s-1}(\Omega)$ is an algebra under pointwise multiplication if, as we assume, $s > \frac{n}{2} + 1$:

PROPOSITION 3.2. *Let $s \in \mathbb{N}$, $s \geq \left[\frac{n}{2}\right] + 2$. Under assumptions (HA1) of Theorem 3.1 (with $\partial_t^s f \in L^2(Q)$ not required), the functions $\{u_k\}$ are well defined at least for $0 \leq k \leq s+1$, with $u_k \in H^{s+1-k}(\Omega)$; moreover, for $2 \leq k \leq s+1$ there exists $C_k > 0$ independent of the data $\{f, u_0, u_1\}$ such that*

$$||u_k||_{s+1-k} \leq C_k \left\{ \sum_{r=0}^{k-2} ||(\partial_t^r f)(\cdot, 0)||_{s-1-r} + \sum_{r=0}^{k-1} ||u_r||_{s+1-r} \right\}. \tag{20}$$

Proposition 3.2 allows us to introduce the following

DEFINITION 3.3. *In the same assumptions of Proposition 3.2, we say that the coefficients $\{a_{ij}\}$ and the data $\{f, u_0, u_1\}$ of* (13) *satisfy the Hyperbolic Compatibility Conditions (HCC) of order s at $\partial\Omega$ for $t = 0$ if*

$$u_k \in H_*^{s+1-k}(\Omega) \qquad \text{for } 0 \leq k \leq s. \tag{21}$$

We remark that in the sequence $\{u_k\}$ defined by (16) we also have a well defined function $u_{s+1} \in L^2(\Omega)$. The HCC (21) are necessary for the solvability of (13) in $X_{s+1}(0,T)$, for if $u \in X_{s+1}(0,T)$ solves (13), then in fact $u_k = (\partial_t^k u)(\cdot,0) \in H^{s+1-k}(\Omega)$ for $0 \le k \le s+1$, so its trace on $\partial\Omega$ is defined at least for $0 \le k \le s$, and must therefore vanishes.

3.3. This last remark allows us to define the HCC also for the initial boundary value problem for the quasilinear equation

$$\varepsilon\, u_{tt} + u_t - \sum_{i,j=1}^{n} b_{ij}(x,t,u,\nabla u,u_t)\partial_i\partial_j u = f(x,t), \qquad (22)$$

for smooth, symmetric coefficients $b_{ij} = b_{ij}(x,t,p,q_1,\dots,q_n,r) : \mathbb{R}^{2n+3} \to \mathbb{R}$. Indeed, assuming that such problem has a local solution $u \in X_{s+1}(0,\tau)$ for some $\tau \in (0,T]$ (as provided e.g. by Theorem 14.3 of Kato, [5]), we can set

$$a_{ij}(x,t) \doteq b_{ij}(x,t,u(x,t),\nabla u(x,t),u_t(x,t))$$

and proceed to compute the functions $\{u_k\}$ as in (16); we realize then that this sequence $\{u_k\}$ is completely determined by the data $\{f,u_0,u_1\}$, and does not require the actual existence of a solution to (22). For example, introducing the vector function $\zeta_0 = \zeta_0(x) \in \mathbb{R}^{2n+3}$ on Ω by $\zeta_0 = \{\cdot,0,u_0,\nabla u_0,u_1\}$, (18) and (19) now take the form

$$\begin{aligned}\varepsilon\, u_2 &= f(\cdot,0) - u_1 + \sum_{i,j=1}^{n} b_{ij}(\zeta_0)\partial_i\partial_j u_0,\\ \varepsilon\, u_3 &= f_t(\cdot,0) - u_2 + \sum_{i,j=1}^{n} b_{ij}(\zeta_0)\partial_i\partial_j u_1 + U_3,\end{aligned}$$

where

$$\begin{aligned}U_3 = \sum_{i,j=1}^{n} \{&(\partial_t b_{ij})(\zeta_0) + (\partial_p b_{ij})(\zeta_0)u_1 + (\nabla_q b_{ij})(\zeta_0)\cdot\nabla u_1 +\\ &+ (\partial_r b_{ij})(\zeta_0)u_2\}\partial_i\partial_j u_0.\end{aligned}$$

We can also show that a corresponding version of Proposition 3.2 still holds if the coefficients $\{b_{ij}\}$ are at least of class $C^{s-1}(\mathbb{R}^{2n+3})$, so that Definition 3.3 again makes sense.

3.4. We now come to the main question of this section, that is, the construction of more regular data for problem (13) that not only approximate the given data $\{f, u_0, u_1\}$, but also satisfy the HCC of higher order. As a motivation, other that the applications we describe in Section 5, we mention that this question most commonly occurs when existence results like Theorem 3.1 are proven by energy methods, first establishing a priori estimates on more regular solutions (which it is possible to differentiate), and then resorting to a density argument. The problem of constructing compatible regularising data was partially addressed in [12], where we showed how to construct data satisfying HCC of order 2 from data satisfying HCC of order 1; here, we generalize the technique presented in [12], to obtain data satisfying HCC of arbitrary order. We claim:

THEOREM 3.4. *Under the same assumptions of Theorem 3.1, there exist sequences* $\{a^\delta_{ij}\}$, $\{f^\delta\} \subset C^\infty([0,T]; H^\infty(\Omega))$ *and* $\{u^\delta_0\}$, $\{u^\delta_1\}$ $\subset H^\infty(\Omega) \cap H^1_0(\Omega)$ *such that* $\{a^\delta_{ij}\}$ *and* $\{f^\delta, u^\delta_0,$ $u^\delta_1\}$ *satisfy the HCC of order* $s+1$, *and*

$$\begin{array}{llll} \text{HC1)} & a^\delta_{ij} \to a_{ij} & \text{in } C^{s-1}(\overline{Q}), & \\ \text{HC2)} & \partial^j_t f^\delta \to \partial^j_t f & \text{in } C([0,T]; H^{s-1-j}(\Omega)), & 0 \le j \le s-1, \\ \text{HC3)} & \partial^s_t f^\delta \to \partial^s_t f & \text{in } L^2(Q_\tau) & \forall\, \tau \in (0,T), \\ \text{HC4)} & u^\delta_0 \to u_0 & \text{in } H^{s+1}(\Omega), & \\ \text{HC5)} & u^\delta_1 \to u_1 & \text{in } H^s(\Omega). & \end{array} \tag{23}$$

Moreover, if u^δ *solves equation* (13) *with coefficients* $\{a^\delta_{ij}\}$ *and data* $\{f^\delta, u^\delta_0, u^\delta_1\}$, *then for all* $\tau \in (0,T)$,

$$u^\delta \to u \quad \text{in } X_s(0,T) \cap X_{s+1}(\tau, T) \qquad \text{as } \delta \to 0. \tag{24}$$

Proof. Following Ikawa, [2], we select sequences $\{a^\delta_{ij}\}$ and $\{h^\delta\}$ from $C^\infty([0,T]; H^\infty(\Omega))$ such that

$$a^\delta_{ij} \to a_{ij}, \qquad h^\delta \to f \qquad \text{in} \quad X_{s-1}(0,T) \cap L^2(Q). \tag{25}$$

By Proposition 2.1 we can also choose, for $0 \le j \le s+1$, sequences $\{u_j^\delta\} \subset H^\infty(\Omega) \cap H_0^1(\Omega)$ such that

$$u_j^\delta \to u_j \qquad \text{in} \qquad H^{s+1-j}(\Omega) \tag{26}$$

(for $j = s+1$ we simply use the density of $C_0^\infty(\Omega)$ in $L^2(\Omega)$). We define then for $0 \le j \le s-1$ the functions $\lambda_j^\delta = \lambda_j^\delta(x)$ on Ω by

$$\lambda_j^\delta \doteq (\partial_t^j h)(\cdot,0) - u_{j+1}^\delta + A_j^\delta[u_0^\delta, \dots, u_j^\delta] - \varepsilon u_{j+2}^\delta, \tag{27}$$

where $A_j^\delta[\dots]$ is defined as in (17), with $L(t)$ replaced by

$$L^\delta(t)u \doteq \sum_{i,j=1}^n a_{ij}^\delta(\cdot,t)\partial_i\partial_j u.$$

Obviously, $\lambda_j^\delta \in C^\infty(\Omega)$; hence, by Proposition 2.2 we can choose functions $\varphi^\delta = \varphi^\delta(x,t)$ such that

$$(\partial_t^j \varphi^\delta)(\cdot,0) = \lambda_j^\delta \qquad \text{for} \qquad 0 \le j \le s-1 : \tag{28}$$

recalling (6), we see that $\varphi^\delta \in C^\infty(\mathbb{R}; H^\infty(\Omega))$. To conclude, we set

$$f^\delta = h^\delta - \varphi^\delta : \tag{29}$$

then, (HC1) follows by Ascoli-Arzela's Theorem, and to verify (HC2) and (HC3), by (29) and the second of (25) it is sufficient to show that

$$\partial_t^j \varphi^\delta \to 0 \ \text{ in } \ C([0,T]; H^{s-1-j}(\Omega)), \quad 0 \le j \le s-1, \tag{30}$$

$$\partial_t^s \varphi^\delta \to 0 \ \text{ in } \ L^2(Q_\tau), \qquad \tau \in (0,T). \tag{31}$$

To this end, we recall estimates (4) and (5) of Proposition 2.2, according to which

$$\sup_{0\le t\le T} ||(\partial_t^j \varphi^\delta)(\cdot,t)||_{s-1-j} \le C \sum_{j=0}^{s-1} ||\lambda_j^\delta||_{s-1-j} \tag{32}$$

for $0 \le j \le s-1$, and, for $\tau \in (0,T)$,

$$\int_\tau^T ||(\partial_t^s \varphi^\delta)(\cdot,t)||^2 dt \le \frac{C}{\tau} \sum_{j=0}^{s-1} ||\lambda_j^\delta||_{s-1-j}^2; \tag{33}$$

next, we see that, by (27), (25) and (26),

$$\lambda_j^\delta \to (\partial_t^j f)(\cdot,0) - u_{j+1} + A_j[u_0,\dots,u_j] - \varepsilon u_{j+2}$$

in $H^{s-1-j}(\Omega)$ as $\delta \to 0$. By the assumed HCC of order s, (16) implies that $\lambda_j^\delta \to 0$ in $H^{s-1-j}(\Omega)$, so that (30) and (31) follow from (32) and (33). Next, (HC4) and (HC5) are true by construction, and to verify that $\{a_{ij}^\delta\}$ and $\{f^\delta, u_0^\delta, u_1^\delta\}$ satisfy the HCC of order $s+1$, according to (16) we define functions $\{z_k^\delta\}$ for $0 \le k \le s+2$ recursively, setting $z_0^\delta = u_0^\delta$, $z_1^\delta = u_1^\delta$ and, for $0 \le k \le s-1$,

$$\varepsilon z_{k+2}^\delta = (\partial_t^k f^\delta)(\cdot,0) - z_{k+1}^\delta + A_k^\delta[z_0^\delta,\dots,z_k^\delta]. \tag{34}$$

We verify then, by induction on k, that $z_k^\delta = u_k^\delta$ for $0 \le k \le s+2$: indeed, this is true for $k=0$ and $k=1$; then, by (34), the induction assumption, (29), (28) and (27) we have

$$\begin{aligned}\varepsilon z_{k+2}^\delta &= (\partial_t^k h^\delta)(\cdot,0) - (\partial_t^k \varphi^\delta)(\cdot,0) - z_{k+1}^\delta + A_k^\delta[z_0^\delta,\dots,z_k^\delta] = \\ &= (\partial_t^k h^\delta)(\cdot,0) - \lambda_k^\delta - u_{k+1}^\delta + A_k^\delta[u_0^\delta,\dots,u_k^\delta] = \varepsilon u_{k+2}^\delta.\end{aligned}$$

Consequently, since $u_k^\delta|_{\partial\Omega} = 0$ by construction, the HCC of order $s+1$ are satisfied. Finally, (24) follows from known results on the strong well-posedness of hyperbolic initial boundary value problems (see e.g. Kato, [5], Chapter 1.5, for the linear problem, and [13] for the quasilinear one). Theorem 3.4 is thus completely proven. □

To conclude this section, it is sufficient to remark that a repeated application of Theorem 3.4 will allow the construction of data satisfying HCC of arbitrary order.

4. Parabolic Compatibility Conditions

4.1. In this section we consider the linear parabolic initial boundary value problem

$$\begin{cases} v_t - \sum_{i,j=1}^n a_{ij}(x,t)\partial_i\partial_j v = g(x,t), \\ v(x,0) = u_0(x), \\ v(\cdot,t)_{|\partial\Omega} = 0, \end{cases} \tag{35}$$

for given data g in Q and v_0 in Ω. Following [11], we consider solutions of (35) in the spaces

$$H_c^m(Q) \doteq \left\{u \in H^{m,m/2}(Q) \mid \partial_t^{m'} u \in C([0,T]; H^{m-2m'-1}(\Omega))\right\},$$

with $m' \doteq \left[\frac{m-1}{2}\right]$ ($[x]$ denoting the integer part of x), and m a sufficiently large integer. We recall from Lions-Magenes, [6], Ch. 4, that

$$H^{m,m/2}(Q) \doteq L^2(0,T; H^m(\Omega)) \cap H^{m/2}(0,T; L^2(\Omega)), \tag{36}$$

and that, setting

$$\tilde{H}^m(Q) \doteq \left\{u \in L^2(0,T; H^m(\Omega)) \mid \partial_t^{m''} u \in L^2(0,T; H^{m-2m''}(\Omega))\right\},$$

with $m'' = \left[\frac{m+1}{2}\right]$, by imbedding and trace theorems we have in fact that

$$\tilde{H}^m(Q) = H_c^m(Q) = H^{m,m/2}(Q) \qquad \text{if } m \text{ even}, \tag{37}$$

$$\tilde{H}^m(Q) \rightharpoonup H_c^m(Q) \rightharpoonup H^{m,m/2}(Q) \qquad \text{if } m \text{ odd}. \tag{38}$$

Assuming again the uniformly strong ellipticity condition (14), from [11] we have

THEOREM 4.1. *Let $s \in \mathbb{N}$, $s \geq \left[\frac{n}{2}\right] + 2$, and assume that*

PA1) a_{ij}, $g \in H_c^s(Q)$, $v_0 \in H_*^{s+1}(\Omega)$,

PA2) *the coefficients $\{a_{ij}\}$ and the data $\{g, v_0\}$ satisfy the PCC of order s described below.*

There exists then a unique $v \in H_c^{s+2}(Q)$, solution of (35).

Again, v is also a classical solution, because of the embedding $H_c^{s+2}(Q) \rightharpoonup C^{2+\alpha,1+\alpha/2}(\overline{Q})$, $\alpha = s - 1 - \frac{n}{2} > 0$ (see e.g. Mazja, [8], or a direct proof in [3], Proposition 3.8, via the imbeddings

$$\begin{aligned} H_c^{s+2}(Q) &\rightharpoonup \left\{u \in L^2(0,T; H^{s+2}(\Omega)) \mid u_{tt} \in L^2(0,T; H^{s-2}(\Omega))\right\} \\ &\rightharpoonup C^{2+\alpha,1+\alpha/2}(\overline{Q})). \end{aligned}$$

4.2. The definition of PCC of order s in Theorem 4.1 is similar to that of the hyperbolic case: namely, we recall the definition (15) of $L(t)$ and, given smooth data $\{g, v_0\}$ and coefficients $\{a_{ij}\}$, starting with v_0 we generate a sequence of functions $\{v_k\}$ on Ω by the recursive definition

$$v_{k+1} \doteq (\partial_t^k g)(\cdot, 0) + A_k[v_0, \dots, v_k], \qquad k \geq 0. \tag{39}$$

We can then prove by induction

PROPOSITION 4.2. *Let $s \in \mathbb{N}$, $s \geq \left[\frac{n}{2}\right] + 2$. Under assumptions (PA1) of Theorem 4.1, the functions $\{v_k\}$ are well defined at least for $0 \leq k \leq \left[\frac{s+1}{2}\right]$, with $v_k \in H^{s+1-2k}(\Omega)$; moreover, for $1 \leq k \leq \left[\frac{s+1}{2}\right]$ there exists $C_k > 0$ independent of the data $\{g, v_0\}$ such that*

$$||v_k||_{s+1-2k} \leq C_k \left\{ \sum_{r=0}^{k-1} ||(\partial_t^r g)(\cdot, 0)||_{s-1-2r} + \sum_{r=0}^{k-1} ||v_r||_{s+1-2r} \right\}. \tag{40}$$

Proposition 4.2 allows us to introduce the following

DEFINITION 4.3. *In the same assumptions of Proposition 4.2, we say that the coefficients $\{a_{ij}\}$ and the data $\{g, v_0\}$ of (35) satisfy the Parabolic Compatibility Conditions (PCC) of order s at $\partial\Omega$ for $t = 0$ if*

$$v_k \in H_*^{s+1-2k}(\Omega) \qquad \text{for } 0 \leq k \leq \left[\frac{s}{2}\right]. \tag{41}$$

We remark that if $s = 2r + 1$ is odd, in the sequence $\{v_k\}$ defined by (39) we also have a well defined function $v_{r+1} \in L^2(\Omega)$. The actual number of conditions in (41) is determined by the requirement that $s + 1 - 2k \geq 1$, which translates into $\left[\frac{s+2}{2}\right]$ conditions. As in the hyperbolic case, the PCC (41) are necessary for the solvability of (35); they can be defined in a similar way for the initial boundary value problem for the quasilinear equation

$$v_t - \sum_{i,j=1}^{n} b_{ij}(x, t, v, \nabla v) \partial_i \partial_j v = g(x, t), \tag{42}$$

with a corresponding version of Proposition 4.2 still holding.

4.3. As for the hyperbolic case, we now show how to construct more regular data for problem (35) that not only approximate the given data $\{g, v_0\}$, but also satisfy the PCC of higher order. We claim:

THEOREM 4.4. *Under the same assumptions of Theorem 4.1, there exist sequences* $\{a^\delta_{ij}\}$, $\{g^\delta\} \subset C^\infty([0,T]; H^\infty(\Omega))$ *and* $\{v^\delta_0\} \subset H^\infty(\Omega)$ $\cap H^1_0(\Omega)$ *such that* $\{a^\delta_{ij}\}$ *and* $\{g^\delta, v^\delta_0\}$ *satisfy the PCC of order* $s+1$, *and*

$$\begin{array}{lll} \text{PC1)} & a^\delta_{ij} \to a_{ij} & \text{in } H^s(Q), \\ \text{PC2)} & g^\delta \to g & \text{in } H^s(Q), \\ \text{PC3)} & v^\delta_0 \to v_0 & \text{in } H^{s+1}(\Omega). \end{array} \tag{43}$$

Moreover, if v^δ *solves equation* (35) *with coefficients* $\{a^\delta_{ij}\}$ *and data* $\{g^\delta, v^\delta_0\}$, *then for all* $\tau \in (0,T)$,

$$v^\delta \to v \quad \text{in } H^{s+2}_c(Q_\tau) \qquad \text{as } \delta \to 0. \tag{44}$$

Proof. The proof is analogous to that of Theorem 3.4; we shall only report the relevant changes. Again, we select sequences $\{a^\delta_{ij}\}$, $\{h^\delta\} \subset C^\infty([0,T]; H^\infty(\Omega))$ and, for $0 \le j \le \left[\frac{s+1}{2}\right]$, $\{v^\delta_j\} \subset H^\infty(\Omega) \cap H^1_0(\Omega)$ such that $a^\delta_{ij} \to a_{ij}$ in $H^s_c(Q)$, $h^\delta \to g$ in $H^s_c(Q)$, and $v^\delta_j \to v_j$ in $H^{s+1-2j}(\Omega)$ (if $s = 2r+1$, for $j = r+1$ we invoke the density of $C^\infty_0(\Omega)$ in $L^2(\Omega)$). We define then for $0 \le j \le \left[\frac{s-1}{2}\right]$

$$\lambda^\delta_j \doteq (\partial^j_t h^\delta)(\cdot, 0) + A^\delta_j[v^\delta_0, \dots, v^\delta_j] - v^\delta_{j+1}, \tag{45}$$

so that $\lambda^\delta_j \in C^\infty(\Omega)$, and, by Proposition 2.2, we determine $\{\varphi^\delta\} \subset C^\infty(\mathbb{R}; H^\infty(\Omega))$ such that

$$(\partial^j_t \varphi^\delta)(\cdot, 0) = \lambda^\delta_j \qquad \text{for } 0 \le j \le \left[\frac{s-1}{2}\right]. \tag{46}$$

Finally, we set $g^\delta = h^\delta - \varphi^\delta$, and verify the conclusions of Theorem 4.4 in the same way as for Theorem 3.4. The only non trivial step is to verify that

$$\varphi^\delta \to 0 \qquad \text{in } H^s_c(Q_\tau), \qquad \tau \in (0,T), \tag{47}$$

which, recalling (37) and (38), we prove by showing that in fact

$$\varphi^\delta \to 0 \qquad \text{in } \tilde{H}^s(Q_\tau). \tag{48}$$

This requires showing that

$$\varphi^\delta \to 0 \quad \text{in} \quad L^2(\tau, T; H^s(\Omega)), \tag{49}$$

$$\partial_t^r \varphi^\delta \to 0 \quad \text{in} \quad L^2(\tau, T; L^2(\Omega)) \quad \text{if } s = 2r, \tag{50}$$

$$\partial_t^{r+1} \varphi^\delta \to 0 \quad \text{in} \quad L^2(\tau, T; H^{-1}(\Omega)) \quad \text{if } s = 2r+1: \tag{51}$$

this is done recalling estimates (10), (11) and (12), with $m = s - 1$, and that, for $0 \leq j \leq \left[\frac{s-1}{2}\right]$, $||\lambda_j^\delta||_{s-1-2j} \to 0$ as $\delta \to 0$, because of the assumed PCC of order s. Finally, (44) follows from the strong well-posedness of parabolic initial-boundary value problems in the spaces $H_c^s(Q)$, which can be proven as in [9] (Theorem 3). Theorem 4.4 is therefore completely proven. □

To conclude this section, it is sufficient to observe that repeated application of Theorem 4.4 allows us to construct approximating data satisfying PCC of arbitrary order.

5. Perturbation of Compatibility Conditions

5.1. In this last section we come to the results that will allow us to show, as mentioned in the Introduction, the equivalency between the global solvability of the initial-boundary value problems for the quasilinear parabolic equation (42) and for its hyperbolic perturbation (22), with coefficients b_{ij} not depending on u_t, and ε small. Namely, given a set of data, i.e. either the "hyperbolic" data $\{f, u_0, u_1\}$ or the "parabolic" ones $\{g, v_0\}$, satisfying the corresponding compatibility conditions, we show how to construct a set of data of the other type (i.e. respectively parabolic or hyperbolic), which not only satisfy the corresponding compatibility conditions, but also approximate the given data, as $\varepsilon \to 0$, in a suitable sense that takes into account the initial layer at $t = 0$, due to the loss of the initial condition on u_t. Again, we will only consider the linear problems; however, it turns out that to carry out these constructions in a way that will be convenient in the quasilinear case, we need to consider more regular data satisfying higher order compatibility conditions: this is an additional application of the results of Theorems 3.4 and 4.4. Still, to avoid unnecessary complications, we shall consider more regular coefficients for both problems (13) and (35), assuming that $a_{ij} \in C^\infty([0, T]; H^\infty(\Omega))$.

5.2. We first consider smooth hyperbolic data $\{f, u_0, u_1\}$ satisfying the HCC of higher order $s+1$; thus, without loss of generality (because of Theorem 3.4), we assume that $f \in C^\infty([0,T]; H^\infty(\Omega))$ and that, if the sequence $\{u_k\}$ is defined recursively by (16),

$$u_j \in H^\infty(\Omega) \cap H_0^1(\Omega) \qquad \text{for } \ 0 \le j \le s+2. \tag{52}$$

We claim:

THEOREM 5.1. *There exist functions $g \in C^\infty([0,T]; H^\infty(\Omega))$ and $v_0 \in H^\infty(\Omega) \cap H_0^1(\Omega)$ such that $\{g, v_0\}$ satisfy the PCC of order $2s+2$, and for all $\tau \in (0,T)$ there exist $M > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$,*

$$||f - g||_{X_{s-1}(\tau,T)} + ||\partial_t^s (f-g)||_{L^2(Q_\tau)} \le M\,\varepsilon. \tag{53}$$

Proof. By Proposition 2.2, we can choose $\phi \in C^\infty(\mathbb{R}; H^\infty(\Omega))$ such that

$$(\partial_t^j \phi)(\cdot, 0) = \varepsilon^{j+1} u_{j+2} \qquad \text{for } \ 0 \le j \le s; \tag{54}$$

we set then $v_0 = u_0$ and

$$g(x,t) = f(x,t) - \phi\left(x, \frac{t}{\varepsilon}\right), \tag{55}$$

and define the functions $\{v_0, \dots, v_{s+1}\}$ recursively by (39), i.e.

$$v_{j+1} = (\partial_t^j g)(\cdot, 0) + A_j[v_0, \dots, v_j], \qquad 0 \le j \le s. \tag{56}$$

We shall show that

$$v_j = u_j \qquad \text{for } \ 0 \le j \le s+1, \tag{57}$$

and therefore, since each u_j vanishes on $\partial\Omega$, the data $\{g, v_0\}$ satisfy the PCC of order $s+2$ (which, recalling the remark after Definition 4.3, calls exactly for $\left[\frac{(2s+2)+2}{2}\right] = s+2$ conditions, as in (57)). We prove (57) as usual, by induction on j: for $j = 0$, (57) is true by construction; then, recalling (56), (16), (55) and (54), we have

$$\begin{aligned} v_{j+1} &= (\partial_t^j f)(\cdot, 0) - \frac{1}{\varepsilon^j} (\partial_t^j \phi)(\cdot, 0) + A_j[u_0, \dots, u_j] = \\ &= \varepsilon u_{j+2} + u_{j+1} - \varepsilon u_{j+2} = u_{j+1}. \end{aligned}$$

Finally, estimate (53) is proven as estimates (4) and (5) of Proposition 2.2; indeed, we shall prove that

$$\sup_{\tau\le t\le T} ||(\partial_t^j \phi)(\cdot, t/\varepsilon)||_{s-1-j} \le C\,\varepsilon^{2-s}\, e^{-\tau/\varepsilon}, \quad 0 \le j \le s-1, \tag{58}$$

$$\int_\tau^T ||(\partial_t^s \phi)(\cdot, t/\varepsilon)||^2 dt \le C\,\varepsilon^{3-2s}\, e^{-2\tau/\varepsilon}, \tag{59}$$

with C independent of ε; hence, to prove (53) it is sufficient to take ε so small that, for instance, $\varepsilon^{-s}\, e^{-\tau/\varepsilon} \le 1$. To prove (58), recalling (54) and the proof of Proposition 2.2 it is sufficient to estimate

$$\begin{aligned}
&\sup_{\tau\le t\le T} ||(\partial_t^j \psi)(\cdot, t/\varepsilon)||^2_{s-1-j} \\
&\qquad \le C \sup_{\tau\le t\le T} \sum_{k,l=0}^{s-1} \int_{\mathbb{R}^n} (1+|\xi|^2)^{s-1-k} \varepsilon^{-2j+2k+2} |\hat{u}_{k+2}(\xi)|^2 \cdot \\
&\qquad\quad \cdot \exp\{-2(l+1)(1+|\xi|^2)^{1/2} t/\varepsilon\} d\xi \\
&\qquad \le C\, e^{-2\tau/\varepsilon} \sum_{k=0}^{s-1} \varepsilon^{2+2k-2j} ||u_{k+2}||^2_{s-1-k},
\end{aligned}$$

whence (58) follows, recalling that $j \le s-1$. Similarly, to show (59) we estimate

$$\begin{aligned}
I &\doteq \sum_{k=0}^{s-1} \int_\tau^T \int_{\mathbb{R}^n} (1+|\xi|^2)^{s-k} \varepsilon^{-2s+2k+2} |\hat{u}_{k+2}(\xi)|^2 \mathrm{e}^{-2(1+|\xi|^2)^{1/2} t/\varepsilon} d\xi\, dt \\
&\le C \sum_{k=0}^{s-1} \int_{\mathbb{R}^n} (1+|\xi|^2)^{s-k-1} |\hat{u}_{k+2}(\xi)|^2 \varepsilon^{-2s+2k+3} (1+|\xi|^2)^{1/2} \cdot \\
&\qquad \cdot \mathrm{e}^{-2(1+|\xi|^2)^{1/2}\tau/\varepsilon} d\xi \\
&\le C \sup_{\xi\in\mathbb{R}^n} \left\{ (1+|\xi|^2)^{1/2} \mathrm{e}^{-2(1+|\xi|^2)^{1/2}\tau/\varepsilon} \right\} \sum_{k=0}^{s-1} \varepsilon^{-2s+2k+3} ||u_{k+2}||^2_{s-k-1}.
\end{aligned}$$

Setting $r = (1+|\xi|^2)^{1/2} \ge 1$ and $\alpha(r) = r\, e^{-2r\tau/\varepsilon}$, we see that α has a maximum at $r_0 = \frac{\varepsilon}{2\tau}$; thus, if ε is sufficiently small, we can have $r_0 < 1$ and therefore $\alpha(r) \le \alpha(1) = e^{-2\tau/\varepsilon}$. Consequently, we can proceed with

$$I \le C\, e^{-2\tau/\varepsilon} \sum_{k=0}^{s-1} \varepsilon^{-2s+2k+3} ||u_k||_{s-k-1} \le C_1\, \varepsilon^{3-2s} e^{-2\tau/\varepsilon},$$

from which (59) follows, ending the proof of Proposition 5.1. □

We remark that if the smooth data $\{f, u_0, u_1\}$ are in fact from a sequence $\{f^\delta, u_0^\delta, u_1^\delta\}$ as in Theorem 3.4, the proof of Proposition 5.1 shows that the constant M in (53) can be chosen independent of δ. Moreover, the writing of (55) as $f(x,t) = g(x,t) + \phi\left(x, \frac{t}{\varepsilon}\right)$ exhibits the well known and expected phenomenon of the splitting of the time variable in the hyperbolic problem (13) into a slow scale variable t and a fast scale variable $\frac{t}{\varepsilon}$. This is due to the loss, in the singular limit problem (35), of the initial condition on u_t, which gives rise to an initial layer at $t = 0$: this is indeed described by estimates (58) and (59). (For more details on this aspect of the singular perturbation, se e.g. Chang-Howles, [1]).

5.3. We now consider smooth parabolic data $\{g, v_0\}$ satisfying the PCC of higher order $2s+2$; thus, without loss of generality (because of Theorem 4.4), we assume that $g \in C^\infty([0,T]; H^\infty(\Omega))$ and that, if the sequence $\{v_k\}$ is defined recursively by (39),

$$v_j \in H^\infty(\Omega) \cap H_0^1(\Omega) \qquad \text{for } 0 \le j \le s+1. \tag{60}$$

We claim:

THEOREM 5.2. *There exist functions $f \in C^\infty([0,T]; H^\infty(\Omega))$ and u_0, $u_1 \in H^\infty(\Omega) \cap H_0^1(\Omega)$ such that $\{f, u_0, u_1\}$ satisfy the HCC of order $s+1$, and*

$$||f - g||_{X_{s-1}(0,T)} + ||\partial_t^s (f-g)||_{L^2(Q_\tau)} \le M\,\varepsilon, \tag{61}$$

with $M > 0$ independent of ε and $\tau \in (0,T)$.

Proof. Again by Proposition 2.2, we can choose $\phi \in C^\infty(\mathbb{R}; H^\infty(\Omega))$ such that

$$(\partial_t^j \phi)(\cdot, 0) = \varepsilon v_{j+2} \qquad \text{for } 0 \le j \le s-1; \tag{62}$$

we set then $f = g + \phi$, $u_0 = v_0$, $u_1 = v_1$, and define the functions $\{u_2, \ldots, u_{s+2}\}$ recursively by (16), i.e.

$$\varepsilon u_{j+2} = (\partial_t^j f)(\cdot, 0) - u_{j+1} + A_j[u_0, \ldots, u_j], \qquad 0 \le j \le s-1. \tag{63}$$

It is then easy to show, again by induction, that

$$u_j = v_j \qquad \text{for } \ 0 \le j \le s+1, \tag{64}$$

and, therefore, that the data $\{f, u_0, u_1\}$ satisfy the HCC of order $s+1$. Finally, estimate (61) follows from estimates (4) and (5), according to which, by (62), we have

$$\begin{aligned}\sup_{0\le t\le T} \|(\partial_t^j \phi)(\cdot,t)\|^2_{s-1-j} &+ \int_\tau^T \|(\partial_t^s \phi)(\cdot,t)\|^2 dt \\ &\le C\left(1+\frac{1}{\tau}\right)^2 \sum_{k=0}^{s-1} \|\varepsilon v_{k+2}\|^2_{s-1-k} \\ &= \varepsilon^2\, C\left(1+\frac{1}{\tau}\right)^2 \sum_{k=2}^{s+1} \|v_k\|^2_{s+1-k}.\end{aligned} \tag{65}$$

□

We remark that, contrary to the previous case, if the smoother data $\{g, v_0\}$ come in fact from a sequence $\{g^\delta, v_0^\delta\}$ as in Theorem 4.4, estimate (65) shows that we can no longer guarantee that the constant M in (61) be bounded independently of δ as $\delta \to 0$; in fact, in general we have that $M = M(\delta)$, with $M(\delta) \to +\infty$ as $\delta \to 0$.

5.4. As a last application, we confirm that the choice of parabolic data $\{g, v_0\}$ provided by Theorem 5.1 is exactly the one that would allow us to try and extend the "only if" part of our equivalency result of [10] to the initial-boundary value problem for the quasi-linear equations (1) and (2) with homogeneous Dirichlet boundary conditions on $\partial\Omega$ (of course, Theorem 5.2 would take similar care of the "if" part). Thus, we assume that the initial-boundary value parabolic problem (2), with the boundary condition $v_{|\partial\Omega} = 0$, can be solved in $H_*^{s+2}(Q_T)$, $s \ge \left[\frac{n}{2}\right] + 2$, for arbitrary $T > 0$ and data $\{g, v_0\}$ satisfying the PCC of order s and, given arbitrary hyperbolic data $\{f, u_0, u_1\}$ satisfying the HCC of order s, we wish to solve the corresponding problem (1) in $X_{s+1}(0,T)$. By Theorem 3.4, we can assume $\{f, u_0, u_1\}$ to be smooth, and to satisfy the HCC of higher order $s+1$; by Theorem 5.1, we select then suitable parabolic data

$\{g, v_0\}$ satisfying the PCC of order $2s+2$. If $v \in H_*^{2s+4}(Q)$ is the corresponding global solution of the parabolic problem, we let $y = u-v$: y would then solve the hyperbolic problem

$$\begin{cases} \varepsilon\, y_{tt} + y_t - \sum_{i,j=1}^n a_{ij}(\nabla v + \nabla y)\partial_i\partial_j y = \Phi \\ \qquad \doteq f - g + \sum_{i,j=1}^n [a_{ij}(\nabla v + \nabla y) - a_{ij}(\nabla v)]\partial_i\partial_j v - \varepsilon\, v_{tt}, \\ y(x,0) = 0, \qquad y_t(0) = 0, \\ y(\cdot,t)_{|\partial\Omega} = 0, \end{cases} \tag{66}$$

where the stated initial conditions are taken according to (57). Since the initial-boundary value hyperbolic problem is solvable at least locally, i.e. in $X_{s+1}(0,\tau)$ for some $\tau \in (0,T]$, so is (66) and, therefore, its data $\{\Phi, 0, 0\}$ must satisfy the HCC of order s; indeed, given (57), we expect that, in fact, $\{\Phi, 0, 0\}$ satisfy the HCC of higher order $s+1$, because

$$y_k = (\partial_t^k(u-v))(\cdot,0) = u_k - v_k = 0 \tag{67}$$

in Ω for $0 \le k \le s+1$. We want now to show that, naturally, conditions (67) hold independently of the actual existence of even a local solution to (66), that is, when the functions $\{y_k\}$ are defined recursively as in (16). Indeed, since obviously

$$\begin{aligned} &\partial_t^k\left([a_{ij}(\nabla v + \nabla y) - a_{ij}(\nabla v)]\partial_i\partial_j v + a_{ij}(\nabla v)\partial_i\partial_j v\right) \\ &\qquad = \partial_t^k\left(a_{ij}(\nabla v + \nabla y)\partial_i\partial_j(v+y) - a_{ij}(\nabla v)\partial_i\partial_j v\right), \end{aligned}$$

in accord with (16) and (66) we define the functions $\{y_k\}$ recursively, setting $y_0 = 0$, $y_1 = 0$ (in accord with (57)), and, for $k \ge 0$,

$$\begin{cases} \varepsilon\, y_{k+2} \doteq (\partial_t^k f)(\cdot,0) - (\partial_t^k g)(\cdot,0) - y_{k+1} - \varepsilon v_{k+2} + \\ \qquad\qquad + A_k[v_0 + y_0, \ldots, v_k + y_k] - A_k[v_0, \ldots, v_k] : \end{cases} \tag{68}$$

it is then immediate to check, by induction on k, that

$$y_k = u_k - v_k \qquad \text{for } 2 \le k \le s+1 \tag{69}$$

and, therefore, $y_k = 0$ by (57), showing the asserted consistency with (67). Indeed, assuming that (69) holds for $2 \le k \le r+1 \le s$, so that

$y_k = u_k - v_k = 0$, recalling (55) we have from (68) that

$$\begin{aligned} \varepsilon\, y_{k+2} &= \frac{1}{\varepsilon^r}(\partial_t^r \phi)(\cdot, 0) - 0 - \varepsilon v_{tt} + \\ &\qquad + A_r[v_0 + 0, \dots, v_k + 0] - A_r[v_0, \dots, v_r] \\ &= \varepsilon u_{r+2} - \varepsilon v_{tt}, \end{aligned}$$

recalling (54).

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