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Global existence and blow-up for a hyperbolic system in three space dimensions

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Summary. - *Using the technique developed by F. John in [7], we study the existence and the nonexistence of global classical solutions to the Cauchy problem for*

$$\partial_t^2 u - \Delta_x u = |v|^p,$$
$$\partial_t^2 v - \Delta_x v = |u|^q,$$

in $\mathbf{R}^3_x \times [0, +\infty[$.

1. Introduction

In this paper we shall consider the following semilinear hyperbolic system

$$\begin{aligned} \partial_t^2 u - \Delta_x u &= |v|^p \\ \partial_t^2 v - \Delta_x v &= |u|^q \end{aligned} \qquad \text{in } \mathbb{R}^n_x \times [0, +\infty[, \tag{1}$$

with initial data

$$\begin{aligned} u(x,0) &= f_1(x), \qquad \partial_t u(x,0) = g_1(x) \\ v(x,0) &= f_2(x), \qquad \partial_t v(x,0) = g_2(x) \end{aligned} \qquad \text{in } \mathbb{R}^n_x, \tag{2}$$

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where $p,\, q > 1$ and $f_1,\, f_2,\, g_1,\, g_2 \in \mathcal{C}_0^\infty(\mathbb{R}^n_x)$. A *global classical solution* to this Cauchy problem is a pair (u, v) of $\mathcal{C}^2$ functions, defined in $\mathbb{R}^n_x \times [0, +\infty[$, satisfying (1), (2). We will use the term *blow–up* to mean the nonexistence of global classical solutions.

The question concerned here is to find sufficient conditions on p and q such that the blow–up for (1), (2) takes place.

In a recent work (see [2]), V. Georgiev, E. Mitidieri and the author have shown that if

$$\max\left\{\frac{p+2+q^{-1}}{pq-1},\ \frac{q+2+p^{-1}}{pq-1}\right\} > \frac{n-1}{2}, \tag{3}$$

and some conditions on initial data hold then the problem (1), (2) has no global classical solutions. In particular, for $n = 2$ or 3, a blow–up result is proved if (3) holds and the functions g_1 and g_2 satisfy

$$\int_{\mathbb{R}^n} g_1(x)\, dx > 0 \quad \text{and} \quad \int_{\mathbb{R}^n} g_2(x)\, dx > 0. \tag{4}$$

This result has been obtained by the so called functional method (see [1, Ch. 2]).

Here we shall prove that in three space dimensions no hypotheses on initial data are needed to produce blow–up when condition (3) holds (Theorem 2.1). The proof of this theorem uses a technique introduced by John (see [7, Th. 1]) and it is based on some particular properties of the solution of the linear nonhomogeneous wave equation in three space dimensions.

It is still an open problem to prove a similar result when $n \neq 3$, for both the semilinear wave equation and the system (1).

In [2] some global existence results for (1), (2) have been proved. More precisely, if $n = 3$ and

$$\max\left\{\frac{p+2+q^{-1}}{pq-1},\ \frac{q+2+p^{-1}}{pq-1}\right\} < 1, \tag{5}$$

with $2 \le p,\ q \le 3$ then (1), (2) has a unique global solution provided the initial data are small in a suitable norm.

In this paper we shall prove that the same result holds without requiring $p,\ q \le 3$ (Theorem 2.3). Moreover if $\min\{p, q\} < 2$ we show that (5) implies the existence of a global continuous solution of the integral problem corresponding to (1), (2) (Theorem 2.2). These improvements are due to a slightly more precise weighted L^∞ estimate for the solution of the linear nonhomogeneous wave equation in three space dimensions (Lemma 4.1), while the iteration technique for the construction of the global solution is the same as in [2].

It is beyond our intentions to give here an exhaustive bibliography on blow–up and global existence results for the solutions of the semilinear wave equation

$$\partial_t^2 u - \Delta_x u = |u|^p \quad \text{in } \mathbf{R}_x^n \times [0, +\infty[. \tag{6}$$

We recall only the cited fundamental article by John [7] in the case $n = 3$, and the papers by Glassey [4], [5], for $n = 2$. The main results in the case $n \geq 4$ are due to Sideris [10] and to Georgiev, Lindblad and Sogge [3]; many other references can be found in [1].

The paper is organized as follows: in Section 1 we state the three main results; the proofs are collected in Sections 2, 3, 4 respectively; the Appendix contains the statements of two uniqueness theorems we use in the proof of Theorem 2.1.

The author thanks with pleasure prof. Enzo Mitidieri and prof. Vladimir Georgiev for many enlightening conversations on the subject of the present work.

2. Results

We begin stating the main blow-up result of the paper.

THEOREM 2.1. *Let f_1, f_2, g_1, $g_2 \in \mathcal{C}_0^\infty(\mathbb{R}_x^3)$ with*

$$\emptyset \neq \operatorname{supp} f_1 \cup \operatorname{supp} f_2 \cup \operatorname{supp} g_1 \cup \operatorname{supp} g_2 \subset \{|x| \leq R\}.$$

Let $T \in]0, +\infty]$ and let $(u, v) \in \mathcal{C}^2(\mathbb{R}_x^3 \times [0, T[)^2$ be the solution of the Cauchy problem

$$\begin{matrix} \partial_t^2 u - \Delta_x u = |v|^p \\ \partial_t^2 v - \Delta_x v = |u|^q \end{matrix} \qquad \text{in } \mathbb{R}_x^3 \times [0, T[, \tag{7}$$

$$\begin{matrix} u(x,0) = f_1(x), & \partial_t u(x,0) = g_1(x) \\ v(x,0) = f_2(x), & \partial_t v(x,0) = g_2(x) \end{matrix} \qquad \text{in } \mathbb{R}_x^3, \tag{8}$$

Suppose that $p,\ q > 1$ *and*

$$\max\left\{\frac{p+2+q^{-1}}{pq-1},\ \frac{q+2+p^{-1}}{pq-1}\right\} > 1. \tag{9}$$

Then $T < +\infty$.

In order to state the global existence results we need to recall some classical properties of the solutions of the linear wave equation in three space dimensions.

Let F be a continuous function defined in $\mathbb{R}^3_x \times [0,+\infty[$. We denote by L the integral operator defined by

$$L(F)(x,t) = \frac{1}{4\pi}\int_0^t (t-s)\,ds \int_{|\eta|=1} F(x+|t-s|\eta, s)\,d\omega_\eta. \tag{10}$$

It is a well known fact that if F is a $\mathcal{C}^2$ function then $L(F)$ is the unique $\mathcal{C}^2$ solution of

$$\partial_t^2 u - \Delta_x u = F \quad \text{in } \mathbb{R}^3_x \times [0,+\infty[, \tag{11}$$

with zero initial data.

Let $f \in \mathcal{C}^1$, $g \in \mathcal{C}$ be two functions defined in $\mathbb{R}^3_x$. We denote by W the integral operator defined by

$$W(f,g)(x,t) = \frac{t}{4\pi}\int_{|\xi|=1} g(x+t\xi)\,d\omega_\xi + \partial_t\Big(\frac{t}{4\pi}\int_{|\xi|=1} f(x+t\xi)\,d\omega_\xi\Big). \tag{12}$$

If $f \in \mathcal{C}^3$ and $g \in \mathcal{C}^2$ then $W(f,g)$ is the unique $\mathcal{C}^2$ solution of

$$\partial_t^2 u - \Delta_x u = 0 \quad \text{in } \mathbb{R}^3_x \times [0,+\infty[, \tag{13}$$

with

$$u(x,0) = f(x), \qquad \partial_t u(x,0) = g(x) \quad \text{in } \mathbb{R}^3_x. \tag{14}$$

Theorem 2.2. *Let* $f_i,\ g_i \in \mathcal{C}_0^\infty(\mathbb{R}^3_x)$ *with* $\operatorname{supp} f_i,\ \operatorname{supp} g_i \subset \{|x| \le R\}$, *for* $i = 1,\ 2$. *Suppose that* $p,\ q > 1$ *and*

$$\max\left\{\frac{p+2+q^{-1}}{pq-1},\ \frac{q+2+p^{-1}}{pq-1}\right\} < 1. \tag{15}$$

Then there exists $\varepsilon_0 > 0$ *such that for each* $\varepsilon \in]0, \varepsilon_0[$ *the problem*

$$\begin{aligned} u &= W(\varepsilon f_1, \varepsilon g_1) + L(|v|^p) \\ v &= W(\varepsilon f_2, \varepsilon g_2) + L(|u|^q) \end{aligned} \qquad \text{in } \mathbb{R}^3_x \times [0, +\infty[, \tag{16}$$

has a solution $(u, v) \in \mathcal{C}(\mathbb{R}^3_x \times [0, +\infty[)^2$.

THEOREM 2.3. *Suppose that the hypotheses of Theorem 2.2 hold. Suppose moreover that* $p,\ q \geq 2$.

Then there exists $\varepsilon_0 > 0$ *such that for each* $\varepsilon \in]0, \varepsilon_0[$ *the Cauchy problem*

$$\begin{aligned} \partial_t^2 u - \Delta_x u &= |v|^p \\ \partial_t^2 v - \Delta_x v &= |u|^q \end{aligned} \qquad \text{in } \mathbb{R}^3_x \times [0, +\infty[, \tag{17}$$

$$\begin{aligned} u(x,0) &= \varepsilon f_1(x), \qquad \partial_t u(x,0) = \varepsilon g_1(x) \\ v(x,0) &= \varepsilon f_2(x) \qquad \partial_t v(x,0) = \varepsilon g_2(x) \end{aligned} \qquad \text{in } \mathbb{R}^3_x, \tag{18}$$

has a unique global classical solution $(u, v) \in \mathcal{C}^2(\mathbb{R}^3_x \times [0, +\infty[)^2$.

3. Proof of Theorem 2.1

In order to prove Theorem 2.1 we adapt to the present situation the proof of John's result [7, Th. 1] (see also [9, Ch. 2]). First of all we remark that as a consequence of Theorem A.2 stated in the Appendix, we have that

$$u(x,t) = v(x,t) = 0 \tag{19}$$

for all $(x, t) \in \mathbb{R}^3 \times [0, T[$ such that $|x| \geq t + R$. Moreover, the condition on the support of the initial data implies that there exists $\bar{x} \in \mathbb{R}^3$ with $|\bar{x}| < 2R$ such that

$$|u(\bar{x}, t)| + |\partial_t u(\bar{x}, t)| + |v(\bar{x}, t)| + |\partial_t v(\bar{x}, t)| \neq 0. \tag{20}$$

For every continuous function φ defined in $\mathbb{R}^3$ we denote by $\bar{\varphi}$ its average on the sphere of center in the origin, i. e.

$$\bar{\varphi}(r) = \frac{1}{4\pi} \int_{|\xi|=1} \varphi(|r|\xi)\, d\omega_\xi,$$

and similarly, for a continuous function F defined in $\mathbb{R}^3 \times [0,T[$ we set

$$\bar{F}(r,t) = \frac{1}{4\pi}\int_{|\xi|=1} F(|r|\xi,t)\,d\omega_\xi.$$

From (7) and (8) we have

$$\left.\begin{aligned} &\partial_t^2(r\bar{u})(r,t) - \partial_r^2(r\bar{u})(r,t) = r\overline{|v|^p}(r,t) \\ &\partial_t^2(r\bar{v})(r,t) - \partial_r^2(r\bar{v})(r,t) = r\overline{|u|^p}(r,t) \end{aligned}\right. \quad \text{in } \mathbb{R}^3 \times [0,T[\tag{21}$$

$$\left.\begin{aligned} &\bar{u}(r,0) = \bar{f}_1(r),\ \partial_t\bar{u}(r,0) = \bar{g}_1(r), \\ &\bar{v}(r,0) = \bar{f}_2(r),\ \partial_t\bar{v}(r,0) = \bar{g}_2(r). \end{aligned}\right. \quad \text{in } \mathbb{R}^3, \tag{22}$$

Consequently

$$\begin{aligned} \bar{u}(r,t) = \frac{1}{2r}&[(r+t)\bar{f}_1(r+t) + (r-t)\bar{f}_1(r-t) + \int_{r-t}^{r+t} \rho\bar{g}_1(\rho)\,d\rho] \\ &+\frac{1}{2r}\int_{Z_{r,t}} \rho\overline{|v|^p}(\rho,\tau)\,d\rho\,d\tau, \\ \bar{v}(r,t) = \frac{1}{2r}&[(r+t)\bar{f}_2(r+t) + (r-t)\bar{f}_2(r-t) + \int_{r-t}^{r+t} \rho\bar{g}_2(\rho)\,d\rho] \\ &+\frac{1}{2r}\int_{Z_{r,t}} \rho\overline{|u|^q}(\rho,\tau)\,d\rho\,d\tau, \end{aligned} \tag{23}$$

where $Z_{r,t}$ denotes the triangle with vertices in (r,t), $(r+t,0)$ and $(r-t,0)$.

Suppose now that $0 \le r \le t - R$. We have

$$\begin{aligned} \bar{u}(r,t) &= \frac{1}{2r}\int_{R_{r,t}} \rho\overline{|v|^p}(\rho,\tau)\,d\rho\,d\tau, \\ \bar{v}(r,t) &= \frac{1}{2r}\int_{R_{r,t}} \rho\overline{|u|^q}(\rho,\tau)\,d\rho\,d\tau, \end{aligned} \tag{24}$$

where $R_{r,t} = \{(\rho,\tau) \in \mathbb{R}^2\ :\ t-r < \tau+\rho < t+r,\ \tau-\rho,\ \tau > 0\}$ (see Fig. 1).

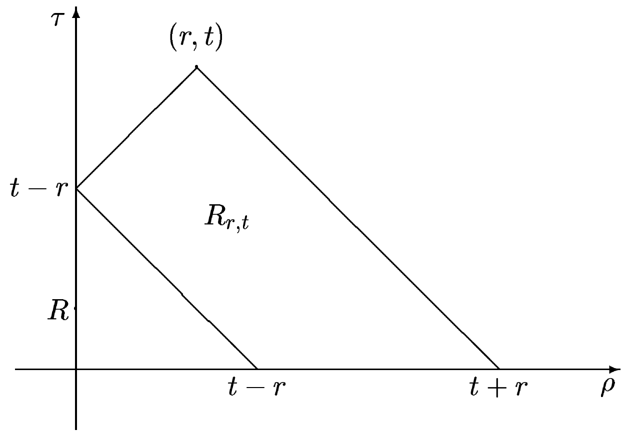

Fig. 1

We claim that there exists $t_0 \in [R, 3R]$ such that

$$|\bar{u}(0,t_0)| + |\bar{v}(0,t_0)| > 0. \tag{25}$$

Indeed if (25) is not satisfied then

$$\bar{u}(0,t) = \bar{v}(0,t) = 0 \tag{26}$$

for all $t \in [R, 3R]$. On the other hand we have

$$\bar{u}(0,t) = \lim_{r\to 0} \frac{1}{2r} \int_{R_{r,t}} \rho \overline{|v|^p}(\rho,\tau)\, d\rho\, d\tau = \int_0^t \rho \overline{|v|^p}(\rho, t-\tau)\, d\rho,$$

and similarly

$$\bar{v}(0,t) = \int_0^t \rho \overline{|u|^q}(\rho, t-\tau)\, d\rho.$$

From (26) we deduce that

$$|\overline{|u|^q}(\rho,\tau)| = |\overline{|v|^p}(\rho,\tau)| = 0, \tag{27}$$

for all $(\rho,\tau) \in \mathbb{R}^2$ such that $R \leq \rho + \tau \leq 3R$. This implies that $u(x,t) = v(x,t) = 0$ for all $(x,t) \in \mathbb{R}^3 \times [0,T[$ such that $R \leq t \leq 3R$ and $R \leq |x| + t \leq 3R$, against (20).

A consequence of (25) is that there exists $t_0 \in [R, 3R]$ such that

$$|\bar{u}(r, t_0 + r)| > 0 \quad \text{and} \quad |\bar{v}(r, t_0 + r)| > 0, \tag{28}$$

for all $r \in \mathbb{R}$. Indeed suppose that $\bar{u}(0, t_0) \neq 0$. Since $u(0, t_0) = \bar{u}(0, t_0)$, we have $u(0, t_0) \neq 0$ and consequently $|u|^q(0, t_0) = \overline{|u|^q}(0, t_0) \neq 0$. Then $\overline{|u|^q}(r, t) > 0$ in a neighborhood of $(0, t_0)$. Finally, noticing that $(0, t_0)$ is a corner of R_{r,t_0+r}, we deduce that

$$\frac{1}{2r} \int_{R_{r,t_0+r}} \rho \overline{|u|^q}(\rho, \tau)\, d\rho\, d\tau = \bar{v}(r, t_0 + r) > 0.$$

In particular this last inequality implies that $\bar{v}(0, t_0) > 0$ and then, by the same argument we obtain the first part of (28).

Let

$$S = \{(\rho, \tau) \in \mathbb{R}^2 \ : \ 6R < \tau + \rho, \ 3R < \tau - \rho < 4R\}$$

$$T = \{(\rho, \tau) \in \mathbb{R}^2 \ : \ 4R < \tau + \rho < 6R, \ R < \tau - \rho < 3R\},$$

be two subset of $\mathbb{R}^2$ (see Fig. 2).

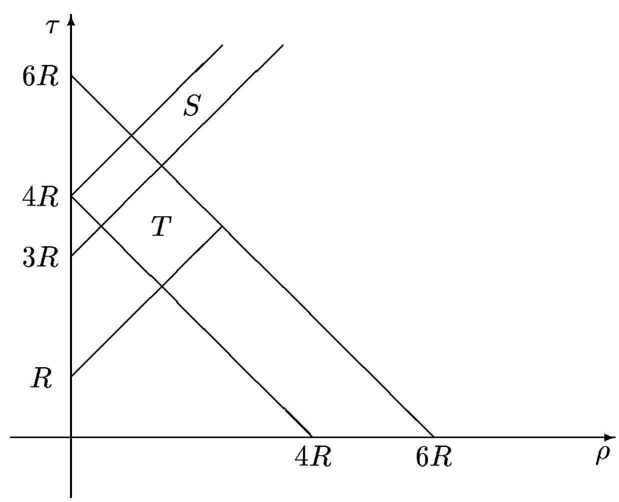

Fig. 2

By Hölder's inequality we know that

$$0 \le |\bar{u}|^q(r,t) \le \overline{|u|^q}(r,t),$$
$$0 \le |\bar{v}|^p(r,t) \le \overline{|v|^p}(r,t). \tag{29}$$

Hence from (28) and (29) we obtain

$$\overline{|u|^q}(r,t_0+r) > 0 \quad \text{and} \quad \overline{|v|^p}(r,t_0+r) > 0$$

for all $r \in \mathbb{R}$. Then

$$c_1 = \frac{1}{2}\int_T \rho\overline{|v|^p}(\rho,\tau)\,d\rho\,d\tau > 0,$$
$$c_2 = \frac{1}{2}\int_T \rho\overline{|u|^q}(\rho,\tau)\,d\rho\,d\tau > 0. \tag{30}$$

Let $(r,t) \in S$. Since $T \subset R_{r,t}$ we deduce from (24) and (30) that

$$\bar{u}(r,t) \ge \frac{c_1}{r}$$
$$\bar{v}(r,t) \ge \frac{c_2}{r} \tag{31}$$

Let $\Sigma = \{(r,t) \in \mathbb{R}^2 \,:\, 0 < r < t - 6R\}$ be a subset of $\mathbb{R}^2$. For $(r,t) \in \Sigma$ we define $S_{r,t} = S \cap R_{r,t}$ and $\Sigma_{r,t} = \Sigma \cap R_{r,t}$ (see Fig. 3).

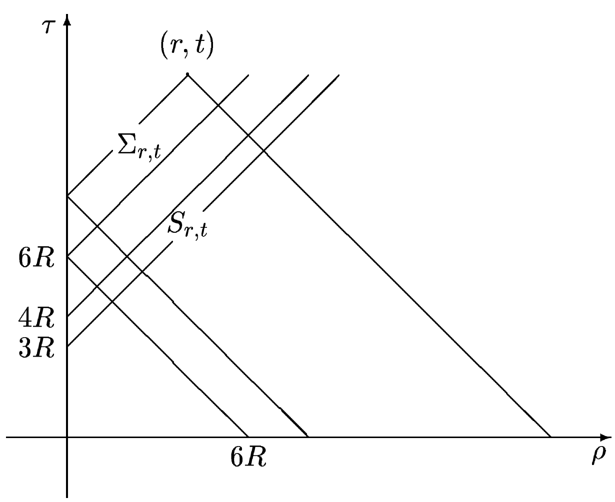

Fig. 3

Let $(r,t) \in \Sigma$. From (29) and (31) we get

$$\overline{|v|^p}(\rho,\tau) \geq |\bar{v}|^p(\rho,\tau) \geq \frac{c_2^p}{\rho^p}$$

for all $(\rho,\tau) \in S_{r,t}$. Then

$$\begin{aligned} \bar{u}(r,t) &\geq \frac{1}{2r}\int_{S_{r,t}} \rho\overline{|v|^p}(\rho,\tau)\,d\rho\,d\tau \geq \frac{1}{2r}c_2^p\int_{S_{r,t}} \rho^{1-p}\,d\rho\,d\tau \\ &\geq \frac{2^{p-2}c_2^pR}{(t+r-3R)^{p-1}} \geq \frac{2^{p-2}c_2^pR}{(t+r)^{p-1}}. \end{aligned} \tag{32}$$

Similarly, for $(r,t) \in \Sigma$ we obtain that

$$\bar{v}(r,t) \geq \frac{2^{q-2}c_1^qR}{(t+r)^{q-1}}. \tag{33}$$

It follows that, for $(r,t) \in \Sigma$ we have

$$\begin{aligned} \bar{u}(r,t) &\geq C_0(t+r)^{-p_0}, \\ \bar{v}(r,t) &\geq \Gamma_0(t+r)^{-q_0}, \end{aligned} \tag{34}$$

where $C_0 = 2^{p-2}c_2^pR$, $\Gamma_0 = 2^{q-2}c_1^qR$ and

$$p_0 = \begin{cases} p-1 & \text{if} \quad q(p-1) \geq 1 \\ 1/q & \text{if} \quad 0 < q(p-1) < 1 \end{cases}$$

$$q_0 = \begin{cases} q-1 & \text{if} \quad p(q-1) \geq 1 \\ 1/p & \text{if} \quad 0 < p(q-1) < 1. \end{cases}$$

We claim that if there exist l, $\lambda > 0$, a, b, α, $\beta \geq 0$ and $C, \Gamma > 0$ such that lq, $\lambda p \geq 1$ and

$$\begin{aligned} \bar{u}(r,t) &\geq C(t+r)^{-l}(t-r-6R)^a(t-r)^b, \\ \bar{v}(r,t) &\geq \Gamma(t+r)^{-\lambda}(t-r-6R)^\alpha(t-r)^\beta, \end{aligned} \tag{35}$$

for all $(r,t) \in \Sigma$, then

$$\begin{aligned} \bar{u}(r,t) &\geq C^*(t+r)^{-1}(t-r-6R)^{a^*}(t-r)^{b^*}, \\ \bar{v}(r,t) &\geq \Gamma^*(t+r)^{-1}(t-r-6R)^{\alpha^*}(t-r)^{\beta^*}, \end{aligned} \tag{36}$$

for all $(r,t) \in \Sigma$ where

$$a^* = \alpha p + 2$$

$$b^* = (\beta + \lambda)p - 1$$

$$C^* \frac{\Gamma^p}{(p\alpha + 2)^2} D_\lambda \quad \text{with} \quad D_\lambda = \frac{1}{4} \min\{\frac{1 - 2^{1-p\lambda}}{2(p\lambda - 1)}, 2^{1-p\lambda}\},$$

and

$$\alpha^* = aq + 2$$

$$\beta^* = (b + l)q - 1$$

$$\Gamma^* \frac{C^q}{(qa + 2)^2} \Delta_l \quad \text{with} \quad \Delta_l = \frac{1}{4} \min\{\frac{1 - 2^{1-ql}}{2(ql - 1)}, 2^{1-ql}\}.$$

Suppose that (35) is satisfied. By (24) and (29) we get

$$\begin{aligned} \bar{u}(r,t) &\geq \frac{1}{2r} \int_{\Sigma_{r,t}} \rho |\bar{v}|^p(\rho,\tau)\, d\rho\, d\tau \\ &\geq \frac{\Gamma^p}{2r} \int_{\Sigma_{r,t}} \rho(\tau + \rho)^{-\lambda p} (\tau - \rho - 6R)^{\alpha p} (\tau - \rho)^{-\beta p}\, d\rho\, d\tau. \end{aligned}$$

Using the new variables $y = \tau + \rho$, $z = \tau - \rho$ we deduce that

$$\bar{u}(r,t) \geq \frac{\Gamma^p}{8r} \int_{6R}^{t-r} dz \int_{t-r}^{t+r} (y - z) y^{-\lambda p} (z - 6R)^{\alpha p} z^{-\beta p}\, dy$$

$$\geq \frac{\Gamma^p}{8r} (t - r)^{-\beta p} \int_{6R}^{t-r} dz \int_{t-r}^{t+r} \frac{(y - z)(z - 6R)^{\alpha p}}{y^{\lambda p}}\, dy$$

$$\geq \frac{\Gamma^p}{8r} \frac{(t - r - 6R)^{\alpha p + 1}}{(\alpha p + 2)^2 (t - r)^{\beta p}} \int_{t-r}^{t+r} \frac{(\alpha p + 2)(y - t + r) + (t - r - 6R)}{y^{\lambda p}}\, dy$$

$$\geq \frac{\Gamma^p}{4} \frac{(t - r - 6R)^{\alpha p + 2}}{(\alpha p + 1)^2 (t - r)^{\beta p}} I(r,t)$$

where $I(r,t) = \frac{1}{2r} \int_{t-r}^{t+r} \frac{dy}{y^{\lambda p}}$. Now it is not difficult to show that (see [7, p. 248])

$$I(r,t) \geq (t + r)^{-1} (t - r)^{1 - \lambda p} \ \min\{\frac{1 - 2^{1-p\lambda}}{2(p\lambda - 1)}, 2^{1-p\lambda}\}.$$

Hence the first inequality of (36) follows. The remaining statement is proved similarly. From (34), using recursively the claim, it follows that for all $(r,t) \in \Sigma$ and $n \in \mathbf{N}$ we have

$$\begin{aligned} \bar{u}(r,t) &\geq C_n(t+r)^{-1}(t-r-6R)^{a_n}(t-r)^{b_n}, \\ \bar{v}(r,t) &\geq \Gamma_n(t+r)^{-1}(t-r-6R)^{\alpha_n}(t-r)^{\beta_n}, \end{aligned} \tag{37}$$

with $a_0 = b_0 = \alpha_0 = \beta_0 = 0$ and, for $m \in \mathbf{N}$,

$$a_{2m+1} = \alpha_{2m+1} = 2(pq)^m + 2(\frac{p+1}{pq-1})((pq)^m - 1),$$

$$a_{2m+2} = \alpha_{2m+2} = 2(p+1)(\frac{(pq)^{m+1}-1}{pq-1}),$$

$$b_{2m+1} = q_0 p(pq)^m - 1, \quad b_{2m+2} = p_0 p(pq)^{m+1} - 1,$$

$$\beta_{2m+1} = p_0 q(pq)^m - 1, \quad \beta_{2m+2} = q_0 p(pq)^{m+1} - 1,$$

C_0, C_1, C_2, Γ_0, Γ_1, $\Gamma_2 > 0$, and

$$C_{m+3} = \frac{C_{m+1}^{pq} D_1 \Delta_1^p}{a_{m+3}^2 \alpha_{m+2}^{2p}}, \quad \Gamma_{m+3} = \frac{\Gamma_{m+1}^{pq} D_1^q \Delta_1}{\alpha_{m+3}^2 a_{m+2}^{2q}}.$$

In particular

$$C_{2m+1} \geq \frac{C_{2m-1}^{pq} D_1 \Delta_1^p}{(2(p+1)(m+1)(pq)^m)^{2p+2}},$$

$$C_{2m+2} \geq \frac{C_{2m}^{pq} D_1 \Delta_1^p}{(2(p+1)(m+1)(pq)^m)^{2p+2}}.$$

Consequently

$$\log(C_{2m+1}) \geq (pq)^m[\log C_1 + \sum_{j=1}^m h_j],$$

$$\log(C_{2m+2}) \geq (pq)^m[\log C_2 + \sum_{j=1}^m h_j],$$

where

$$h_j = \frac{\log D_1 + p\log\Delta_1 - 2(p+2)[\log(2(p+1)(j+1)) + j\log(pq)]}{(pq)^j}.$$

Since the series $\sum_j h_j$ is convergent, we finally deduce that there exists $H \in \mathbb{R}$ such that

$$C_{2m+1} \geq \exp(H(pq)^m) \quad \text{and} \quad C_{2m+2} \geq \exp(H(pq)^m)$$

for all $m \in \mathbf{N}$.

By (37), for all $n \in \mathbf{N}$ we have

$$\bar{u}(r,t) \geq \varphi(r,t)\exp((pq)^n[H + \tfrac{2(pq+p)}{pq-1}\log(t-r-6R) + \\ - q_0 p\log(t-r)])$$

and

$$\bar{v}(r,t) \geq \varphi(r,t)\exp((pq)^n[H + \tfrac{2(p^2q+pq)}{pq-1}\log(t-r-6R) + \\ - p_0 pq\log(t-r)]).$$

where

$$\varphi(r,t) \geq \frac{t-r}{(t+r)(t-r-6R)^{\frac{2(p+1)}{pq-1}}}.$$

It is now easy to verify that if (9) holds then there exists $(\bar{r},\bar{t}) \in \Sigma$ such that

$$H + \frac{2(pq+p)}{pq-1}\log(\bar{t}-\bar{r}-6R) - q_0 p\log(\bar{t}-\bar{r}) > 0$$

or

$$H + \frac{2(p^2q+pq)}{pq-1}\log(\bar{t}-\bar{r}-6R) - p_0 pq\log(\bar{t}-\bar{r}) > 0.$$

This means that $T < \bar{t}$. The proof of Theorem 2.1 is complete.

4. Proof of Theorem 2.2

Let

$$\tau_+ = 1 + |t + |x||, \qquad \tau_- = 1 + |t - |x||$$

be defined in $\mathbb{R}^3 \times [0,+\infty[$. The principal tool in the proof of Theorem 2.2 will be the inequality established in the following lemma (see [7, Lemma III]).

LEMMA 4.1. *Let $R > 1$. Let α, β, γ, δ be real parameters. Suppose that one of the following conditions holds:*

i) $\alpha = 1$, $\beta = \gamma - 2$, $\gamma > 2$ *and* $\delta > 1$;

ii) $\gamma - 1 > \alpha \geq 0$, $\beta = 0$, $2 \geq \gamma \geq 1$ *and* $\delta > 1$;

iii) $\alpha = 1$, $\gamma + \delta - 3 > \beta > 0$, $\gamma > 3 - \delta$ *and* $1 \geq \delta \geq 0$.

Then there exists $C > 0$ such that

$$\|\tau_+^\alpha \tau_-^\beta L(F)\|_{L^\infty(\mathbb{R}_x^3 \times [0,+\infty[)} \leq C \|\tau_+^\gamma \tau_-^\delta F\|_{L^\infty(\mathbb{R}_x^3 \times [0,+\infty[)}, \tag{38}$$

for all $F \in \mathcal{C}(\mathbb{R}_x^3 \times [0,+\infty[)$ such that $\operatorname{supp} F \subset \{(x,t) \,:\, |x| \leq t + R\}$ *and $\tau_+^\gamma \tau_-^\delta F \in L^\infty(\mathbb{R}_x^3 \times [0,+\infty[)$.*

In proving this lemma we shall follow some ideas contained in [7, Lemma III]. We present the proof in detail.
Proof. Let $\tilde{F}$ be defined by

$$\tilde{F}(x,t) = \sup_{|x|=r} |F(x,t)|. \tag{39}$$

By using the identity for iterated spherical means ([6, p. 81]), it is not difficult to show that if $|x| = r$ then

$$|L(F)(x,t)| \leq \int_0^t ds \int_{|r-t+s|}^{r+t-s} \frac{\lambda}{2r} \tilde{F}(\lambda, s)\, d\lambda = \int_{R_{r,t}} \frac{\lambda}{2r} \tilde{F}(\lambda, s)\, ds\, d\lambda, \tag{40}$$

where $R_{r,t} = \{(\lambda, s) \in \mathbb{R}^2 \,:\, t-r < s+\lambda < t+r,\ s-\lambda < t-r,\ s > 0\}$. In order to prove (38) it is sufficient to show that if for all $(\lambda, s) \in [0,+\infty[\times[0,+\infty[$ we have

$$0 \leq \tilde{F}(\lambda, s) \leq \frac{K}{(1+s+\lambda)^\gamma (1+|s-\lambda|)^\delta} \tag{41}$$

then

$$\int_{R_{r,t}} \frac{\lambda}{2r} \tilde{F}(\lambda, s)\, d\lambda\, ds \leq \frac{CK}{(1+t+r)^\alpha (1+|t-r|)^\beta} \tag{42}$$

for all $(r,t) \in [0,+\infty[\times[0,+\infty[$. *case i).* Suppose that $t - R \leq r \leq$

$t + R$. Then $1 \leq 1 + |t - r| \leq 1 + R$ and consequently

$$\frac{(1 + |t - r|)^\beta}{(1 + R)^\beta} \leq 1. \tag{43}$$

Moreover we have

$$1 + 2t - R \leq 1 + t + r \leq 1 + 2t + R \leq 2R(1 + t). \tag{44}$$

From (41) we deduce that

$$\tilde{F}(\lambda, s) \leq \frac{K}{(1 + s + \lambda)^\gamma} \leq \frac{K}{(1 + s)^\gamma},$$

and recalling that $\tilde{F}(\lambda, s) = 0$ for $\lambda > R + s$ we obtain

$$\int_{R_{r,t}} \frac{\lambda}{2r} \tilde{F}(\lambda, s)\, ds\, d\lambda \leq \int_0^t \frac{K}{(1 + s)^\gamma} k(s)\, ds, \tag{45}$$

where

$$k(s) = \frac{1}{2r} \int_{|r-t+s|}^{\max\{r+t+s, R+s\}} \lambda\, d\lambda.$$

A simple computation shows that $k(s) \leq 12R^2(s + 1)/(t + 1)$ (see [7, p. 255]). Hence

$$\int_{R_{r,t}} \frac{\lambda}{2r} \tilde{F}(\lambda, s)\, ds\, d\lambda \leq \frac{12R^2 K}{t + 1} \int_0^{+\infty} \frac{ds}{(1 + s)^{\gamma - 1}} = \frac{12R^2 K}{(\gamma - 2)(t + 1)}. \tag{46}$$

Finally from (43), (44) and (46) we obtain

$$\int_{R_{r,t}} \frac{\lambda}{2r} \tilde{F}(\lambda, s)\, ds\, d\lambda \leq \frac{24R^3(1 + R)^\beta}{\gamma - 2} \frac{K}{(1 + t + r)(1 + |t - r|)^\beta}. \tag{47}$$

and (42) follows.

Suppose on the contrary that $0 \leq r \leq t - R$. We introduce the new variables $y = s + \lambda$, $z = s - \lambda$. We have

$$\int_{R_{r,t}} \frac{\lambda}{2r} \tilde{F}(\lambda, s)\, ds\, d\lambda \leq A_1 + A_2,$$

where

$$A_1 = \frac{K}{8r}\int_{-R}^{0} dz \int_{t-r}^{t+r} \frac{y-z}{(1+y)^{\gamma}(1-z)^{\delta}}\, dy,$$
$$A_2 = \frac{K}{8r}\int_{0}^{t-r} dz \int_{t-r}^{t+r} \frac{y-z}{(1+y)^{\gamma}(1+z)^{\delta}}\, dy, \tag{48}$$

and

$$A_1 \le \frac{KR}{8r}\int_{-R}^{0} dz \int_{t-r}^{t+r} \frac{dy}{(1+y)^{\gamma-1}} \le \frac{KR^2}{8r}\int_{t-r}^{t+r} \frac{dy}{(1+y)^{\gamma-1}}, \tag{49}$$

$$A_2 \le \frac{KR}{8r}\int_{0}^{+\infty} \frac{dz}{(1+z)^{\delta}} \int_{t-r}^{t+r} \frac{dy}{(1+y)^{\gamma-1}} \le \frac{K}{8(\delta-1)r}\int_{t-r}^{t+r} \frac{dy}{(1+y)^{\gamma-1}}. \tag{50}$$

If $1+t-r \le (1+t+r)/2$ then

$$\frac{1}{r} \le \frac{4}{1+t+r}.$$

On the other hand

$$\int_{t-r}^{t+r} \frac{dy}{(1+y)^{\gamma-1}} \le \frac{1}{\gamma-2}\frac{1}{(1+t-r)^{\gamma-2}},$$

hence

$$A_1 + A_2 \le \frac{1}{2(\gamma-2)}(R^2 + \frac{1}{\delta-1})\frac{K}{(1+t+r)(1+t-r)^{\gamma-2}}. \tag{51}$$

If $1+t-r > (1+t+r)/2$, by using the mean value theorem we deduce that

$$\frac{1}{2r}\int_{t-r}^{t+r} \frac{dy}{(1+y)^{\gamma-1}} \le \frac{1}{(1+t-r)^{\gamma-1}} \le \frac{2}{(1+t+r)(1+t-r)^{\gamma-2}},$$

hence (51) and consequently (42) follow. *case ii)*. Suppose that $t-R \le r \le t+R$. Proceeding as in the *case i)* we get

$$\int_{R_{r,t}} \frac{\lambda}{2r}\tilde{F}(\lambda,s)\, ds\, d\lambda \le \frac{12R^2K}{t+1}\int_0^t \frac{ds}{(1+s)^{\gamma-1}}.$$

Using (44) and the fact that $\alpha < \gamma - 1$ we deduce that

$$\begin{aligned}\frac{1}{t+1}\int_0^t \frac{ds}{(1+s)^{\gamma-1}} &\leq \frac{1}{t+1}\int_0^t \frac{ds}{(1+s)^{\alpha}}\\ &\leq \frac{1}{1-\alpha}\frac{1}{(t+1)^{\alpha}} \leq \frac{2R}{1-\alpha}\frac{1}{(1+t+r)^{\alpha}},\end{aligned}$$

and consequently

$$\int_{R_{r,t}} \frac{\lambda}{2r}\tilde{F}(\lambda,s)\,ds\,d\lambda \leq \frac{24R^2}{1-\alpha}\frac{K}{(1+t+r)^{\alpha}}.$$

Suppose then $0 \leq r \leq t - R$. As in the previous *case i)* we have

$$\int_{R_{r,t}} \frac{\lambda}{2r}\tilde{F}(\lambda,s)\,ds\,d\lambda \leq A_1 + A_2,$$

with A_1 and A_2 defined in (48). Hence inequalities (49) and (50) hold.

If $1+t-r \leq (1+t+r)/2$ we deduce that $1+t+r \leq 4r$ and then

$$\begin{aligned}\int_{t-r}^{t+r} \frac{dy}{(1+y)^{\gamma-1}} &\leq \int_{t-r}^{t+r} \frac{dy}{(1+y)^{\alpha}}\\ &\leq \frac{(1+t+r)^{1-\alpha}}{1-\alpha} \leq \frac{4r}{1-\alpha}\frac{1}{(1+t+r)^{\alpha}}.\end{aligned}$$

This last inequality implies that

$$\int_{R_{r,t}} \frac{\lambda}{2r}\tilde{F}(\lambda,s)\,ds\,d\lambda \leq \frac{1}{2(1-\alpha)}(R^2 + \frac{1}{\delta-1})\frac{K}{(1+t+r)^{\alpha}}. \qquad (52)$$

If $1+t-r > (1+t+r)/2$, the mean value theorem implies

$$\int_{t-r}^{t+r} \frac{dy}{(1+y)^{\gamma-1}} \leq \frac{4r}{(1+t+r)^{\alpha}},$$

and (52) i.e. (42) follows.

case iii). If $t - R \leq r \leq t + R$ the computation is exactly the same as in the *case i).*

Suppose that $0 \le r < t - R$. As before

$$\int_{R_{r,t}} \frac{\lambda}{2r} \tilde{F}(\lambda, s)\, ds\, d\lambda \le A_1 + A_2,$$

with A_1 and A_2 defined in (48). Let β^* be a real number such that $\beta < \beta^* < \gamma + \delta - 3$. Since $\beta^* + 1 < \gamma + \delta - 2 \le \gamma - 1$, we have

$$A_1 \le \frac{KR^2}{8r} \int_{t-r}^{t+r} \frac{dy}{(1+y)^{\gamma-1}} \le \frac{KR^2}{8r} \int_{t-r}^{t+r} \frac{dy}{(1+y)^{\beta^*+1}}. \tag{53}$$

Moreover if $\delta^* = \gamma + \delta - 2 - \beta^* > 1 \ge \delta$ then

$$\begin{aligned} A_2 &\le \frac{K}{8r} \int_0^{t-r} dz \int_{t-r}^{t+r} \frac{dy}{(1+y)^{\gamma-1}(1+z)^{\delta}} \\ &\le \frac{K}{8r} \int_0^{t-r} dz \int_{t-r}^{t+r} \frac{dy}{(1+y)^{\beta^*+1}(1+z)^{\delta^*}}, \end{aligned}$$

and thus we obtain

$$A_2 \le \frac{K}{8(\delta^* - 1)r} \int_{t-r}^{t+r} \frac{dy}{(1+y)^{\beta^*+1}}. \tag{54}$$

By (53) and (54) we get

$$\int_{R_{r,t}} \frac{\lambda}{2r} \tilde{F}(\lambda, s)\, ds\, d\lambda \le \frac{K}{8r}(R^2 + \frac{1}{\delta^* - 1}) \int_{t-r}^{t+r} \frac{dy}{(1+y)^{\beta^*+1}}. \tag{55}$$

If $1+t-r \le (1+t+r)/2$ we have that $1/r \le 1/(1+t+r)$, hence

$$\begin{aligned} \frac{1}{2r} \int_{t-r}^{t+r} \frac{dy}{(1+y)^{\beta^*+1}} &\le \frac{1}{2(\beta^*+1)} \frac{1}{r(1+t-r)^{\beta^*}} \\ &\le \frac{2}{(\beta^*+1)} \frac{1}{(1+t+r)(1+t-r)^{\beta}}. \end{aligned} \tag{56}$$

If on the contrary $1+t-r > (1+t+r)/2$, again by using the mean value theorem we obtain

$$\frac{1}{2r} \int_{t-r}^{t+r} \frac{dy}{(1+y)^{\beta^*+1}} \le \frac{1}{(1+t-r)^{\beta^*+1}} \le \frac{2}{(1+t+r)(1+t-r)^{\beta}}. \tag{57}$$

(55), (56) and (57) give (42) in this case. The proof of Lemma 4.1 is complete.

Proof of Theorem 2.2. In view of the symmetry of the problem, we suppose that $1 < p \leq q$. We set

$$U_{-1} = W(\varepsilon f_1, \varepsilon g_1), \quad V_{-1} = W(\varepsilon f_2, \varepsilon g_2), \tag{58}$$

and, for all $j \in \mathbf{N}$,

$$\begin{aligned} U_j &= W(\varepsilon f_1, \varepsilon g_1) + L(|V_{j-1}|^p), \\ V_j &= W(\varepsilon f_2, \varepsilon g_2) + L(|U_{j-1}|^q). \end{aligned} \tag{59}$$

A consequence of strong Huygen's principle is that (see [7, p. 253]) for all $\sigma > 0$ there exists $C_\sigma > 0$, C_σ independent of ε, such that

$$\|\tau_+ \tau_-^\sigma W(\varepsilon f_1, \varepsilon g_1)\|_{L^\infty} \leq \varepsilon C_\sigma, \tag{60}$$

for all $\varepsilon > 0$.

First suppose that

$$pq - 2q - 1 > 0. \tag{61}$$

Since $1 < p \leq q$, (61) implies that $p > 2$. From (60) we deduce that there exists $C_1 \geq 1$, C_1 not depending on ε such that

$$\|\tau_+ \tau_-^{p-2} U_{-1}\|_{L^\infty} + \|\tau_+ \tau_-^{q-2} V_{-1}\|_{L^\infty} \leq \varepsilon C_1. \tag{62}$$

We claim that there exists $\varepsilon_0 \in]0,1[$ such that for all $\varepsilon \in [0, \varepsilon_0[$ and $j \in \mathbf{N}$ we have

$$\|\tau_+ \tau_-^{p-2} U_j\|_{L^\infty} + \|\tau_+ \tau_-^{q-2} V_j\|_{L^\infty} \leq 2\varepsilon C_1. \tag{63}$$

In order to prove (63) we shall use the induction principle. Suppose that (63) holds for $j = j_0$. Then

$$\|\tau_+^q \tau_-^{q(p-2)} |U_{j_0}|^q\|_{L^\infty} + \|\tau_+^p \tau_-^{p(q-2)} |V_{j_0}|^p\|_{L^\infty} \leq 2^{q+1} C_1^q \varepsilon^p. \tag{64}$$

Since p, $q > 2$, $pq - 2q > 1$ and $pq - 2p > 1$, the hypotheses of Lemma 4.1, *case i)* are satisfied. As a consequence, (38) implies

$$\|\tau_+ \tau_-^{q-2} L(|U_{j_0}|^q)\|_{L^\infty} + \|\tau_+ \tau_-^{p-2} L(|V_{j_0}|^p)\|_{L^\infty} \leq 2^{q+1} C C_1^q \varepsilon^p, \tag{65}$$

and from (62) and (65) we have

$$\|\tau_+\tau_-^{q-2}U_{j_0+1}\|_{L^\infty} + \|\tau_+\tau_-^{p-2}V_{j_0+1}\|_{L^\infty} \le \varepsilon C_1(1 + 2^{q+1}CC_1^{q-1}\varepsilon^{p-1}). \tag{66}$$

To prove (63) it is enough to to choose ε_0 such that

$$1 + 2^{q+1}CC_1^{q-1}\varepsilon_0^{p-1} \le 2. \tag{67}$$

Now we consider the case

$$2 < p < q, \quad p^2q - pq - 3p - 1 > 0 \quad \text{and} \quad pq - 2q - 1 \le 0. \tag{68}$$

By choosing $\mu < 1$ such that

$$\mu p^2 q - pq - 3p - 1 > 0, \tag{69}$$

and using (60) it follows that there exists $C_1 \ge 1$ satisfying

$$\|\tau_+\tau_-^{p-2}U_{-1}\|_{L^\infty} + \|\tau_+\tau_-^{\mu pq-q-3}V_{-1}\|_{L^\infty} \le \varepsilon C_1. \tag{70}$$

We claim that there exists $\varepsilon_0 \in]0,1[$ such that for all $\varepsilon \in [0,\varepsilon_0[$ and $j \in \mathbf{N}$ we have

$$\|\tau_+\tau_-^{p-2}U_j\|_{L^\infty} + \|\tau_+\tau_-^{\mu pq-q-3}V_j\|_{L^\infty} \le 2\varepsilon C_1. \tag{71}$$

Suppose that (71) holds for $j = j_0$. Then

$$\|\tau_+^q\tau_-^{q(p-2)}|U_{j_0}|^q\|_{L^\infty} + \|\tau_+^p\tau_-^{p(\mu pq-q-3)}|V_{j_0}|^p\|_{L^\infty} \le 2^{q+1}C_1^q\varepsilon^p. \tag{72}$$

By Lemma 4.1 *case iii)* we have

$$\|\tau_+\tau_-^{\mu pq-q-3}L(|U_{j_0}|^q)\|_{L^\infty} + \le 2^{q+1}CC_1^q\varepsilon^p, \tag{73}$$

and by Lemma 4.1 *case i)* we obtain

$$\|\tau_+\tau_-^{p-2}L(|V_{j_0}|^p)\|_{L^\infty} \le 2^{q+1}CC_1^q\varepsilon^p. \tag{74}$$

From (70), (73) and (74) we get (71) for $j = j_0 + 1$ taking ε_0 as in (67).

Let us finally consider the case

$$p \le 2, \quad p^2q - pq - 3p - 1 > 0 \quad \text{and} \quad pq - 2q - 1 \le 0. \tag{75}$$

We choose $0 < \mu,\ \nu < 1$ such that

$$\nu < p - 1 \quad \text{and} \quad \nu\mu pq - 3p > 1. \tag{76}$$

From (60) we deduce that there exists $C_1 \geq 1$ satisfying

$$\|\tau_+^{\nu} U_{-1}\|_{L^\infty} + \|\tau_+ \tau_-^{\nu\mu q - 3} V_{-1}\|_{L^\infty} \leq \varepsilon C_1. \tag{77}$$

By the usual recursive argument we prove that there exists $\varepsilon_0 \in]0, 1[$ such that for all $\varepsilon \in [0, \varepsilon_0[$ and $j \in \mathbf{N}$ we have

$$\|\tau_+^{\nu} U_j\|_{L^\infty} + \|\tau_+ \tau_-^{\nu\mu q - 3} V_j\|_{L^\infty} \leq 2\varepsilon C_1, \tag{78}$$

Indeed if (78) holds for $j = j_0$ then

$$\|\tau_+^{q\nu} |U_{j_0}|^q\|_{L^\infty} + \|\tau_+^{p} \tau_-^{p(\nu\mu q - 3)} |V_{j_0}|^p\|_{L^\infty} \leq 2^{q+1} \varepsilon^p C_1^q. \tag{79}$$

Consequently, by Lemma 4.1 *case iii)* and by Lemma 4.1 *case ii)* we have respectively

$$\|\tau_+ \tau_-^{\nu\mu q - 3} L(|U_{j_0}|^q)\|_{L^\infty} \leq 2^{q+1} \varepsilon^p C_1^q, \tag{80}$$

$$\|\tau_+^{\nu} L(|V_{j_0}|^p)\|_{L^\infty} \leq 2^{q+1} \varepsilon^p C_1^q. \tag{81}$$

As a consequence, from (80), (81) and (62), inequality (78) follows for $j = j_0 + 1$.

The claims (63), (71) and (78) imply that the sequences (U_j), (V_j) are bounded in some weighted L^∞ space. Our goal is to show that these sequences are indeed convergent in the same space if ε is sufficiently small.

Let us suppose that (61) holds. From (63) we deduce that there exists $C_2 \geq 1$ such that

$$\|\tau_+ \tau_-^{p-2} (U_0 - U_{-1})\|_{L^\infty} + \|\tau_+ \tau_-^{q-2} (V_0 - V_{-1})\|_{L^\infty} \leq \varepsilon C_2. \tag{82}$$

We claim that there exists $\varepsilon_1 \in]0, \varepsilon_0[$ such that for all $\varepsilon \in [0, \varepsilon_1[$ and $j \in \mathbf{N}$ we have

$$\|\tau_+ \tau_-^{p-2} (U_{j+1} - U_j)\|_{L^\infty} + \|\tau_+ \tau_-^{q-2} (V_{j+1} - V_j)\|_{L^\infty} \leq \frac{\varepsilon C_2}{2^j}. \tag{83}$$

Again we shall argue by induction. Suppose that (83) holds for $j = j_0$. By (63) we have that

$$\begin{aligned} &\|\tau_+\tau_-^{p-2}(U_{j_0+2} - U_{j_0+1})\|_{L^\infty} \\ &\quad = \|\tau_+\tau_-^{p-2} L(|V_{j_0+1}|^p - |V_{j_0}|^p)\|_{L^\infty} \\ &\quad \le C\|\tau_+^p\tau_-^{p(q-2)}(|V_{j_0+1}|^p - |V_{j_0}|^p)\|_{L^\infty} \\ &\quad \le C\|(\tau_+\tau_-^{q-2}|V_{j_0+1}|)^p - (\tau_+\tau_-^{q-2}|V_{j_0}|^p)\|_{L^\infty} \\ &\quad \le Cp\|\tau_+\tau_-^{q-2}(V_{j_0+1} - V_{j_0})\|_{L^\infty}\cdot \\ &\qquad\qquad \cdot(\|\tau_+\tau_-^{q-2}V_{j_0+1}\|_{L^\infty}^{p-1} + \|\tau_+\tau_-^{q-2}V_{j_0}\|_{L^\infty}^{p-1}) \\ &\quad \le Cp\frac{\varepsilon C_2}{2^{j_0}}(2^p\varepsilon^{p-1}C_1^{p-1}), \end{aligned} \tag{84}$$

and similarly

$$\|\tau_+\tau_-^{q-2}(V_{j_0+2} - V_{j_0+1})\|_{L^\infty} \le Cq\frac{\varepsilon C_2}{2^{j_0}}(2^q\varepsilon^{q-1}C_1^{q-1}). \tag{85}$$

Claim (83) for $j = j_0 + 1$ is a consequence of (84) and (85) provided ε_1 is chosen such that

$$2^{q+2}qCC_1^{q-1}\varepsilon_1^{p-1} < 1.$$

When (68) or (75) holds the argument is similar. It is immediate to verify that the limit of $((U_j, V_j))$ is the solution of the problem (16). The proof of Theorem 2.2 is complete.

5. Proof of Theorem 2.3

Let (U_j), (V_j) be the sequences defined in (58), (59). As seen in the proof of the Theorem 2.2 these sequences are convergent in a weighted L^∞ space. Let u and v be respectively the limit of (U_j) and (V_j). By the proof of Theorem 2.2 we know that (u, v) is the solution of the integral equation associated to (17) and (18). We prove now that if p, $q \ge 2$ and ε_0 is sufficiently small, then the x–derivatives of order ≤ 2 of u and v exist and are continuous. By [7, Lemma I], this fact implies that (u, v) is the unique $\mathcal{C}^2$ solution of (17), (18).

Since U_{-1}, $V_{-1} \in \mathcal{C}^\infty$ and p, $q \geq 2$ we have U_j, $V_j \in \mathcal{C}^2$ for all $j \in \mathbf{N}$ and

$$\begin{aligned}
\partial_{x_k} U_j &= \partial_{x_k} U_{-1} + L(p|V_{j-1}|^{p-2} V_{j-1} \partial_{x_k} V_{j-1}),\\
\partial_{x_k} V_j &= \partial_{x_k} V_{-1} + L(p|U_{j-1}|^{p-2} U_{j-1} \partial_{x_k} U_{j-1}),\\
\partial_{x_k}\partial_{x_h} U_j &= \partial_{x_k}\partial_{x_h} U_{-1} + L(p(p-1)|V_{j-1}|^{p-2} \partial_{x_k} V_{j-1} \partial_{x_h} V_{j-1} +\\
&\quad + p|V_{j-1}|^{p-2} V_{j-1} \partial_{x_k}\partial_{x_h} V_{j-1}),\\
\partial_{x_k}\partial_{x_h} V_j &= \partial_{x_k}\partial_{x_h} V_{-1} + L(p(p-1)|U_{j-1}|^{p-2} \partial_{x_k} U_{j-1} \partial_{x_h} U_{j-1} +\\
&\quad + p|U_{j-1}|^{p-2} U_{j-1} \partial_{x_k}\partial_{x_h} U_{j-1}).
\end{aligned}$$

Without loss of generality we suppose that $1 < p \leq q$.

Let first consider the case (61). By using the explicit form of U_{-1} and V_{-1} together with (60), we infer that there exists $C_3 > 0$, C_3 not depending on ε such that

$$\|\tau_+ \tau_-^{p-2} \partial_{x_k} U_{-1}\|_{L^\infty} + \|\tau_+ \tau_-^{q-2} \partial_{x_k} V_{-1}\|_{L^\infty} \leq \varepsilon C_3, \tag{86}$$

and

$$\|\tau_+ \tau_-^{p-2} \partial_{x_k}\partial_{x_h} U_{-1}\|_{L^\infty} + \|\tau_+ \tau_-^{q-2} \partial_{x_k}\partial_{x_h} V_{-1}\|_{L^\infty} \leq \varepsilon C_3. \tag{87}$$

We claim that there exists $\varepsilon_2 \in]0, \varepsilon_1[$ such that for all $\varepsilon \in [0, \varepsilon_2[$ and $j \in \mathbf{N}$ we have

$$\|\tau_+ \tau_-^{p-2} \partial_{x_k} U_j\|_{L^\infty} + \|\tau_+ \tau_-^{q-2} \partial_{x_k} V_j\|_{L^\infty} \leq 2\varepsilon C_3, \tag{88}$$

and

$$\|\tau_+ \tau_-^{p-2} \partial_{x_k}\partial_{x_h} U_j\|_{L^\infty} + \|\tau_+ \tau_-^{q-2} \partial_{x_k}\partial_{x_h} V_j\|_{L^\infty} \leq 2\varepsilon C_3. \tag{89}$$

We argue by induction. Let (88) be true for $j = j_0$. By Lemma 4.1 *case i)* we have

$$\begin{aligned}
&\|\tau_+ \tau_-^{p-2} L(p|V_{j_0}|^{p-2} V_{j_0} \partial_{x_k} V_{j_0})\|_{L^\infty}\\
&\qquad \leq C\|\tau_+^p \tau_-^{p(q-2)} (p|V_{j_0}|^{p-1} \partial_{x_k} V_{j_0})\|_{L^\infty}\\
&\qquad \leq pC\|\tau_+ \tau_-^{q-2} |V_{j_0}|\|_{L^\infty}^{p-1} \|\tau_+ \tau_-^{q-2} \partial_{x_k} V_{j_0}\|_{L^\infty},
\end{aligned} \tag{90}$$

and

$$\begin{aligned}\|\tau_+\tau_-^{q-2}L(q|U_{j_0}|^{q-2}U_{j_0}\partial_{x_k}U_{j_0})\|_{L^\infty} \\ \leq qC\|\tau_+\tau_-^{p-2}|U_{j_0}|\|_{L^\infty}^{q-1}\|\tau_+\tau_-^{p-2}\partial_{x_k}U_{j_0}\|_{L^\infty}.\end{aligned} \tag{91}$$

From (90) and (91), by using (63) and (86), for all $\varepsilon \in]0,\varepsilon_1[$, we have that

$$\begin{aligned}\|\tau_+\tau_-^{p-2}\partial_{x_k}U_{j_0+1}\|_{L^\infty} + \|\tau_+\tau_-^{q-2}\partial_{x_k}V_{j_0+1}\|_{L^\infty} \\ \leq \varepsilon C_3(1+\varepsilon^{p-1}2^{q+2}qCC_1^{q-1}),\end{aligned}$$

hence (88) follows for $j = j_0+1$, by choosing ε_2 sufficiently small. Suppose now that (89) holds for $j = j_0$. We have

$$\begin{aligned}&\|\tau_+\tau_-^{p-2}L(p(p-1)|V_{j_0}|^{p-2}\partial_{x_k}V_{j_0}\partial_{x_h}V_{j_0} + p|V_{j_0}|^{p-2}V_{j_0}\partial_{x_k}\partial_{x_h}V_{j_0})\|_{L^\infty} \\ &\leq \|\tau_+\tau_-^{p-2}L(p(p-1)|V_{j_0}|^{p-2}\partial_{x_k}V_{j_0}\partial_{x_h}V_{j_0})\|_{L^\infty} \\ &\qquad + \|\tau_+\tau_-^{p-2}L(p|V_{j_0}|^{p-1}\partial_{x_k}\partial_{x_h}V_{j_0})\|_{L^\infty} \\ &\leq p(p-1)C\|\tau_+^p\tau_-^{p(q-2)}|V_{j_0}|^{p-2}\partial_{x_k}V_{j_0}\partial_{x_h}V_{j_0}\|_{L^\infty} \\ &\qquad + pC\|\tau_+^p\tau_-^{p(q-2)}|V_{j_0}|^{p-1}\partial_{x_k}\partial_{x_h}V_{j_0}\|_{L^\infty} \\ &\leq p(p-1)C\|\tau_+\tau_-^{q-2}V_{j_0}\|_{L^\infty}^{p-2}\|\tau_+\tau_-^{q-2}\partial_{x_k}V_{j_0}\|_{L^\infty}\|\tau_+\tau_-^{q-2}\partial_{x_h}V_{j_0}\|_{L^\infty} \\ &\qquad + pC\|\tau_+\tau_-^{q-2}V_{j_0}\|_{L^\infty}^{p-1}\|\tau_+\tau_-^{q-2}\partial_{x_k}\partial_{x_h}V_{j_0}\|_{L^\infty}.\end{aligned} \tag{92}$$

and

$$\begin{aligned}&\|\tau_+\tau_-^{q-2}L(q(q-1)|U_{j_0}|^{q-2}\partial_{x_k}U_{j_0}\partial_{x_h}U_{j_0} + q|U_{j_0}|^{q-2}U_{j_0}\partial_{x_k}\partial_{x_h}U_{j_0})\|_{L^\infty} \\ &\leq q(q-1)C\|\tau_+\tau_-^{p-2}U_{j_0}\|_{L^\infty}^{q-2}\|\tau_+\tau_-^{p-2}\partial_{x_k}U_{j_0}\|_{L^\infty}\|\tau_+\tau_-^{p-2}\partial_{x_h}U_{j_0}\|_{L^\infty} \\ &\qquad + qC\|\tau_+\tau_-^{p-2}U_{j_0}\|_{L^\infty}^{q-1}\|\tau_+\tau_-^{p-2}\partial_{x_k}\partial_{x_h}U_{j_0}\|_{L^\infty}.\end{aligned} \tag{93}$$

Finally from (92) and (93), by using (63), (87) and (88) with $j = j_0+1$, we get

$$\begin{aligned}\|\tau_+\tau_-^{p-2}\partial_{x_k}\partial_{x_h}U_{j_0+1}\|_{L^\infty} + \|\tau_+\tau_-^{q-2}\partial_{x_k}\partial_{x_h}V_{j_0+1}\|_{L^\infty} \\ \leq \varepsilon C_3(1+\varepsilon^{p-1}(2^{q+1}(q(q-1)C_1^{q-2}C_3 + qC_1^{q-1}))),\end{aligned}$$

and by choosing a smaller value for ε_2, (89) follows for $j = j_0 + 1$. In order to complete the proof of Theorem 2.3 when (61) holds, we need only to check that the sequences $(\partial_{x_k} U_j)$, $(\partial_{x_k} V_j)$ $(\partial_{x_k}\partial_{x_h} U_j)$ $(\partial_{x_k}\partial_{x_h} V_j)$ are convergent. This will be a consequence of the following fact: there exists $C_4 > 0$ and $\varepsilon_3 \in]0, \varepsilon_2[$ such that for all $\varepsilon \in]0, \varepsilon_3[$ and $j \in \mathbf{N}$ we have

$$\begin{aligned} &\|\tau_+ \tau_-^{p-2}(\partial_{x_k} U_j - \partial_{x_k} U_{j+1})\|_{L^\infty} + \\ &\quad + \|\tau_+ \tau_-^{q-2}(\partial_{x_k} V_j - \partial_{x_k} V_{j+1})\|_{L^\infty} \leq \frac{\varepsilon C_4}{2^j}, \end{aligned} \tag{94}$$

and

$$\begin{aligned} &\|\tau_+ \tau_-^{p-2}\partial_{x_k}(\partial_{x_h} U_j - \partial_{x_k}\partial_{x_h} U_{j+1})\|_{L^\infty} + \\ &\quad + \|\tau_+ \tau_-^{q-2}(\partial_{x_k}\partial_{x_h} V_j - \partial_{x_k}\partial_{x_h} V_{j+1})\|_{L^\infty} \leq \frac{\varepsilon C_4}{2^j}. \end{aligned} \tag{95}$$

If (68) and (69) are satisfied, inequalities (88), (89), (94), (95) follow, with the choice $\tau_+ \tau_-^{p-2}$ and $\tau_+ \tau_-^{\mu pq - q - 3}$.

Finally if $p = 2$ and $q > 7/2$ then (88), (89), (94) and (95) are proved with the weights $\tau_+ \tau_-^{p-2}$ and $\tau_+ \tau_-^{q-2}$ replaced in by τ_+^{ν} and $\tau_+ \tau_-^{\mu\nu q - 3}$ respectively.

The remaining part of the proof consists in a quite long and straightforward computation. We omit the details.

A. Appendix

For completeness we state here two uniqueness results for a class of hyperbolic systems. The interested reader may refer to [8, Th. 4, Th. 4a] for the proofs of these results in the case of the scalar equation. The proofs in the vector valued case proceed similarly.

THEOREM A.1. *Let $\tau > 0$ and $x_0 \in \mathbb{R}^n$. Let F and G be two real valued $\mathcal{C}^1$ functions such that*

$$F(0) = F'(0) = G(0) = G'(0) = 0. \tag{96}$$

Let (u, v) be a $\mathcal{C}^2$ solution of

$$\begin{aligned} \partial_t^2 u - \Delta_x u &= F(v) \\ \partial_t^2 v - \Delta_x v &= G(u) \end{aligned} \tag{97}$$

in the conic region $R_{x_0,\tau} = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} \,:\, t+|x-x_0| \leq \tau,\ t \geq 0\}$.

Suppose that

$$u(x,0) = \partial_t u(x,0) = v(x,0) = \partial_t v(x,0) = 0 \tag{98}$$

for all $x \in \mathbb{R}^n$ *such that* $|x - x_0| \leq \tau$.

Then $u(x_0,\tau) = v(x_0,\tau) = 0$.

THEOREM A.2. *Let* σ, $\tau > 0$. *Let* F *and* G *be two real valued* $\mathcal{C}^1$ *functions satisfying (96). Let* (u,v) *be a* $\mathcal{C}^2$ *solution of (97) in the strip* $S_\tau = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} \,:\, 0 \leq t \leq \tau\}$.

Suppose that (98) holds in the set $\{x \in \mathbb{R}^n \,:\, |x| \geq \sigma\}$.

Then $u(x,t) = v(x,t) = 0$ *for all* $(x,t) \in S_\tau$ *such that* $|x| \geq \sigma + t$.

REFERENCES

[1] ALINHAC S., "Blowup for Nonlinear Hyperbolic Equations", PNLDE **17**, Birkhäuser, Boston 1995.

[2] DEL SANTO D., GEORGIEV V. and MITIDIERI E., *Global existence of the solutions and formation of singularities for a class of hyperbolic systems*, in "Geometrical Optics and Related Topics", F. Colombini and N. Lerner eds., PNLDE **32**, Birkhäuser, Boston 1997, pp. 117–140.

[3] GEORGIEV V., LINDBLAD H. and SOGGE C., *Weighted Strichartz estimate and global existence for semilinear wave equation*, Amer. J. Math., to appear.

[4] GLASSEY R., *Finite–time blow–up for solutions of nonlinear wave equations*, Math. Z. **177** (1981), 323–340.

[5] GLASSEY R., *Existence in the large for* $\Box u = F(u)$ *in two space dimensions*, Math. Z. **178** (1981), 233–261.

[6] JOHN F., "Plane Waves and Spherical Means Applied to Partial Differential Equations" Interscience, New York, 1955.

[7] JOHN F., *Blow–up of solutions of nonlinear wave equations in three space dimensions*, Manuscripta Math. **28** (1979), 235–268.

[8] JOHN F., *Blow–up for quasi–linear wave equations in three space dimensions*, Comm. Pure Appl. Math. **34** (1981), 29–51.

[9] JOHN F., "Nonlinear Wave Equations, Formation of Singularities" University lecture series, American Mathematical Society, Providence, 1990.

[10] SIDERIS T., *Nonexistence of global solutions to semilinear wave equations in high dimensions*, J. Differential Eq. **52** (1984), 378–406.

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