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Semicontinuity of vectorial functionals in Orlicz-Sobolev spaces

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Summary. - *We study integral vectorial functionals*

$$F(u,\Omega) - \int_{\Omega} f(x,u(x),Du(x))dx$$

*where f satisfies quasi-convexity assumption and its growth is controlled in term of N-functions. We obtain semicontinuity results in the weak * topology of Orlicz-Sobolev spaces.*

1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^n$ with $\partial\Omega$ lipschitzian, consider a function $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn} \to \mathbb{R}$ and the variational integral

$$F(u,\Omega) = \int_{\Omega} f(x,u(x),Du(x))\,dx \tag{1}$$

where $u : \Omega \to \mathbb{R}^N$.

Assume that $f = f(x,s,z)$ is a *Carathéodory* function, i.e. f is measurable in x for every $(s,z) \in \mathbb{R}^N \times \mathbb{R}^{Nn}$ and continuous in (s,z) for almost every $x \in \Omega$, and that it is also *quasi-convex* in z, i.e. for every $(x_0,s_0,z_0) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn}$ and $\varphi \in C_0^1(\Omega,\mathbb{R}^N)$

$$f(x_0,s_0,z_0)|\Omega| \le \int_{\Omega} f(x_0,s_0,z_0 + D\varphi(x))\,dx.$$

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Moreover, suppose that f satisfies the growth condition

$$-c_1\{1+\Phi_1(|s|)+\Phi_1(|z|)\} \leq f(x,s,z) \leq c_2\{1+\Phi_2(|s|)+\Phi_2(|z|)\} \quad (2)$$

for every $(x,s,z) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn}$, where c_1, c_2 are non negative constants and Φ_i, $i = 1,2$, is a *N-function*, i.e. a positive continuous convex function such that $\Phi_i(0) = 0$, $\lim\limits_{t\to 0}\Phi_i(t)/t = 0$ and $\lim\limits_{t\to+\infty} \Phi_i(t)/t = +\infty$ (see Section 2 for definitions).

If in (2) $\Phi_i(t) = t^p$, $i = 1,2$, $p > 1$, Fusco [15] proved the weak semicontinuity of (1) in the ordinary Sobolev space $W^{1,p+\varepsilon}(\Omega,\mathbb{R}^N)$, $\varepsilon > 0$. This result was improved by Acerbi and Fusco in [2] showing weak semicontinuity of (1) in $W^{1,p}(\Omega,\mathbb{R}^N)$ if f is non negative and $\Phi_2(t) = t^p$, and by Marcellini in [20] under less restrictive growth conditions. If $\Phi_2(t) = t^p$ and if f satisfies some additional structure conditions, the weak semicontinuity of (1) was proved by Marcellini [21] in $W^{1,q}(\Omega,\mathbb{R}^N)$ with $q > \frac{n}{n+1}p$, by Fonseca and Marcellini [14] for $q > p-1$ and by Malý [19] for $q \geq p-1$. Recently, Fonseca and Malý [13] and Malý [18] proved the lower semicontinuity of (1) for $q > \frac{n-1}{n}p$. Finally, if (1) is poli-convex and $n = N$, Dacorogna and Marcellini [6] proved a semicontinuity result for $q > n-1$, while the borderline case $q = n-1$ was established by Acerbi and Dal Maso [1] and by Dal Maso and Sbordone [8]. An elementary approach was found by Fusco and Hutchinson [16].

In this paper, we obtain, for quasi-convex integrals satisfying the non-standard growth condition (2), some semicontinuity results in the weak* topology of Orlicz-Sobolev space $W^{1,\Phi_2}(\Omega,\mathbb{R}^N)$ (see Section 2 for definitions).

In Section 2 we introduce the definitions and some properties of N-functions, Orlicz and Orlicz-Sobolev spaces.

In Section 3, Theorem 3.1, we show that if $f = f(z)$, Φ_2 belongs to class Δ_2 (see Section 2 for definitions) and Φ_1 is suitably related to it, then (1) is sequentially lower semicontinuous in the weak* topology of the Orlicz-Sobolev space $W^{1,\Phi_2}(\Omega,\mathbb{R}^N)$. The proof generalizes the technique developed by Marcellini in [20]. Moreover, in this case, we prove an existence theorem.

In Section 4, Theorem 4.1, we consider functionals depending on $f = f(x,s,z)$ and satisfying (2) with $\Phi_1 = \Phi_2 = \Phi$. We succeed in proving a semicontinuity result in $W^{1,\Gamma}(\Omega,\mathbb{R}^N)$ with Γ a suitable N-function related to Φ following Marcellini and Sbordone [22].

Finally in Section 5, we exhibit some examples of non trivial applications of the semicontinuity Theorems 3.1, 4.1 and of the existence Theorem 3.3.

We observe that Ball in [5] considered some variational problems in the framework of Orlicz-Sobolev spaces obtaining some semicontinuity and existence results for poli-convex integrals.

2. N-Functions and Orlicz Spaces

In this section we recall some definitions and well known properties on N-functions and Orlicz spaces (see for references [3], [17], [25]). A continuous and convex function $\Phi : [0,+\infty) \to [0,+\infty)$ is called *N-function* if it satisfies

$$\Phi(0) = 0, \quad \Phi(t) > 0, \quad t > 0,$$

$$\lim_{t\to 0} \Phi(t)/t = 0, \quad \lim_{t\to+\infty} \Phi(t)/t = +\infty.$$

A N-function Φ has an integral representation

$$\Phi(t) = \int_0^t p(s)\, ds \qquad t \in [0,+\infty),$$

where $p : [0,+\infty) \to [0,+\infty)$ is nondecreasing, right continuous and it satisfies

$$p(0) = 0, \quad p(s) > 0 \quad s > 0, \quad \lim_{s\to+\infty} p(s) = +\infty.$$

The function p is the *right derivative* of Φ.

What is important in the definition of a N-function is the behaviour at infinity, in fact, a continuous convex function $Q : [0,+\infty[\to [0,+\infty[$ satisfying

$$Q(t)/t \to +\infty \qquad t \to +\infty$$

is such that there exist a N-function Φ and $t_0 > 0$ such that for every $t \geq t_0$ we get

$$Q(t) = \Phi(t).$$

Such a function Q is called *principal part* of the N-function Φ.

Let Φ be a N-function, for $t \geq 0$ consider the function

$$\Psi(t) = \max_{s>0}\{st - \Phi(s)\},$$

it is easy to show that Ψ is a N-function, Ψ is called the *complementary N-function* of Φ. By the very definition of Ψ it is obvious that the pair Φ, Ψ satisfies *Young's inequality*:

$$st \leq \Phi(s) + \Psi(t) \qquad s, t \in \mathbb{R},$$

with equality holding if $s = p(t)$ or $t = q(s)$, where q is the right derivative of Ψ.

In the sequel we will deal with a particular class of N-functions. We say that a N-function Φ belongs to the *class* Δ_2, denoted by $\Phi \in \Delta_2$, if there exist $k > 1$ and $t_0 \geq 0$ such that

$$tp(t) \leq k\Phi(t) \qquad t \geq t_0. \tag{3}$$

It is not difficult to check that definition (3) is equivalent to the classical one, i.e. $\Phi \in \Delta_2$ if and only if there exist $k > 1$ and $t_0 \geq 0$ such that for every $t \geq t_0$

$$\Phi(2t) \leq 2^k \Phi(t).$$

For related properties of N-functions of class Δ_2 see [7].

Let Ω be an open bounded subset of $\mathbb{R}^n$, the *Orlicz class* $K^\Phi(\Omega, \mathbb{R}^N)$ is the set of all (equivalence classes modulo equality a.e. in Ω of) measurable functions $u : \Omega \to \mathbb{R}^N$ satisfying

$$\int_\Omega \Phi(|u(x)|)\, dx < +\infty.$$

The *Orlicz space* $L^\Phi(\Omega, \mathbb{R}^N)$ associated with the N-function Φ and the open set Ω, is the linear hull of $K^\Phi(\Omega, \mathbb{R}^N)$. The equality $K^\Phi(\Omega, \mathbb{R}^N) \equiv L^\Phi(\Omega, \mathbb{R}^N)$ holds if and only if $\Phi \in \Delta_2$.

The functional $\| u \|_{\Phi,\Omega} : L^\Phi(\Omega, \mathbb{R}^N) \to \mathbb{R}$, simply denoted by $\| u \|_\Phi$, defined by

$$\| u \|_{\Phi,\Omega} = \inf\left\{\lambda > 0 : \int_\Omega \Phi\left(\frac{|u(x)|}{\lambda}\right) dx \leq 1\right\}, \tag{4}$$

is a norm and $L^\Phi(\Omega,\mathbb{R}^N)$ is a Banach space with respect to it.

In the sequel we will denote with $s - L^\Phi(\Omega,\mathbb{R}^N)$ the norm convergence in $L^\Phi(\Omega,\mathbb{R}^N)$.

Many relevant properties of the Orlicz spaces are related to class Δ_2, for instance:

PROPOSITION 2.1. *If $\Phi \in \Delta_2$ and $\{u_a\}_{a\in I} \subset L^\Phi(\Omega,\mathbb{R}^N)$ then*

$$\sup_I \| u_a \|_{\Phi,\Omega} < +\infty \text{ if and only if } \sup_I \int_\Omega \Phi(|u_a(x)|)\,dx < +\infty.$$

The closure in the norm topology of $C_0^\infty(\Omega,\mathbb{R}^N)$ in $L^\Phi(\Omega,\mathbb{R}^N)$ is denoted by $E^\Phi(\Omega,\mathbb{R}^N)$. We have $E^\Phi(\Omega,\mathbb{R}^N) \subseteq K^\Phi(\Omega,\mathbb{R}^N) \subseteq L^\Phi(\Omega,\mathbb{R}^N)$, with equality holding if and only if $\Phi \in \Delta_2$.

Moreover the Orlicz space associated with a N-function Φ is separable if and only if the generating N-function belongs to class Δ_2. The separability result is a consequence of the following approximation theorem which generalizes an analogous property of L^p spaces (see [12]).

THEOREM 2.2. *Let Ω be a bounded open set in $\mathbb{R}^n$, and let $\Phi \in \Delta_2$. For a natural number r, let $\{Q_{i,r}\}$ be a family of open cubes satisfying*

$$\operatorname{diam} Q_{i,r} \le \frac{1}{r}; \quad Q_{i,r} \cap Q_{j,r} = \phi \quad i \ne j; \quad \bigcup_i \overline{Q}_{i,r} = \overline{\Omega}.$$

For $u \in L^\Phi(\Omega,\mathbb{R}^N)$ define the functions

$$u_r(x) = \sum_i \left\{ \frac{1}{|Q_{i,r}|} \int_{Q_{i,r}} u(y)\,dy \right\} \chi_{Q_{i,r}}(x).$$

Then $\{u_r\} \subset E^\Phi(\Omega,\mathbb{R}^N)$ and moreover $\{u_r\} \to u$ s-$L^\Phi(\Omega,\mathbb{R}^N)$.

In the sequel we will use the following result (see [17]).

PROPOSITION 2.3. *Let Φ, Γ be N-functions such that*

$$\lim_{t\to+\infty} \frac{\Phi(t)}{\Gamma(t)} = +\infty,$$

and let H *be a mean bounded family of functions in* $L^\Phi(\Omega,\mathbb{R}^N)$*, i.e.* $\sup_H \int_\Omega \Phi(|u(x)|)\,dx < +\infty$*, then the set of functions* $G = \{\Gamma(|u|) : u \in H\}$ *has equi-absolutely continuous integrals on* Ω.

The *Orlicz-Sobolev space* $W^{1,\Phi}(\Omega,\mathbb{R}^N)$ consists of all functions u in $L^\Phi(\Omega,\mathbb{R}^N)$ whose distributional derivatives belong to $L^\Phi(\Omega,\mathbb{R}^N)$. As in the case of ordinary Sobolev spaces, $W_0^{1,\Phi}(\Omega,\mathbb{R}^N)$ is taken to be the closure in the norm topology of $C_0^\infty(\Omega,\mathbb{R}^N)$ in $W^{1,\Phi}(\Omega,\mathbb{R}^N)$.

The following embedding theorem holds (see [3], [4], [10]).

Theorem 2.4. *Let* Ω *be an open subset of* $\mathbb{R}^n$ *with* $\partial\Omega$ *lipschitzian, let* $\Phi \in \Delta_2$*, then the imbedding*

$$W^{1,\Phi}(\Omega,\mathbb{R}^N) \to L^\Phi(\Omega,\mathbb{R}^N)$$

is compact.

We now introduce the weak * convergence in $L^\Phi(\Omega,\mathbb{R}^N)$. The space $L^\Phi(\Omega,\mathbb{R}^N)$ can be regarded as the dual space of $E^\Psi(\Omega,\mathbb{R}^N)$ (see [3], [17], [25]), so it is possible to characterize the convergence of sequences in the weak * topology of $L^\Phi(\Omega,\mathbb{R}^N)$ in the following way: $\{u_r\} \to u \;\; *w\text{-}L^\Phi(\Omega,\mathbb{R}^N)$ if and only if for every $\nu \in E^\Psi(\Omega,\mathbb{R}^N)$

$$\lim_r \int_\Omega u_r(x)v(x)\,dx = \int_\Omega u(x)v(x)\,dx.$$

Since this, weak * convergence is often called E^Ψ*-convergence.*

By means of the Hahn-Banach theorem we characterize the weak* convergence in the space $W^{1,\Phi}(\Omega,\mathbb{R}^N) : \{u_r\} \to u \; *w\text{-}W^{1,\Phi}(\Omega,\mathbb{R}^N)$ if and only if $\{u_r\}$ and $\{D_i u_r\}$, $1 \le i \le n$, converge to u, $D_i u \; *w\text{-}L^\Phi(\Omega,\mathbb{R}^N)$, respectively. Finally if $\Phi \in \Delta_2$ we get $\left[L^\Phi(\Omega,\mathbb{R}^N)\right]' \sim L^\Psi(\Omega,\mathbb{R}^N)$.

3. Semicontinuity theorem: the case $f = f(z)$

Theorem 3.1. *Let* Ω *be an open bounded subset of* $\mathbb{R}^n$ *with* $\partial\Omega$ *lipschitzian, let* $u : \Omega \to \mathbb{R}^N$*, consider the functional*

$$F(u,\Omega) = \int_\Omega f(Du(x))\,dx,$$

where $f : \mathbb{R}^{Nn} \to \mathbb{R}$ *is a quasi-convex function such that for every* $z \in \mathbb{R}^{Nn}$ *we have*

$$-c\{1 + \Phi_1(|z|)\} \leq f(z) \leq c\{1 + \Phi(|z|)\} \tag{5}$$

with c *positive constant,* $\Phi \in \Delta_2$ *and* Φ_1 N*-function such that*

$$\lim_{t \to +\infty} \frac{\Phi(t)}{\Phi_1(t)} = +\infty. \tag{6}$$

Then F *is sequentially lower semicontinuous in* $*w\text{-}W^{1,\Phi}(\Omega, \mathbb{R}^N)$, *i.e. for every sequence* $\{u_r\} \to u$ $*w\text{-}W^{1,\Phi}(\Omega, \mathbb{R}^N)$ *we have*

$$\liminf_r \int_\Omega f(Du_r)\, dx \geq \int_\Omega f(Du)\, dx.$$

In the sequel we will use the following result which generalizes a proposition given by Marcellini [20].

PROPOSITION 3.2. *Let* $g : \mathbb{R}^{Nn} \to \mathbb{R}$ *be a function separately convex in each variable, such that there exist a* N*-function* $\Gamma \in \Delta_2$ *and a positive constant* c *such that for every* $z \in \mathbb{R}^{Nn}$

$$|g(z)| \leq c\{1 + \Gamma(|z|)\}. \tag{7}$$

Then g *is continuous, besides, denoted by* h *the right derivative of* Γ*, we have*

$$|g(z) - q(w)| \leq c_1\{1 + h(1 + |z| + |w|)\}|z - w| \tag{8}$$

for every $z, w \in \mathbb{R}^{Nn}$ *with* c_1 *positive constant.*

Proof. For $z, w \in \mathbb{R}^{Nn}$ consider the vectors

$$a^k = (w_1, \ldots, w_k, z_{k+1}, \ldots, z_{Nn}) \quad 0 \leq k \leq Nn,$$

then using the convexity of g in each variable, we have for $t \geq 1$

$$g(a^{k+1}) - g(a^k) \leq \frac{g(a^k + t(a^{k+1} - a^k)) - g(a^k)}{t}.$$

By the very definition of a^k it follows that, for every k and t, we have

$$|a^k + t(a^{k+1} - a^k)| \leq 1 + |z| + |w| + t|z - w|,$$

so if we choose $\bar{t} = \dfrac{1 + |z| + |w|}{|z - w|} > 1$ we get

$$|a^k + \bar{t}(a^{k+1} - a^k)| \leq 2\{1 + |z| + |w|\}.$$

By (7) we have

$$|g(a^k)| \leq c\{1 + \Gamma(1 + |z| + |w|)\}.$$

and also, using assumption $\Gamma \in \Delta_2$ we get

$$|g(a^k + \bar{t}(a^{k+1} - a^k))| \leq c_1\{1 + \Gamma(1 + |z| + |w|)\}.$$

Thus we have

$$\begin{aligned} g(a^{k+1}) - g(a^k) &\leq c_1 \frac{1 + \Gamma(1 + |z| + |w|)}{1 + |z| + |w|}|z - w| \\ &\leq c_1\{1 + h(1 + |z| + |w|)\}|z - w|, \end{aligned}$$

adding up on k we get the inequality

$$g(w) - g(z) \leq c_2\{1 + h(1 + |z| + |w|)\}|z - w|,$$

reversing the role of z and w we get (8). □

Proof of Theorem 3.1. We assume first that $u \in W^{1,\Phi}(\Omega, \mathbb{R}^N)$ is an affine function, i.e. there exists $z_0 \in \mathbb{R}^{Nn}$ such that for every $x \in \mathbb{R}^n$ it holds $Du(x) \equiv z_0$.
Denote with $\{u_r\}$ a sequence such that $\{u_r\} \to u$ $*w$-$W^{1,\Phi}(\Omega, \mathbb{R}^N)$. If u, u_r have the same boundary values, i.e. $(u_r - u) \in W_0^{1,\Phi}(\Omega, \mathbb{R}^N)$, for every r, the result follows easily by quasi-convexity. In fact, by (5), the functional F is continuous in s-$W^{1,\Phi}(\Omega, \mathbb{R}^N)$, then the quasi-convexity inequality holds for test functions in $W_0^{1,\Phi}(\Omega, \mathbb{R}^N)$ and so we get semicontinuity inequality.

In the general case we change the boundary data of u_r using a method developed by De Giorgi [9]. Let Ω_0 be an open set compactly contained in Ω and fix $k = \frac{1}{2}\text{dist}\,(\overline{\Omega}_0, \partial\Omega)$, for $h \in \mathbf{N}$ define the open sets

$$\Omega_i = \left\{x \in \Omega : \ \text{dist}\,(x, \Omega_0) < \frac{i}{h}k\right\} \qquad 1 \le i \le h$$

and consider a family of functions $\phi_i \in C_0'(\Omega_i)$ such that

$$0 \le \phi_i \le 1; \quad \phi_i \equiv 1 \quad \Omega_{i-1}; \quad \phi_i \equiv 0 \quad \Omega \setminus \Omega_i; \quad |D\phi_i| \le \frac{h+1}{k}.$$

For every r, let $\nu_r = u_r - u$, then $\{\nu_r\} \to 0$ $*w\text{-}W^{1,\Phi}(\Omega, \mathbb{R}^N)$, now define the functions

$$\nu_{i,r}(x) = \phi_i(x)\nu_r(x),$$

since $\nu_{i,r} \in W_0^{1,\Phi}(\Omega, \mathbb{R}^N)$ for every i and r we have

$$\begin{aligned} F(u,\Omega) \le F(u+\nu_{i,r},\Omega) &= \int_\Omega f(z_0 + Dv_{i,r})\,dx \\ &= \int_{\Omega_{i-1}} f(Du_r)\,dx + \int_{\Omega_i \setminus \Omega_{i-1}} f(z_0 + D\nu_{i,r})dx + \int_{\Omega\setminus\Omega_i} f(z_0)dx \\ &= \int_\Omega f(Du_r)dx - \int_{\Omega - \Omega_{i-1}} f(Du_r)dx + \int_{\Omega_i\setminus\Omega_{i-1}} f(z_0 + D\nu_{i,r})dx + \\ &\quad + |\Omega \setminus \Omega_0| f(z_0) - \int_{\Omega_i \setminus \Omega_0} f(Du)dx. \end{aligned} \tag{9}$$

Since $\{\nu_r\}$ is weakly $*$ convergent, then $\{D\nu_r\}$ is bounded in norm $L^\Phi(\Omega, \mathbb{R}^N)$, and then, by Proposition 2.1, there exists a positive constant c_1 such that

$$\sup_r \int_\Omega \Phi(|D\nu_r|)\,dx \le c_1.$$

Therefore there is $0 \le j \le h$ such that

$$\limsup_r \int_{\Omega_j \setminus \Omega_{j-1}} \Phi(|D\nu_r|)\,dx \le \frac{c_1}{h}.$$

Since the imbedding $W^{1,\Phi} \to L^{\Phi}$ is compact we obtain that $\{\nu_r\} \to 0$ s-$L^{\Phi}(\Omega, \mathbb{R}^N)$ and then

$$\lim_r \int_\Omega \Phi(|\nu_r|)\, dx = 0.$$

Now we estimate the integrals in (9), we have

$$\begin{aligned} \int_{\Omega_j \setminus \Omega_{j-1}} & f(z_0 + D\nu_{j,r}) dx \\ & \leq c \int_{\Omega_j \setminus \Omega_{j-1}} \{1 + \Phi(|z_0| + |\phi_j|\,|D\nu_r| + |D\phi_j|\,|\nu_r|)\} dx \qquad (10) \\ & \leq c_2\, |\Omega \setminus \Omega_0| + \frac{c_4}{h} + c_3 \left(\frac{h+1}{k}\right)^m \int_\Omega \Phi(|v_r|)\, dx, \end{aligned}$$

besides by (5) we get

$$- \int_{\Omega \setminus \Omega_{j-1}} f(Du_r) dx - \int_{\Omega_j \setminus \Omega_0} f(Du) dx \leq c_5 \int_{\Omega \setminus \Omega_0} \{1 + \Phi_1(|Du_r|) + \Phi_1(|Du|)\} dx. \qquad (11)$$

Using Proposition 2.3 we obtain that the functions $\Phi_1(|Du_r|)$ have equi-absolutely continuous integrals, so that the right term of (11) goes to zero if the measure of $\Omega \setminus \Omega_0$ does.

So, by (10) and (11), (9) becomes

$$\begin{aligned} F(u, \Omega) \leq F(u_r, \Omega) + c_3 \left(\frac{h+1}{k}\right)^m \int_\Omega \Phi(|\nu_r|)\, dx + \frac{c_4}{h} + \\ + c_5 \int_{\Omega \setminus \Omega_0} \{\Phi_1(|Du_r|) + \Phi_1(|Du|)\}\, dx + c_6 |\Omega \setminus \Omega_0|, \end{aligned}$$

the assertion follows passing to the limit as $|\Omega \setminus \Omega_0| \to 0$, $r \to +\infty$ and $h \to +\infty$.

Passing to the general case let $u \in W^{1,\Phi}(\Omega, \mathbb{R}^N)$ and $\{u_r\}$ be a sequence such that $\{u_r\} \to u$ $*w$-$W^{1,\Phi}(\Omega, \mathbb{R}^N)$. Consider a family of open cubes $\{Q_{i,m}\}$ as in Theorem 2.2, and define on every cube $Q_{i,m}$ the functions

$$\nu_{r,m} = u_r - u + < (Du)_{i,m}, x >$$

where

$$(Du)_{i,m} = \frac{1}{|Q_{i,m}|} \int_{Q_{i,m}} Du(y)\, dy.$$

Then

$$D\nu_{r,m} = Du_r - Du + (Du)_m$$

where

$$(Du)_m(x) = \sum_i (Du)_{i,m} \chi_{Q_{i,m}}(x).$$

Fix $0 < \varepsilon < 1$, we prove that for suitable m we have

$$|F(u_r, \Omega) - F(\nu_{r,m}, \Omega)| \leq \varepsilon.$$

Let p be the right derivative of Φ, by Proposition 3.2 and Young's inequality we get

$$\begin{aligned}
&|F(u_r, \Omega) - F(\nu_{r,m}, \Omega)| \\
&\quad \leq \sum_i \int_{Q_{i,m}} |f(Du_r) - f(D\nu_{r,m})|\, dx \\
&\quad \leq c_1 \sum_i \int_{Q_{i,m}} \{1 + p(1 + |Du_r| + |D\nu_{r,m}|)\} |Du_r - D\nu_{r,m}|\, dx \\
&\quad \leq c_\varepsilon \int_\Omega \Phi(|Du - (Du)_m|) dx + \\
&\qquad + c_1 \varepsilon \sum_i \int_{Q_{i,m}} \{\Psi(1) + \Psi(p(1 + |Du_r| + |D\nu_{r,m}|))\} dx \\
&\quad = I_1 + I_2.
\end{aligned}$$

For a suitable m, by Theorem 2.2, we obtain

$$I_1 \leq \varepsilon.$$

Since $\Phi \in \Delta_2$, there exist $k > 1$ and $t_0 \geq 0$ such that for every $t \geq t_0 \geq 0$: $\Psi(p(t)) \leq k\Phi(t)$, then

$$I_2 \leq c_2\varepsilon + c_3\varepsilon \int_\Omega \{\Phi(|Du_r|) + \Phi(|Du - (Du)_m|)\}\, dx,$$

therefore by Proposition 2.1 and Theorem 2.2 we have

$$|F(u_r, \Omega) - F(\nu_{r,m}, \Omega)| \leq c_5\varepsilon.$$

In a similar way we can show that

$$\left| F(u, \Omega) - \int_\Omega f((Du)_m)\, dx \right| \leq \varepsilon.$$

Fix $M \in \mathbf{N}$ and set $\Omega_M = \bigcup_{i=1}^{M} Q_{i,m}$, since $\{\nu_{r,m}\} \to < (Du)_{i,m}, x >$ $*w\text{-}W^{1,\Phi}(Q_{i,m}, \mathbb{R}^N)$ for every i by the first part of the proof we have

$$\int_{\Omega_M} f((Du)_m)\, dx \leq \liminf_r \int_{\Omega_M} f(D\nu_{r,m})\, dx.$$

Using (5) and the convexity of Φ, it is easy to prove that for suitable M it holds

$$\int_{\Omega \setminus \Omega_M} f((Du)_m)\, dx \leq \varepsilon.$$

Moreover, as the integrals of functions $\Phi_1(|Du_r|)$ are equi-absolutely continuous, we get

$$-\int_{\Omega \setminus \Omega_M} f(D\nu_{r,m})\, dx \leq \varepsilon.$$

We can conclude that

$$\begin{aligned} F(u, \Omega) &\leq \int_\Omega f((Du)_m)\, dx + \varepsilon \\ &\leq \int_{\Omega_M} f((Du)_m)\, dx + 2\varepsilon \leq \liminf_r \int_{\Omega_M} f(D\nu_{r,m})\, dx + 2\varepsilon \\ &\leq \liminf_r \int_\Omega f(D\nu_{r,m})\, dx + 3\varepsilon \leq \liminf_r \int_\Omega f(Du_r)\, dx + c_8\varepsilon. \end{aligned}$$

Finally the semicontinuity follows. □

By the previous semicontinuity result, we are able to state the following existence theorem in the context of Orlicz-Sobolev spaces, using the Direct Methods of the Calculus of Variations.

THEOREM 3.3. *Let $f : \mathbb{R}^{Nn} \to \mathbb{R}$ be a quasi-convex function satisfying*

$$c_1\{\Phi(|z|) - 1\} \le f(z) \le c_2\{\Phi(|z|) + 1\} \tag{12}$$

for every $z \in \mathbb{R}^{Nn}$, where c_1 and c_2 are positive constants and $\Phi \in \Delta_2$.

Let Ω be an open bounded subset of $\mathbb{R}^n$ with $\partial\Omega$ lipschitzian, let ν be a function in $W^{1,\Phi}(\Omega, \mathbb{R}^N)$, consider the Dirichlet's class

$$V = \nu + W_0^{1,\Phi}(\Omega, \mathbb{R}^N),$$

then the problem $m = \inf_V F(u, \Omega)$ has solution.

Proof. Functional F is lower bounded and coercive in the strong topology of V.

In fact for every $w \in L^\Phi(\Omega, \mathbb{R}^N)$ it holds

$$\| w \|_\Phi \le 1 + \int_\Omega \Phi(|w|)\, dx,$$

then by (12) we have

$$F(u, \Omega) \ge c_3\{\| Du \|_\Phi - 1\},$$

so F is lower bounded on V, i.e. $m > -\infty$.

On the other hand from

$$\| Du \|_\Phi \ge \| D(u - \nu) \|_\Phi - \| D\nu \|_\Phi$$

it follows

$$F(u, \Omega) \ge c_4\{\| D(u - \nu) \|_\Phi - 1\},$$

and, as in $W_0^{1,\Phi}(\Omega, \mathbb{R}^N)$ the norm of the gradient and the usual one are equivalent, F is coercive with respect to the strong topology of V.

Let $\{u_r\}$ be a minimizing sequence of F on V, i.e. $\lim_r F(u_r,\Omega) = m$, then, by coercitivity of F, $\{u_r\}$ is bounded in norm. Thus, there exist a subsequence of $\{u_r\}$, which we still denote by $\{u_r\}$, and a function $u \in \nu + W_0^{1,\Phi}(\Omega,\mathbb{R}^N)$ such that $\{u_r\} \to u$ $*w$-$W^{1,\Phi}(\Omega,\mathbb{R}^N)$.

By Theorem 3.1 F is sequentially lower semicontinuous in $*w$-$W^{1,\Phi}(\Omega,\mathbb{R}^N)$ then

$$F(u,\Omega) \leq \liminf_r F(u_r,\Omega) = m,$$

since this we get $m = F(u,\Omega)$. □

4. Semicontinuity theorem: the general case

THEOREM 4.1. *Let Ω be an open bounded subset of $\mathbb{R}^n$ with $\partial\Omega$ lipschitzian, let $f(x,s,z)$, defined on $\Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn}$ with real values, be a Carathéodory function quasi-convex in z such that there exist positive constants c_0, c_1, c_2 and Φ_1, Φ_2 N-functions belonging to class Δ_2 such that*

$$|f(x,s,z)| \leq c_0 + c_1\Phi_1(|s|) + c_2\Phi_2(|z|) \tag{13}$$

for every $(x,s,z) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn}$.

Then the functional

$$F(u,\Omega) = \int_\Omega f(x,u(x),Du(x))\,dx$$

is sequentially lower semicontinuous in $*w$-$W^{1,\Gamma}(\Omega,\mathbb{R}^N)$ for every N-function $\Gamma \in \Delta_2$ such that

$$\lim_{t\to+\infty} \frac{\Gamma(t)}{\Phi_i(t)} = +\infty \qquad i = 1,2. \tag{14}$$

REMARK 4.2. The assumptions $\Gamma \in \Delta_2$ and (14) imply that the following embeddings are compact

$$W^{1,\Gamma}(\Omega,\mathbb{R}^N) \;\to\; L^{\Phi_i}(\Omega,\mathbb{R}^N) \qquad i = 1,2.$$

REMARK 4.3. If $f = f(z)$ Theorem 4.1 is a consequence of Theorem 3.1.

The following result, due to Scorza Dragoni (see [11]), characterizes the Carathéodory functions.

Proposition 4.4. *$g : \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^{Nn} \to \mathbb{R}$ is a Carathéodory function if and only if for every compact subset $C \subset \mathbb{R}^n$ and every $\gamma > 0$ there exists a compact subset $C_\gamma \subset C$ such that $|C \setminus C_\gamma| \le \gamma$ and that the restriction of g to $C_\gamma \times \mathbb{R}^N \times \mathbb{R}^{Nn}$ is continuous.*

Proof of Theorem 4.1. Let $\tau > 0$, then there exist a positive integer m and a finite number of open cubes $\{Q_{i,m}\}$, whose sides have length $1/m$, satisfying

$$Q_{i,m} \subset\subset \Omega, \quad Q_{i,m} \cap Q_{j,m} = \phi \quad i \neq j, \quad \left| \Omega \setminus \bigcup_{i \le m} Q_{i,m} \right| \le \tau.$$

Let $t > 0$ and ν be a function, define

$$\Omega_{\nu,t} = \{x \in \Omega : |\nu(x)| > t\}$$

and set

$$\nu_t(x) = \nu(x) \chi_{\Omega \setminus \Omega_{\nu,t}}(x).$$

Fix m and i, define

$$\nu_{i,m} = \frac{1}{|Q_{i,m}|} \int_{Q_{i,m}} \nu(x)\, dx,$$

then consider

$$\nu_m(x) = \begin{cases} \sum_i \nu_{i,m} \chi_{Q_{i,m}}(x) & x \in Q_m \\ 0 & x \in \Omega \setminus Q_m \end{cases},$$

and set $\nu_{t,m} = (\nu_t)_m$.

Let $u \in W^{1,\Gamma}(\Omega, \mathbb{R}^N)$ and $\{u_r\}$ be a sequence convergent to u in $*w$-$W^{1,\Gamma}(\Omega, \mathbb{R}^N)$.

By Remark 4.2, $\{u_r\}$ converges to u in s-$L^{\Phi_1}(\Omega, \mathbb{R}^N)$ and then it is convergent to u almost everywhere in Ω. Moreover, by Proposition 2.3, the functions $\Phi_1(|u|), \Phi_1(|u_r|), \Phi_2(|Du|)$ and $\Phi_2(|Du_r|)$ have equi-absolutely continuous integrals on Ω.

Consider $\{x_m\}$, $x_m = [\mathrm{Id}_{\mathbb{R}^n}]_m$, $\{u_{t,m}\}$ and $\{[Du]_{t,m}\}$, by Theorem 2.2 they are convergent almost everywhere in Ω to $\mathrm{Id}_{\mathbb{R}^n}, u_t, [Du]_t$ respectively.

We get

$$\int_\Omega f(x,u,Du)\,dx$$

$$= \int_{\Omega\setminus Q_m} f(x,u,Du)\,dx + \tag{I$_1$}$$

$$+ \int_{Q_m} \{f(x,u,Du) - f(x,u_t,[Du]_t)\}\,dx + \tag{I$_2$}$$

$$+ \int_{Q_m} \{f(x,u_t,[Du]_t) - f(x_m,u_{t,m},[Du]_{t,m})\}\,dx + \tag{I$_3$}$$

$$+ \int_{Q_m} \{f(x_m,u_{t,m},[Du]_{t,m}) - f(x_m,u_{t,m},[Du]_{t,m} + D(u_r - u))\}\,dx + \tag{I$_4$}$$

$$+ \int_{Q_m} \{f(x_m,u_{t,m},[Du]_{t,m} + D(u_r - u)) - f(x_m,u_{t,m},[Du]_{t,m} + [Du_r]_t - [Du]_t)\}dx + \tag{I$_5$}$$

$$+ \int_{Q_m} \{f(x_m,u_{t,m},[Du]_{t,m} + [Du_r]_t - [Du]_t) - f(x,u_t,[Du_r]_t)\}\,dx + \tag{I$_6$}$$

$$+ \int_{Q_m} \{f(x,u_t,[Du_r]_t) - f(x,[u_r]_t,[Du_r]_t)\}\,dx + \tag{I$_7$}$$

$$+ \int_{Q_m} \{f(x,[u_r]_t,[Du_r]_t) - f(x,u_r,Du_r)\}\,dx + \tag{I$_8$}$$

$$- \int_{\Omega\setminus Q_m} f(x,u_r,Du_r)\,dx + \tag{I$_9$}$$

$$+ \int_{\Omega} f(x, u_r, Du_r)\, dx. \tag{I$_{10}$}$$

By Proposition 2.3 it follows from (13) and (14) the equi-absolute continuity of integrals $F(u, \Omega)$ and $F(u_r, \Omega)$, thus if t is large enough we get

$$I_2 + I_5 + I_8 \leq \varepsilon.$$

Fixing a suitable m and using (13) we get

$$I_1 + I_9 \leq \varepsilon.$$

Consider I_3, I_6 and I_7, by Egorov theorem, Proposition 4.4 and equi-absolute continuity of integrals it follows that the sum of these addenda is less than ε.

Finally I_4 has non positive inferior limit by Remark 4.3.

So we have

$$\int_{\Omega} f(x, u, Du)\, dx \leq c\varepsilon + \int_{\Omega} f(x, u_r, Du_r)\, dx$$

and finally the result follows passing to inferior limits for $r \to +\infty$.

□

5. Examples

In this section we exhibit some examples of applications of the semicontinuity Theorems 3.1, 4.1 and of the existence Theorem 3.3. The first example deals with Theorem 4.1. We are interested in the case $N = n = 2$, completely solved in [1], [6] and [8] for positive poli-convex functionals, so we consider a suitable modification of a family of quasi-convex functions, introduced by Šverák [26], with sub-quadratic growth at infinity, which, then, are neither convex nor poli-convex.

Let $A, B \in M^{2\times 2}$ such that

$$\operatorname{rank}(A - B) \geq 2,$$

then $K = \{A, B\}$ is compact and non convex.

For $p > 1$ define the function

$$d_p(z) = [d(z)]^p,$$

where $z \in M^{2\times 2}$ and $d(z)$ denotes the distance of z from K.

Šverák in [26] proved that the quasi-convex envelope Qd_p of d_p satisfies

$$Qd_p(z) > 0 \quad \text{for every } z \in M^{2\times 2} \setminus K,$$

moreover if $1 < p < 2$ then Qd_p is quasi-convex but not poli-convex.

For $1 < p < 2$ define the function $f_p : M^{2\times 2} \to \mathbb{R}$ by

$$f_p(z) = d_p(z) \ln(e + d(z)).$$

Since $d_p \leq f_p$ we have $(Qf_p)^{-1}(0) = K$, then Qf_p is not convex, and not even poli-convex since it has sub-quadratic growth at infinity.

Moreover, since $d(z)$ has linear growth at infinity, we obtain

$$0 \leq Qf_p(z) \leq c_1\{1 + |z|^p \ln(e + |z|)\}. \tag{15}$$

Let $a : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ be a non negative measurable function belonging to $L^\infty(\Omega)$, define the function $g_p(x, s, z) = a(x, s)Qf_p(z)$. Then, by (15), g_p satisfies growth conditions of type (13) with the N-function $\Phi_p(t) = t^p \ln(e + t) \in \Delta_2$, thus by Theorem 4.1 the functional

$$G_p(u, \Omega) = \int_\Omega g_p(x, u, Du)\, dx \tag{16}$$

is sequentially lower semicontinuous in $*w$-$W^{1,\Phi_{\alpha,p}}(\Omega, \mathbb{R}^2)$ with $\Phi_{\alpha,p}(t) = t^p \ln^\alpha(e + t)$, $\alpha > 1$.

Finally, observe that applying Theorem 2.4 of [2], Proposition 1 of [15] and Theorem 1.1 of [20] we obtain the weak lower semicontinuity of (16) in $W^{1,p+\varepsilon}(\Omega, \mathbb{R}^2)$ for every $\varepsilon > 0$, which is a proper subspace of $W^{1,\Phi_{\alpha,p}}(\Omega, \mathbb{R}^2)$ for every $\alpha > 1$, $\varepsilon > 0$.

The following example is obtained by applying a result of Zhang who developed in [27] a method to construct quasi-convex functions with linear growth at infinity from known quasi-convex functions.

Consider, as before, $A, B \in M^{N\times n}$ such that

$$\text{rank}\,(A - B) \geq 2,$$

and set $K = \{A, B\}$, then K is compact and non convex.

Let $z \in M^{N\times n}$ and denote with $d(z)$ the distance of z from K, in [27] Zhang proved that the quasi-convex envelope Qd of d satisfies

$$Qd(z) > 0 \qquad \text{for every } z \in M^{N\times n} \setminus K.$$

Thus Qd is a quasi-convex function with linear growth at infinity, i.e. there exist c_i, $1 \le i \le 4$, non negative constants satisfying for every $z \in M^{N\times n}$

$$-c_1 + c_2|z| \le Qd(z) \le c_3 + c_4|z|, \tag{17}$$

and Qd is not convex since $(Qd)^{-1}(0) = K$.

Consider $\Phi_{a,b}(t) = t^{a+b\sin(\ln(\ln t))}$, which is the principal part of a N-function of class Δ_2 if $a > 1 + b\sqrt{2}$, then the function

$$h_{a,b}(z) = (\Phi_{a,b} \circ Qd)(z)$$

is quasi-convex but not convex. In fact $h_{a,b}$ is the composition of a N-function with a quasi-convex function, then, since $\Phi_{a,b}(t) = 0$ if and only if $t = 0$, it follows $(h_{a,b})^{-1}(0) = K$.

Since $\Phi_{a,b} \in \Delta_2$, by (17), there exist c_5, c_6 non negative constants such that for every $z \in M^{N\times n}$

$$0 \le h_{a,b}(z) \le \Phi_{a,b}(c_3 + c_4|z|) \le c_5 + c_6\Phi_{a,b}(|z|). \tag{18}$$

By (18) we get that $h_{a,b}$ satisfies

$$0 \le h_{a,b}(z) \le c_5 + c_6|z|^{a+b}, \tag{19}$$

moreover by (17) and the continuity of Qd it is easy to show that the power $a + b$ in (19) is sharp.

By Theorem 3.1, it follows the sequential lower semicontinuity of the functional

$$H_{a,b}(u, \Omega) = \int_\Omega h_{a,b}(Du)\, dx \tag{20}$$

in $*w$-$W^{1,\Phi_{a,b}}(\Omega, \mathbb{R}^N)$. Moreover, Theorem 3.3 gives the existence of minimizers for the Dirichlet problem $\min\limits_V H_{a,b}(u, \Omega)$, where $V = \nu + W_0^{1,\Phi_{a,b}}(\Omega, \mathbb{R}^N)$ and $\nu \in W^{1,\Phi_{a,b}}(\Omega, \mathbb{R}^N)$.

We remark that Theorem 3.1 applied to (20) gives a different result with respect to semicontinuity theorems known in ordinary Sobolev spaces. Let $N = n \geq 3$, Theorems 2.4 of [2] and 1.1 of [20], assure the weak lower semicontinuity of (20) in $W^{1,a+b}(\Omega, \mathbb{R}^N)$ which is a proper subspace of $W^{1,\Phi_{a,b}}(\Omega, \mathbb{R}^N)$. Moreover, the results in [13] and [18] give the lower semicontinuity of (20) in w-$W^{1,p}(\Omega, \mathbb{R}^N)$ for $p > \frac{N-1}{N}(a+b)$, and taking in account [19] we get semicontinuity for $p \geq a+b-1$. Let $a+b < N$, then $\frac{N-1}{N}(a+b) > a+b-1$, if we assume $a+b-1 > a-b$, i.e. $b > \frac{1}{2}$, since $\Phi_{a,b}(t) = t^{a-b}$ for infinite $t \in \mathbb{R}$, we can conclude that Theorem 3.1 states semicontinuity in a different space with respect to previous results. We observe explicitly that there exist positive constants a, b satisfying $a > 1 + b\sqrt{2}$, $a + b < N$ and $b > \frac{1}{2}$, e.g. for $N = 3$ take $a = 2$ and $b = \frac{5}{8}$.

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References

[1] Acerbi E. and Dal Maso G., *New lower semicontinuity results for poli-convex integrals case*, Preprint SISSA **52/M**, Trieste, 1993.

[2] Acerbi E. and Fusco N., *Semicontinuity problems in the calculus of variations*, Arch. Rational Mech. Anal. **86** (1984), 125–145.

[3] Adams R.A., "Sobolev Spaces", Academic Press, New York, 1975.

[4] Adams R.A., *On the Orlicz-Sobolev imbedding theorem*, J. Funct. Anal. **24** (1977), 241–257.

[5] Ball J.M., *Convexity conditions and existence theorems in nonlinear elasticity*, Arch. Rational Mech. Anal. **63** (1977), 337–403.

[6] Dacorogna B. and Marcellini P., *Semicontinuité pour des intégrandes polyconvexes sans continuité des déterminants*, C. R. Acad. Sci. Paris **311** (1990), 393–396.

[7] Dall'Aglio A., Mascolo E. and Papi G., *Regularity for local minima of functionals with nonstandard growth conditions*, to appear.

[8] Dal Maso G. and Sbordone C., *Weak lower semicontinuity of policonvex integrals: a borderline case*, Preprint SISSA **45/M**, Trieste, 1993

[9] De Giorgi E., "Teoremi di semicontinuità nel calcolo ‘ delle variazioni", INdAM, Roma, 1968-69.

[10] Donaldson T.K. and Trudinger N.S., *Orlicz-Sobolev spaces and embedding theorems*, J. Funct. Anal. **8** (1971), 42–75.

[11] EKELAND I. and TEMAM R., "Convex analysis and ' variationals problems", North-Holland, Amsterdam 1976.

[12] FOCARDI M., "Semicontinuità di funzionali vettoriali nella classe di Orlicz-Sobolev"', Tesi di Laurea, Università di Firenze, 1996.

[13] FONSECA I. and MALÝ J., *Relaxation of multiple integrals in Sobolev spaces below the growth exponent for the energy density*, Ann. Inst. H. Poincarè (Analyse non lineaire), to appear.

[14] FONSECA I. and MARCELLINI P., *Relaxation of multiple integrals in subcritical Sobolev spaces*, J. Geom. Anal., to appear.

[15] FUSCO N., *Quasi convessità e semicontinuità per integrali multipli di ordine superiore*, Ricerche di Mat. **29** (1980), 307–323.

[16] FUSCO N. and HUTCHINSON J.E., *A direct proof for lower semicontinuity of poli-convex functionals*, Preprint, 1994.

[17] KRASNOSEL'SKIJ M.A. and RUTICKII YA.B., "Convex function and Orlicz spaces", Noordhoff, Groningen, 1961.

[18] MALÝ J., *Weak lower semicontinuity of poli-convex and quasi-convex integrals*, Vortragsreihe, Bonn, 1993, to appear.

[19] MALÝ J., *Lower semicontinuity of quasi-convex integrals*, Manus. Math. **85** (1994), 419–428.

[20] MARCELLINI P., *Approximation of quasiconvex functions and lower semicontinuity of multiple integrals*, Manus. Math. **51** (1985), 1–28.

[21] MARCELLINI P., *On the definition and the lower semicontinuity of certain quasiconvex integrals*, Ann. Inst. H. Poincarè (Analyse non lineaire) **3** (1986), 391–409.

[22] MARCELLINI P. and SBORDONE C., *On the existence of minima of multiple integrals in the calculus of variations*, J. Math. Pures Appl. **62** (1983), 1–10.

[23] MEYERS N.G., *Quasiconvexity and the semicontinuity of multiple variational integrals of any order*, Trans. Amer. Math. Soc. **119** (1965), 125–149.

[24] MORREY C.B.JR., *Quasiconvexity and the lower semicontinuity of multiple integrals*, Pacific J. Math. **2** (1952), 25–53.

[25] RAO M.M. and REN Z.D., "Theory of Orlicz spaces", Pure and Appl. Math., Marcel Dekker, New York, 1991.

[26] ŠVERÁK V., *Quasi-convex functions with sub-quadratic growth*, Proc. Roy. Soc. London **433A** (1991), 723–725.

[27] ZHANG K., *A construction of quasi-convex functions with linear growth at infinity*, Ann. Scuola Norm. Sup. Pisa **19** (1992), 313–326.

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