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ADDENDUM to the paper The Bounded-Open Topology and its Relatives

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SUMMARY. - *Addendum to the paper The Bounded-Open Topology and its Relatives*, Rendiconti dell'Istituto di Matematica della Università di Trieste **27** (1995), 61–77.

In this short note, we would like to provide detailed clarification of a few claims and rectification of two minor errors made in the aforesaid paper cited as [1]. Throughout this note X stands for a Tychonoff space and $C(X)$ is the set of all real-valued continuous functions on X while $C^*(X) = \{f \in C(X) : f \text{ is bounded }\}$. A subset $A \subset X$ is called bounded if $f(A)$ is a bounded subset of $\mathbb{R}$ (the real line with the usual topology) for each $f \in C(X)$. Let $\mathcal{G}$ be a collection of some bounded subsets of X satisfying the condition: if $A, B \in \mathcal{G}$, there exists a set $C \in \mathcal{G}$ such that $A \cup B \subseteq C$ holds. To define the $\mathcal{G}$-open topology on $C(X)$, we take the subbasic open sets of the form

$$[A, V] = \{f \in C(X) : \overline{f(A)} \subseteq V\} \text{ where } A \in \mathcal{G} \text{ and } V \text{ is open in } \mathbb{R}.$$

We denote the space $C(X)$ with $\mathcal{G}$-open topology by $C_{\mathcal{G}}(X)$. At the end of the first paragraph of [1, p. 64], it has been noted that for the point-open and compact-open topologies, V can always be taken

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as a bounded open interval. In general for $\mathcal{G}$-open topologies this property can actually be assumed if we put a mild restriction on $\mathcal{G}$. First we need the following result.

LEMMA 1.1. *Let $f \in [A, V]$ where $A \in \mathcal{G}$ and V is open in $\mathbb{R}$. Then there exist bounded subsets A_i of X and bounded open intervals W_i in $\mathbb{R}$ $(1 \leq i \leq n)$ such that $f \in \bigcap_{i=1}^{n} [A_i, W_i] \subseteq [A, V]$.*

Proof. Let $z \in \overline{f(A)}$. There exists $\epsilon_z > 0$ such that $z \in (z - \epsilon_z, z + \epsilon_z) \subseteq [z - 2\epsilon_z, z + 2\epsilon_z] \subseteq V$. Since $\overline{f(A)}$ is compact, there exist $i = 1, 2, \ldots, n$ such that $\overline{f(A)} \subseteq \bigcup_{i=1}^{n} (z_i - \epsilon_{z_i}, z_i + \epsilon_{z_i}) \subseteq \bigcup_{i=1}^{n} [z_i - 2\epsilon_{z_i}, z_i + 2\epsilon_{z_i}] \subseteq V$.

Let $V_i = (z_i - \epsilon_{z_i}, z_i + \epsilon_{z_i})$, $W_i = (z_i - 2\epsilon_{z_i}, z_i + 2\epsilon_{z_i})$ and $A_i = \mathrm{cl}_A(A \cap f^{-1}(V_i))$ for $i = 1, 2, \ldots, n$. It is routine to check that $A = \bigcup_{i=1}^{n} A_i$, and $f \in \bigcap_{i=1}^{n} [A_i, W_i] \subset [A, V]$. □

A family $\mathcal{G}$ of bounded subsets of X is said to be hereditary with respect to closed domains if it satisfies the following condition: whenever $A \in \mathcal{G}$ and B is a closed subdomain of A, then $B \in \mathcal{G}$ as well.

COROLLARY 1.2. *Suppose $\mathcal{G}$ is a family of bounded subsets of X hereditary with respect to closed domains.*
Then the collection $\{[A, V] : A \in \mathcal{G},\ V$ a bounded open interval in $\mathbb{R}\}$ forms a subbase for $C_{\mathcal{G}}(X)$.

Now we are going to give a clarification on Theorem 2.5 in [1]. The proof of this result (with a correction yet to be made) given in [1] can only work if we put a restriction on $\mathcal{G}$.

THEOREM 1.3. *Suppose $\mathcal{G}$ is a family of bounded subsets of X hereditary with respect to closed domains. Then $C^*(X)$ is dense in $C_{\mathcal{G}}(X)$.*

Proof. Let $\bigcap_{i=1}^{n} [A_i, V_i]$ be a basic open set in $C_{\mathcal{G}}(X)$ containing f where V_i's are bounded open intervals in $\mathbb{R}$ and $A_i \in \mathcal{G}$. Then there exists a bounded open interval (a, b) such that $\bigcup_{i=1}^{n} \overline{f(A_i)} \subseteq \bigcup_{i=1}^{n} V_i \subseteq (a, b)$. The rest of the proof is as mentioned in the proof of Theorem 2.5 in [1]. □

We really do not need $\mathcal{G}$ to be hereditary with respect to the closed domain in order that $C^*(X)$ to be dense in $C_{\mathcal{G}}(X)$, but we need to change the proof.

THEOREM 1.4. *For any space X, $C^*(X)$ is dense in $C_{\mathcal{G},u}(X)$. (Note $C_{\mathcal{G},u}(X)$ is the space $C(X)$ equipped with the topology of uniform convergence on $\mathcal{G}$).*

Proof. We show that $< f, A, \epsilon > \cap C^*(X) \neq \emptyset$ for all $f \in C(X)$, for all $A \in \mathcal{G}$ and for all $\epsilon > 0$. Let $f \in C(X)$. Then f has a continuous extension f^ν from νX (Hewitt-realcompactification of X) into $\mathbb{R}$. Since A is bounded, $\mathrm{cl}_{\beta X} A \subseteq \nu X$ (βX is the Stone-Čech compactification of X). Let $A_1 = \mathrm{cl}_{\beta X} A$ and $f_1 = f^\nu|_{A_1} =$ the restriction of f^ν to A_1. Note $f_1(A_1)$ is compact in $\mathbb{R}$ and hence $f_1(A_1) \subseteq [a, b]$ for some closed bounded interval $[a, b]$ in $\mathbb{R}$. Now there exists a continuous function $F : \beta X \to [a, b]$ such that $F|_{A_1} = f_1$. Let $g = F|_X$. It is easy to check that $g \in C^*(X) \cap < f, A, \epsilon >$. □

Since $C_{\mathcal{G}}(X) \leq C_{\mathcal{G},u}(X)$ (see [1, Theorem 3.1]), we have the following result.

COROLLARY 1.5. *For any space X, $C^*(X)$ is dense in $C_{\mathcal{G}}(X)$.*

Now we would like to clarify the status of Proposition 2.1 in [1]. This result should be modified to the following version.

PROPOSITION 1.6. *Suppose $\mathcal{G}$ is a family of bounded subsets of X hereditary with respect to closed domains. Then $C_{\mathcal{G}}(X)$ is completely regular. If in addition $\mathcal{G}$ is a network, then $C_{\mathcal{G}}(X)$ is also Hausdorff.*

Proof. The fact that $C_{\mathcal{G}}(X)$ is completely regular can be proved in a manner similar to the proof in [2, Lemma 5.1]. □

Finally we would like to make a remark on the Examples 3.14, 3.15, 3.17 of [1]. The proof of the result that $X = \beta\mathbb{N} - \{p\}$ (where p belongs to $\mathbb{N}^* = \beta\mathbb{N} - \mathbb{N}$) is not normal assumes the Continuum Hypothesis (CH). See [3] for the details. But Jack Porter has recently communicated to the authors that Sapirovskii (around 1988 but reference unknown) proved the following without Continuum Hypothesis: there exists a point p in $\beta\mathbb{N}$ such that $\beta\mathbb{N} - \{p\}$ is not normal. Consequently, it should be emphasized that in these examples the CH can be avoided.

References

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