Rend. Istit. Mat. Univ. Trieste
Vol. XXIX, 45–54 (1997)

Joint Entropy and Gaussian Functions

Eduardo Pascali and Carlo Sempi (*)

Summary. - *An old conjecture of Hirschmann's claimed that the absolute maximum of the "joint entropy" functional E occurs for the real gaussian functions; here we show that these functions are only saddle points for the functional E.*

Introduction

In an old paper [1], Hirschman showed that for every function $\phi : \mathbf{R} \to \mathbf{C}$ such that

$$\int_{\mathbf{R}} |\phi(x)|^2 \, dx = 1$$

the functional

$$E(\phi) := \int_{\mathbf{R}} |\phi(x)|^2 \, \log|\phi(x)|^2 \, dx + \int_{\mathbf{R}} |\hat{\phi}(x)|^2 \, \log|\hat{\phi}(x)|^2 \, dx$$

has a non–positive value provided its definition has a meaning. Here $\hat{\phi}$ is the Fourier transform of ϕ. That author also conjectured that the absolute maximum of the functional E occurs for the real gaussian functions.

The functional E is encountered in Information Theory where it is called "joint entropy".

Later Leipnik [2] claimed that the maximum for E is achieved by the complex gaussian functions

$$\gamma_{\alpha,\beta}(x) := c(\alpha) \, e^{-\alpha x^2 + i\beta x}, \tag{1}$$

(*) Authors' address: Dipartimento di Matematica, Università di Lecce, 73100 Lecce (Italy).

where $\alpha > 0$ and $c(\alpha)$ is a normalization constant.

Below we show that the above claim is not true and that the real gaussian functions

$$\phi_\alpha(x) = c(\alpha)\, e^{-\alpha x^2} \tag{2}$$

give only a saddle point for the functional E. In retrospect, one can see that, in [2], only the stationarity of the functional E was investigated and not its maximum properties. Moreover there is no difference in considering the functions $\gamma_{\alpha,\beta}$ of (1) instead of the real gaussian functions ϕ_α, since $|\gamma_{\alpha,\beta}|^2 = |\phi_\alpha|^2$ and $|\hat{\gamma}_{\alpha,\beta}|^2 = |\hat{\phi}_\alpha|^2$. Thus the claim of [2] could have been replaced by the other one that the absolute maximum of the functional E occurs for the real gaussian functions (2). This would have proved Hirschman's conjecture. However, we shall prove that this is not the case. We still do not know which functions provide the absolute maximum of the functional E. This remains, to the best of our knowledge, an open question.

The main result

On the measure space $(\mathbf{R}, \mathcal{L}, \lambda)$ where λ is the Lebesgue measure on the Lebesgue measurable sets $\mathcal{L}$ of $\mathbf{R}$, we consider the complex–valued functions $\phi \in L^1 \cap L^2$; these have a Fourier transform $\hat{\phi}$ given by

$$\hat{\phi}(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} \phi(x)\, e^{itx}\, dx$$

We recall that L^2 is a Hilbert space with inner product and norm defined respectively by

$$\langle \phi, \psi \rangle := \int_{\mathbf{R}} \phi(x)\, \overline{\psi}(x)\, dx$$

and by

$$\|\phi\|_2^2 := \langle \phi, \phi \rangle\,.$$

The orthogonality condition $\langle \phi, \psi \rangle = 0$ is equivalent to the other one

$$\int_{\mathbf{R}} \Re\left(\phi\, \overline{\psi}\right)\, d\lambda = 0.$$

Let us introduce the subset M of $L^1 \cap L^2$ defined by

$$M := \left\{ \phi \in L^1 \cap L^2 : \|\phi\|_2 = 1 \right\},$$

let us denote by $E(\cdot, M)$ the restriction of the functional E to M and by $\mathcal{A}$ the set of *admissible* functions, i.e. the subset of M on which both terms in the definition of E are finite,

$$\mathcal{A} := \{\phi \in \mathcal{M} : |\mathcal{E}(\phi)| < +\infty\}.$$

LEMMA 1. *For every sufficiently regular $\phi, \psi \in \mathcal{A}$ such that $\langle \phi, \psi \rangle = 0$, let $\epsilon := \|\psi\|_2^2$ and $\beta(t) := 1 - \sqrt{1 - \epsilon\, t^2}$; then $(1 - \beta(t))\, \phi + t\, \psi \in M$ and*

$$\begin{aligned} &\frac{d}{dt} E\left((1 - \beta(t))\, \phi + t\, \psi, M\right)\Big|_{t=0} \\ &= \int_{\mathbf{R}} \left(\phi(x)\overline{\psi}(x) + \overline{\phi}(x)\psi(x)\right) \log |\phi(x)|^2 \, dx \\ &\quad + \int_{\mathbf{R}} \left(\hat{\phi}(x)\overline{\hat{\psi}}(x) + \overline{\hat{\phi}}(x)\hat{\psi}(x)\right) \log |\hat{\phi}(x)|^2 \, dx \end{aligned}$$

Proof. We shall only consider the first term E_1 in the definition of E and exploit the symmetry between the first and the second term of E.

It is immediate to check that, with the given expressions for ϵ and β, $(1 - \beta(t))\, \phi + t\, \psi$ belongs to M. Notice that

$$\beta'(t) = \frac{\epsilon\, t}{\sqrt{1 - \epsilon\, t^2}}, \quad \beta''(t) = \frac{\epsilon}{(1 - \epsilon\, t^2)^{3/2}}$$

and hence $\beta(0) = \beta'(0) = 0$ and $\beta''(0) = \epsilon$.

Setting

$$h(t) := |(1 - \beta(t))\, \phi + t\psi|^2 \quad \text{and} \quad g(t) := h(t)\, \log h(t),$$

one has

$$\begin{aligned} h(t) &= (1 - \beta(t))^2\, |\phi|^2 + t^2\, |\psi|^2 + 2t\, (1 - \beta(t))\, \Re(\phi\, \overline{\psi}), \\ h'(t) &= -2\, (1 - \beta(t))\, \beta'(t)\, |\phi|^2 + 2t\, |\psi|^2 \\ &\quad + \left(\phi\overline{\psi} + \overline{\phi}\psi\right)\, (1 - \beta(t) - t\, \beta'(t)), \\ g'(t) &= h'(t)\, (1 + \log h(t)) \\ g'(0) &= \left(\phi\overline{\psi} + \overline{\phi}\psi\right)\, (\log |\phi|^2 + 1). \end{aligned}$$

The condition $\langle \phi, \psi \rangle = 0$ implies, by the Plancherel formula (see, e.g., [3]), $\langle \hat{\phi}, \hat{\psi} \rangle = 0$; moreover, differentiating under the integral sign

$$\frac{d}{dt} E_1\left((1-\beta(t))\,\phi + t\,\psi, M\right)\Big|_{t=0} = \int_{\mathbf{R}} g'(0)\,dx$$

$$= \int_{\mathbf{R}} \left(\phi(x)\overline{\psi}(x) + \overline{\phi}(x)\psi(x)\right)\,\log|\phi(x)|^2\,dx.$$

□

Remark 1. *If $\phi = \overline{\phi}$, $\hat{\phi} = \overline{\hat{\phi}}$, $\psi = \overline{\psi}$, and $\psi(x) = -\psi(-x)$, then*

$$\frac{d}{dt} E\left((1-\beta(t))\,\phi + t\,\psi, M\right)\Big|_{t=0} = 2\int_{\mathbf{R}} \left(\phi(x)\,\psi(x)\right)\,\log|\phi(x)|^2\,dx$$

in order to see this it suffices to note that, in this conditions, $\hat{\psi} + \overline{\hat{\psi}} = 0$.

Remark 2. *With the previous notation we have:*

$$h''(t) = -2\,|\phi|^2\,\left(\beta'' - \beta\beta'' - \beta'^2\right) + 2\,|\psi|^2 - \left(\phi\,\overline{\psi} + \overline{\phi}\,\psi\right)\,(2\,\beta' + t\,\beta''),$$

and

$$g''(t) = -\left\{2\,|\phi|^2\,\left(\beta'' - \beta\,\beta'' - \beta'^2\right) + 2\,|\psi|^2 - \left(\phi\,\overline{\psi} + \overline{\phi}\,\psi\right)\,(2\,\beta' + t\,\beta'')\right\}\cdot$$
$$\cdot\,(1 + \log h(t))$$
$$+ \frac{\left(-2\,|\phi|^2\,(\beta' - \beta\,\beta') + 2t\,|\psi|^2 - \left(\phi\,\overline{\psi} + \overline{\phi}\,\psi\right)\,(1 - \beta + t\,\beta')\right)^2}{|(1-\beta(t))\,\phi + t\,\psi|^2}.$$

Lemma 2. *For every pair of sufficiently regular functions $\phi, \psi \in \mathcal{A}$ such that $\langle \phi, \psi \rangle = 0$, one has*

$$\frac{d^2}{dt^2} E\left((1-\beta(t))\,\phi + t\,\psi, M\right)\Big|_{t=0} = -2\,\|\psi\|_2^2\,E(\phi, M) +$$
$$+ 2\left(\int_{\mathbf{R}} |\psi(x)|^2\,\log|\phi(x)|^2\,dx + \int_{\mathbf{R}} |\hat{\psi}(x)|^2\,\log|\hat{\phi}(x)|^2\,dx\right) +$$
$$+ \int_{\mathbf{R}} \frac{\left(\phi(x)\,\overline{\psi}(x) + \overline{\phi}(x)\,\psi(x)\right)^2}{|\phi(x)|^2}\,dx + \int_{\mathbf{R}} \frac{\left(\hat{\phi}(s)\,\overline{\hat{\psi}}(s) + \overline{\hat{\phi}}(s)\,\hat{\psi}(s)\right)^2}{|\phi(s)|^2}\,ds.$$

Proof. We shall consider the functional E_1 as in the proof of Lemma 1, from which we have

$$\begin{aligned}
&\frac{d}{dt} E_1\left((1-\beta(t))\ \phi + t\,\psi, M\right) \\
&\quad = \int_{\mathbf{R}} \left\{2\ (\beta\,\beta' - \beta')\ |\phi|^2 + 2t\,|\psi|^2 + (1-\beta-t\,\beta')\right\}\cdot \\
&\quad\quad \cdot\left(\phi\overline{\psi} + \psi\overline{\phi}\right)(1+\log h(t))\ dx \\
&\quad = \int_{\mathbf{R}} \left\{2\ (\beta\,\beta' - \beta')\ |\phi|^2 + 2t\,|\psi|^2 + (1-\beta-t\,\beta')\right\}\cdot \\
&\quad\quad \cdot\left(\phi\overline{\psi} + \psi\overline{\phi}\right)\log h(t)\,dx + 2\int_{\mathbf{R}} \left\{(\beta\,\beta' - \beta')\ |\phi|^2 + t\,|\psi|^2\right\}\,dx.
\end{aligned}$$

Hence

$$\begin{aligned}
&\frac{d^2}{dt^2} E_1\left((1-\beta(t))\,\phi + t\,\psi, M\right)\Big|_{t=0} \\
&= \int_{\mathbf{R}} \left\{-2\,|\phi|^2\ \left(\beta''(t) - \beta(t)\beta''(t) - \beta'^2(t)\right) + 2\,|\psi|^2\right\}\,\log h(t)\,dx\Big|_{t=0} + \\
&\quad - \int_{\mathbf{R}} \left(\phi\overline{\psi} + \psi\overline{\phi}\right)\ (2\beta'(t) + t\,\beta''(t))\ \log h(t)\,dx\Big|_{t=0} + \\
&\quad + \int_{\mathbf{R}} \rho(t,x)\ |(1-\beta(t))\ \phi + t\,\psi|^{-2}\ dx\Big|_{t=0} + \\
&\quad + \int_{\mathbf{R}} \left\{-2\,|\phi|^2\ \left(\beta''(t) - \beta(t)\beta''(t) - \beta'^2(t)\right) + 2\,|\psi|^2\right\}\,dx\Big|_{t=0}
\end{aligned}$$

where

$$\rho(t,x) = \left\{-2\,|\phi|^2\ (\beta' - \beta\beta') + 2\,|\psi|^2 - \left(\phi\overline{\psi} + \psi\overline{\phi}\right)\ (1-\beta+t\,\beta')\right\}^2.$$

Then

$$\begin{aligned}
&\frac{d^2}{dt^2}E_1\left((1-\beta(t))\,\phi+t\,\psi,M\right)\Big|_{t=0}\\
&=\int_{\mathbf{R}}\left\{-2|\phi(x)|^2\,\|\psi\|_2^2+2\,|\phi(x)|\right\}\,\log|\phi(x)|^2\,dx+\\
&\quad+\int_{\mathbf{R}}\left\{-2|\phi(x)|^2\,\|\psi\|_2^2+2\,|\phi(x)|\right\}\,dx+\\
&\quad+\int_{\mathbf{R}}\frac{\left(\phi(x)\overline{\psi}(x)+\psi(x)\overline{\phi}(x)\right)^2}{|\phi(x)|^2}\,dx\\
&=-2\,\|\psi\|_2^2\int_{\mathbf{R}}|\phi(x)|^2\,\log|\phi(x)|^2\,dx+\\
&\quad+2\int_{\mathbf{R}}|\psi(x)|^2\,\log|\phi(x)|^2\,dx+\\
&\quad+\int_{\mathbf{R}}\frac{\left(\phi(x)\overline{\psi}(x)+\psi(x)\overline{\phi}(x)\right)^2}{|\phi(x)|^2}\,dx
\end{aligned}$$

This proves the assertion. □

REMARK 3. *If* $\phi=\overline{\phi}$, $\hat{\phi}=\overline{\hat{\phi}}$, $\psi=\overline{\psi}$ *and* $\phi(x)=-\phi(-x)$, *then*

$$\begin{aligned}
&\frac{d^2}{dt^2}E\left((1-\beta(t))\;\phi+t\,\psi,M\right)\Big|_{t=0}=2\,\|\psi\|_2^2\;(2-E(\phi,M))+\\
&\quad+2\int_{\mathbf{R}}|\psi(x)|^2\,\log|\phi(x)|^2\,dx+2\int_{\mathbf{R}}|\hat{\psi}(x)|^2\,\log|\hat{\phi}(x)|^2\,dx.
\end{aligned}$$

We now apply the two previous lemmata to the real gaussian functions

$$\phi_\alpha(x)=c(\alpha)\,e^{-\alpha x^2}\quad\text{where}\quad c(\alpha)=\left(\frac{2\,\alpha}{\pi}\right)^{1/4}\quad(\alpha>0). \tag{2}$$

Their Fourier transforms are

$$\hat{\phi}_\alpha(t)=b(\alpha)\,e^{-t^2/4\alpha}\quad\text{with}\quad b(\alpha)=(2\alpha\pi)^{-1/4};$$

both ϕ_α and $\hat{\phi}_\alpha$ belong to M. An easy calculation shows that, for every $\alpha>0$, $E(\phi_\alpha)=-\log(e\pi)$, which does not depend on the parameter α.

The following lemma establishes a sort of "stationarity" property for ϕ_α with respect to $E_1(\cdot, M)$.

LEMMA 3. *For every real gaussian function ϕ_α and for every function $\psi \in M$ that is real, odd and sufficiently regular, one has*

$$\frac{d}{dt} E\left((1-\beta(t))\ \phi_\alpha + t\,\psi, M\right)\Big|_{t=0} = 0.$$

Proof. Using the results of Lemma 1 and of Remark 1, one has

$$\begin{aligned}
&2 \int_{\mathbf{R}} \phi_\alpha(x)\,\psi(x)\ \log|\phi(x)|^2\,dx \\
&\quad = 2 \int_{\mathbf{R}} \phi_\alpha(x)\,\psi(x)\ \log\left(c^2(\alpha)\,e^{-2\alpha\,x^2}\right)\,dx \\
&\quad = 4\ \log\left(c(\alpha)\right) \int_{\mathbf{R}} \phi_\alpha(x)\,\psi(x)\,dx - 4 \int_{\mathbf{R}} x^2\,\phi_\alpha(x)\,\psi(x)\,dx \\
&\quad = -4\,\alpha\,c(\alpha) \int_{\mathbf{R}} x^2\,\psi(x)\,e^{-\alpha\,x^2}\,dx = 0,
\end{aligned}$$

since the integrand is odd. □

LEMMA 4. *For every gaussian function ϕ_α and for every sufficiently regular real odd function $\psi \in L^1 \cap L^2$ one has*

$$\begin{aligned}
&\frac{d^2}{dt^2} E\left((1-\beta(t))\ \phi_\alpha + t\,\psi, M\right)\Big|_{t=0} \\
&\qquad = 6\,\|\psi\|_2^2 - 4\alpha \int_{\mathbf{R}} x^2\,\psi^2(x)\,dx - \frac{1}{\alpha} \int_{\mathbf{R}} \psi'^2(x)\,dx
\end{aligned}$$

Proof. One has

$$\begin{aligned}
&\frac{d^2}{dt^2} E((1-\beta(t))\ \phi_\alpha + t\,\psi, M)\Big|_{t=0} \\
&\quad = 2\,\|\psi\|_2^2\ \{2 - E\left(\phi_\alpha, M\right)\} + 2 \int_{\mathbf{R}} \psi^2(x)\ \log\psi^2(x)\,dx + \\
&\qquad + 2 \int_{\mathbf{R}} |\hat{\psi}^2(x)|\ \log\phi^2(x)\,dx = 2\,\|\psi\|_2^2\ (3 + \log\pi) + 2\,I \\
&\quad = 6\,\|\psi\|_2^2 - 4\alpha \int_{\mathbf{R}} x^2\,\psi^2(x)\,dx - \frac{1}{\alpha} \int_{\mathbf{R}} \psi'^2(x)\,dx
\end{aligned}$$

where I stands for the expression

$$\|\psi\|_2^2 \log c^2(\alpha) + \int_{\mathbf{R}} \left(-2\alpha x^2\right) |\psi(x)|^2 \, dx + \\ + \|\psi\|_2^2 \log b^2(\alpha) + \frac{1}{2\alpha} \int_{\mathbf{R}} x^2 \, |\hat{\psi}(x)|^2 \, dx.$$

□

We shall now need the following family of auxiliary functions

$$\psi_{m,\xi}(x) := \begin{cases} -\dfrac{\xi}{m}(x+2m), & \text{if } x \in [-2m, -m], \\ \dfrac{\xi}{m}\, x, & \text{if } x \in [-m, m], \\ -\dfrac{\xi}{m}(x-2m), & \text{if } x \in [m, 2m], \\ 0, & \text{if } x \notin [-2m, 2m], \end{cases}$$

where m and ξ are strictly positive. The functions $\psi_{m,\xi}$ are real and odd so that to them one can apply Lemmata 3 and 4. We have

$$\|\psi_{m,\xi}\|_2^2 = \frac{4}{3}\, m\xi^2.$$

An easy calculation yields

$$A := 4\,\alpha \int_{\mathbf{R}} x^2 \, \psi^2(x)\, dx + \frac{1}{\alpha} \int_{\mathbf{R}} \psi'^2(x)\, dx = \frac{88}{15}\, \alpha \xi^2 m^3 + \frac{4\xi^2}{\alpha m}.$$

For every $\alpha > 0$, $m > 0$ and $\xi > 0$, we have from Lemma 4

$$\left.\frac{d^2}{dt^2} E\left((1-\beta(t))\ \phi_\alpha + t\, \psi_{m,\xi}, M\right)\right|_{t=0} = 4m\xi^2 \left[2 - \frac{22}{15}\, \alpha m^2 - \frac{1}{\alpha m^2}\right].$$

We are now ready to prove the main result of this note.

PROPOSITION 1. *For every $\alpha > 0$ the real gaussian function ϕ_α is a saddle point for the functional $E(\cdot, M)$.*

Proof. One has

$$\left.\frac{d^2}{dt^2} E\left((1-\beta(t))\ \phi_\alpha + t\, \psi_{m,\xi}, M\right)\right|_{t=0} = f_{\alpha,\xi}(m)$$

with
$$f_{\alpha,\xi}(m) := 4m\xi^2 \left[2 - \frac{22}{15}\,\alpha m^2 - \frac{1}{\alpha m^2}\right].$$
For every ξ and for every α, one has
$$\lim_{m\to 0^+} f_{\alpha,\xi}(m) = -\infty \quad \text{and} \quad \lim_{m\to +\infty} f_{\alpha,\xi}(m) = -\infty.$$
Thus the functional $E(\cdot, M)$ cannot take its maximum value when evaluated at the gaussian function ϕ_α, because
$$\frac{d^2}{dt^2} E\left((1-\beta(t))\ \phi_\alpha + t\,\psi_{m,\xi}, M\right)\Big|_{t=0} < 0$$
when $m > 0$ is close to 0. Furthermore the function $t \mapsto 2 - (22/15)t - 1/t$ takes it maximum value at the point $t_0 = \sqrt{15/22}$ and
$$f_{\alpha,\xi}\left(\sqrt{\frac{15}{22}}\right) = 8\sqrt{\frac{15}{22}}\,\xi\,\left(1 - \sqrt{\frac{15}{22}}\right) > 0.$$
Therefore there exist t_1 and t_2 with $0 < t_1 < t_2$ such that $f_{\alpha,\xi}(t) > 0$ for every $t \in [t_1, t_2]$. As a consequence, for every $\alpha > 0$ one can choose $m > 0$ in such a way that
$$\frac{d^2}{dt^2} E\left((1-\beta(t))\ \phi_\alpha + t\,\psi_{m,\xi}, M\right)\Big|_{t=0} > 0.$$
What precedes shows that every real gaussian function ϕ_α is a saddle point for $E(\cdot, M)$. □

It is now clear that Leipnik's claim cannot be true.

Finally, we have been somewhat vague in stating the assumptions of Lemmata of this paper in that we have requested the functions to be "sufficiently regular", but all the steps and results we have used are justified for the functions to which, in the final propositions, we have applied them.

References

[1] Hirschman I.I.jr., *A note on entropy*, Amer. J. Math. **79** (1957), 152–156.

[2] LEIPNIK R., *Entropy and the Uncertainty Principle*, Information and Control **2** (1959), 64–79.
[3] RUDIN W., *Real and Complex Analysis*, McGraw–Hill, New York, 1966.

Received July 19, 1997.