Rend. Istit. Mat. Univ. Trieste
Suppl. Vol. XXVIII, 421–451 (1997)

On Commutative Sums of Generators

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Pierre Grisvard in memoriam

SUMMARY. - *The theorem by Da Prato and Grisvard on resolvent commutative sums of linear m-dissipative operators is recast and extended in the framework of integrated semigroups.*

1. Introduction

In their classical paper, *Da Prato* and *Grisvard* (1975) proved the following result among many others:

THEOREM 1.1 (*Da Prato, Grisvard,* Theorem 3.3). *Let A and B be m-dissipative resolvent commutative operators on a Banach space X. If at least one of them is densely defined, then $A+B$ (with domain $D(A)\cap D(B)$) has an m-dissipative closure.*

Using *Arendt's* (1987a) characterization of m-dissipative operators in terms of l.L.c. (locally Lipschitz continuous) integrated semigroups, this result can be reformulated as follows:

THEOREM 1.2. *Let A be the generator of a C_0-semigroup S and B the generator of a l.L.c. integrated semigroup Ψ such that Ψ and S commute. Then $A+B$ has a closure that generates a l.L.c. integrated semigroup Ξ.*

In this paper we give an alternative proof of Da Prato's and Grisvard's theorem 3.3 by directly constructing the l.L.c. integrated

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Research partially supported by NSF grant DMS-9403884

semigroup Ξ in Theorem 1.2,

$$\Xi(t)x = \int_0^t \Psi(dr)S(r)x, \tag{1.1}$$

and by studying its properties. The integral in (3.1) is a Stieltjes integral. Actually we prove a more general version of Theorem 1.2 which has also the following corollary.

THEOREM 1.3. *Let X be an abstract M space. Let A be the generator of a positive C_0-semigroup S and B the generator of an increasing integrated semigroup Ψ such that Ψ and S commute. Then $A + B$ has a closed extension that generates an increasing integrated semigroup.*

Extending formula (1.1) we prove the following dual version of the above theorem by Da Prato and Grisvard:

THEOREM 1.4. *Let A be a densely defined m-dissipative operator on the Banach space X and B be an m-dissipative operator on the dual space X^* such that A^* and B are resolvent commutative. Then $A^* + B$ has an m-dissipative closure.*

Actually we prove the following more general result.

THEOREM 1.4. *Let A and B be resolvent commutative generators of l.L.c. integrated semigroups on a Banach space X. Assume that the domain of A coincides with its generalized domain (or Favard class) and that there exists some $N > 0$ such that*

$$\|x\| \le N \limsup_{\lambda\to\infty} \|\lambda(\lambda - A)^{-1}x\| \quad \forall x \in X.$$

Then $A + B$ has a closure that generates a l.L.c. semigroup.

A result concerning bounded commutative perturbations of integrated semigroups can be found in *Kellermann, Hieber* (1989), Proposition 3.1.

While the paper by Da Prato and Grisvard presents applications with A essentially being the time derivative, there has also been interest in applying the theory of commutative sums to evolution equations (Cauchy problems)

$$(d/dt)v(t) = (A+B)v(t) + f(t), \quad v(0) = x. \tag{1.2}$$

See *Huyer* (preprint), e.g. The integrated semigroup approach provides a result in this direction.

THEOREM 1.5. *Let A and B be such that the assumptions of any of the Theorems 1.1 to 1.4 are satisfied. Let $x \in D(B)$ and $x, Bx \in \overline{D(A)}$, and $f : [0,\infty) \to X$ be continuous such that at least one of the following conditions is satisfied:*

(i) f takes values in $D(A)$ and $Af(t)$ is continuous in t.

(ii) f takes values in $D(B)$ and $Bf(t)$ is continuous in t.

Then there exists an integral solution to (1.1), i.e., a continuous function $v : [0,\infty) \to X$ such that $\int_0^t v(s)ds$ takes values in $D(A) \cap D(B)$ and

$$v(t) = x + (A+B)\int_0^t v(s)ds + \int_0^t f(s)ds.$$

The function v is given by

$$v(t) = \Xi'(t)x + \int_0^t \Xi(dr) f(t-r)$$

where Ξ is the integrated semigroup whose generator extends $A+B$ and the integral is a Stieltjes integral.

2. Semigroups, dual semigroups, and integrated semigroups

In the tradition of *Hille, Phillips* (1957, Def. 8.3.5), a (one-parameter) semigroup on a vector space X is a family of linear transformations $S(t), t > 0$, satisfying

$$S(t+r) = S(t)S(r) \quad \forall t, r > 0. \tag{2.1}$$

2.1 B_0-semigroups

All semigroups S we are going to consider here operate on a Banach space X and will satisfy the extra condition

$$\limsup_{t \searrow 0} \|S(t)x\| < \infty \quad \forall x \in X. \tag{2.2}$$

We call these semigroups B_0-*semigroups*. It follows from the uniform boundedness principle and from (2.1) that any B_0-semigroup is exponentially bounded, i.e., there exist $M \geq 1, \omega \in \mathbf{R}$ such that

$$\|S(t)\| \leq Me^{\omega t} \quad \forall t > 0.$$

The (exponential) *growth bound* (or *type*) of a B_0-semigroup S, $\omega(S)$, is defined as the infimum of numbers $\omega \in \mathbf{R}$ such that, for some $M \geq 1$,

$$\|S(t)\| \leq Me^{\omega t} \quad \forall t > 0.$$

The following relation holds (*Clément*, 1987):

$$\omega(S) = \inf_{t>0} \frac{1}{t} \ln \|S(t)\| = \lim_{t \searrow \infty} \frac{1}{t} \ln \|S(t)\|.$$

2.2 C_0-semigroups

A semigroup S is called a C_0-semigroup if

$$\|S(t)x - x\| \to 0, \quad t \searrow 0, x \in X. \tag{2.3}$$

It is then convenient to extend $S(t)$ to $[0, \infty)$ by

$$S(0)x = x, \quad x \in X. \tag{2.4}$$

(2.1) then holds for all $t, r \geq 0$ and $S(t)$ is strongly continuous in $t \geq 0$. For C_0-semigroups the infinitesimal generator A is defined by

$$Ax = \lim_{t \searrow 0} (1/t)(S(t)x - x) \tag{2.5}$$

with $D(A)$ consisting of all elements $x \in X$ for which this limit exists.

Any C_0-semigroup is a B_0-semigroup and so is exponentially bounded. If S is a B_0-semigroup on a Banach space X, one introduces the space

$$X_\circ = \{x \in X;\ \|S(t)x - x\| \to 0,\ t \searrow 0\}. \tag{2.6}$$

$X_\circ$ is a closed subspace of X that is invariant under $S(t)$ for all $t \geq 0$. The restriction of S to $X_\circ$, $S_\circ$, is a C_0-semigroup on $X_\circ$.

If S is a C_0-semigroup on X, we can consider the dual semigroup $S^*(t) = (S(t))^*$ on the dual Banach space X^*. The uniform boundedness principle implies that S^* is a B_0-semigroup. The space $(X^*)_\circ$ defined by (2.6) for S^* rather than S coincides with $\overline{D(A^*)}$ and the symbol $X^\odot$ is used:

$$X^\odot = \overline{D(A^*)} = (X^*)_\circ.$$

The restriction of the dual semigroup S^* to $X^\odot$ is denoted by $S^\odot$ and the infinitesimal generator of the C_0-semigroup $S^\odot$, $A^\odot$, is the part of A^* in $X^\odot$. One can continue this procedure and consider $X^{\odot *}$ and its closed subspace $X^{\odot\odot} = \overline{D(A^{\odot *})}$ and the C_0-semigroup $S^{\odot\odot}$ on $X^{\odot\odot}$ generated by the part of $A^{\odot *}$ in $X^{\odot\odot}$, $A^{\odot\odot}$. See *van Neerven* (1992) for details and references.

2.3 Integrated semigroups

(Once) integrated semigroups Φ are motivated by formally defining

$$\Phi(t) = \int_0^t S(s)ds,$$

with a semigroup S, and discovering that

$$\begin{aligned} \Phi(t)\Phi(r) &= \int_0^{t+r} \Phi(s)ds - \int_0^t \Phi(s)ds - \int_0^r \Phi(s)ds, \\ &\qquad\qquad t, r \geq 0, \\ \Phi(0) &= 0. \end{aligned} \tag{2.7}$$

As topological property of Φ one generally chooses that $\Phi(t)$ is strongly continuous in $t \geq 0$, i.e., $\Phi(t)x$ is a continuous function of

$t \geq 0$ for any $x \in X$. We mention that n times integrated semigroups have been considered (see *Arendt* (1987a), *Neubrander* (1988) for two pioneering papers, *Ahmed* (1991) for a short account available in a text-book and *Hieber* (thesis) for the case that n is a positive, but not necessarily natural number). One is mainly interested in non-degenerate integrated semigroups, i.e., $\Phi(t)x = 0$ for all $t > 0$ occurs only for $x = 0$. The *generator* A of a non-degenerate integrated semigroups is given by requiring that, for $x, y \in X$,

$$x \in D(A), y = Ax \iff \Phi(t)x - tx = \int_0^t \Phi(s)y ds \quad \forall t \geq 0. \quad (2.8)$$

Notice that this definition makes sense and defines a closed operator A, even if S is not an integrated semigroup. Actually one has the following result:

THEOREM 2.1. *Let $\Phi(t), t \geq 0$, be a non-degenerate strongly continuous family of bounded linear operators on X and let the closed linear operator A be defined by* (2.8). *Then Φ is an integrated semigroup if and only if $\int_0^t \Phi(s)ds \in D(A)$ for all $t \geq 0$ and*

$$A \int_0^t \Phi(s)ds = \Phi(t)x - tx \quad \forall t \geq 0.$$

Proof. The "only if" part follows from *Thieme* (1990a, Lemma 3.5). The "if part" part follows from the proof of *Thieme* (1990a, Theorem 6.2).

If $\Phi(t)$ is *exponentially bounded*, i.e., there exist $M, \omega > 0$ such that

$$\|\Phi(t)\| \leq Me^{\omega t} \quad \forall t \geq 0,$$

one has the following useful relation between the Laplace transforms of the integrated semigroup and the resolvent of the generator. It follows by combining Theorem 3.1 in *Arendt* (1987a) and Proposition 3.10 in *Thieme* (1990a).

THEOREM 2.2. *Let $\Phi(t), t \geq 0$, be a strongly continuous exponentially bounded family of bounded linear operators on X and $A : D(A) \to X$ be a linear operator in X. Then Φ is a non-degenerate* *integrated semigroup and A its generator if and only if there exists some $\omega > 0$ such that any $\lambda > \omega$ is contained in the resolvent set of A and*

$$(\lambda - A)^{-1} = \lambda \hat{\Phi}(\lambda). \tag{2.9}$$

Here the Laplace transform $\hat{\Phi}(\lambda)$ is defined pointwise: $\hat{\Phi}(\lambda)x = \int_0^\infty e^{-\lambda t}\Phi(t)x dt$. Actually formula (2.9) can be used to define the generator A in the case of exponentially bounded integrated semigroups (*Arendt*, 1987a, p. 338; *Neubrander*, 1988, Definition 4.1).

A particularly interesting family of (once) integrated semigroups are the locally Lipschitz continuous ones (*Arendt*, 1987a; *Kellermann, Hieber*, 1989; *Lumer*, 1991a,b; *Thieme*, 1990b, 1991, 1996, preprint a, b).

THEOREM 2.3. *The following statements (i), (ii), and (iii) are equivalent for a linear closed operator A in a Banach space X:*

(i) *A is the generator of a non-degenerate integrated semigroup Φ that is locally Lipschitz continuous in the sense that, for any $b > 0$, there exists a constant $\Lambda > 0$ such that*

$$\|\Phi(t) - \Phi(r)\| \le \Lambda|t - r|; \quad 0 \le r, t \le b.$$

(ii) *A is the generator of a non-degenerate integrated semigroup Φ and there exist constants $M \ge 1, \omega \in \mathbf{R}$ such that*

$$\|\Phi(t) - \Phi(r)\| \le M \int_r^t e^{\omega s} ds, \quad 0 \le r \le t < \infty.$$

(iii) *A is a Θ operator, i.e., there exist constants $M \ge 1, \omega \in \mathbf{R}$ such that (ω, ∞) is contained in the resolvent set of A and*

$$\|(\lambda - A)^{-n}\| \le M(\lambda - \omega)^{-n}, \quad n = 1, 2, \ldots$$

The constants M, ω in (ii), (iii) *can be chosen to be identical.*

• *Moreover, if one (and then all) of* (i), (ii), (iii) *holds, $\overline{D(A)}$ coincides with those $x \in X$ for which $\Phi(t)x$ is continuously differentiable. The derivatives $S_\circ(t) = \Phi'(t)x, t \ge 0, x \in \overline{D(A)}$, provide* *bounded linear operators* $S_\circ(t)$ *from* $X_\circ = \overline{D(A)}$ *into itself forming a* C_0*-semigroup on* $X_\circ$ *which is generated by the part of* A *in* $X_\circ$, $A_\circ$. *Finally* $\Phi(t)$ *maps* X *into* $X_\circ$ *and*

$$\Phi'(r)\Phi(t) = \Phi(t+r) - \Phi(r), \quad r,t \geq 0. \tag{2.10}$$

Proof. The statements follow from combining the results by *Arendt* (1987a, Theorem 4.1) with those by *Kellermann, Hieber* (1989, Proposition 2.2, Theorem 2.4, and their proofs).

Θ-operators have also been considered without making the explicit connection to integrated semigroups, see *van Neerven* (1992) and *Sinestrari* (1994) for surveys and references. After equivalent renormalization, Θ-operators are quasi-m-dissipative operators that have a rich (nonlinear) theory including the celebrated *Crandall-Liggett* theorem (see *Bénilan, Wittbold*, 1994, for a survey and references).

REMARK 2.4. $X_\circ = \overline{D(A)}$ can be characterized in various ways:

$$\begin{aligned} X_\circ = &\ \{x \in X;\ \|\lambda(\lambda - A)^{-1}x - x\| \to 0, \quad \lambda \to \infty\} \\ = &\ \{x \in X; \|(1/h)\Phi(h)x - x\| \to 0,\ h \searrow 0\}. \end{aligned} \tag{2.11}$$

We also mention the following equalities for the spectra of A and $A_\circ$

$$\sigma(A) = \sigma(A_\circ), \quad \sigma_p(A) = \sigma_p(A_\circ). \tag{2.12}$$

The second equality is obvious. $\rho(A) \subseteq \rho(A_\circ)$ follows from the fact that $(\lambda - A_\circ)^{-1}$ is obtained by restricting $(\lambda - A)^{-1}$ to $X_\circ$. $\rho(A_\circ) \subseteq \rho(A)$ follows from the fact that $\rho(A)$ is not empty and, if $\mu \in \rho(A)$ and $\lambda \in \rho(A_\circ)$, we find the resolvent of A as

$$(\lambda - A)^{-1} = (\mu - A)^{-1} + (\mu - \lambda)(\lambda - A_\circ)^{-1}(\mu - A)^{-1}.$$

We define

$$X^\odot = \{x^* \in X^*;\ \|\lambda(\lambda - A)^{-1*}x^* - x^*\| \to 0,\ \lambda \to \infty\}. \tag{2.13}$$

If we want to emphasize the dependence of $X^\odot$ on the generator A, we write $X^\odot_A$. The resolvent identity implies that, for $\lambda \in \rho(A)$, $(\lambda - A)^{-1*}$ maps X^* into $X^\odot$ and actually

$$X^\odot = \overline{(\lambda - A)^{-1*} X^*}.$$

Notice that $X^\odot$ separates points in X and norms $X_\circ$:

$$\|x_\circ\| \leq M \sup\{|\langle x_\circ, x^\odot\rangle|;\ x^\odot \in X^\odot, \|x^\odot\| \leq 1\}.$$

Vice versa, $X_\circ$ norms $X^\odot$. The restriction of $(\lambda - A)^{-1*}$ to $X^\odot$ forms a family of pseudoresolvents that is actually the resolvent of a closed linear operator $A^\odot$ in $X^\odot$. It is easy to show that $A^\odot$ is densely defined in $X^\odot$ and, of course, a Θ operator, and thus the infinitesimal generator of a C_0-semigroup $S^\odot$ on $X^\odot$. We have the following relations:

$$\begin{aligned} X^\odot =&\ \{x^* \in X^*;\ \|(1/h)\Phi^*(h)x^* - x^*\| \to 0, \quad h \searrow 0\} \\ \Phi^*(t)x^\odot =&\ \int_0^t S^\odot(r)x^\odot dr, \quad t \geq 0, x^\odot \in X^\odot, \\ \langle S_\circ(t)x_\circ, x^\odot\rangle =&\ \langle x_\circ, S^\odot(t)x^\odot\rangle, \quad t \geq 0, x_\circ \in X_\circ, x^\odot \in X^\odot. \end{aligned} \tag{2.14}$$

We also mention that A acts like a dual operator for $A^\odot$: If $x, y \in X$, then

$$x \in D(A), Ax = y \iff \langle A^\odot x^\odot x\rangle = \langle x^\odot, y\rangle \quad \forall x^\odot \in D(A^\odot). \tag{2.15}$$

Following Section 2.2, we can consider the dual semigroup $S_\circ^*(t)$ on $X_\circ^*$ and its restriction $S_\circ^\odot(t)$ to $X_\circ^\odot = \overline{D(A_\circ^*)}$. It can be shown that the mapping

$$\ell : X^* \to X_\circ^*, \qquad \langle x_\circ, \ell x^*\rangle = \langle x_\circ, x^*\rangle, \quad x_\circ \in X_\circ, x^* \in X^*, \tag{2.16}$$

induces a continuous isomorphism from $X^\odot$ onto the space $X_\circ^\odot$,

$$\ell S^\odot(t)x^\odot = S_\circ^\odot(t)\ell x^\odot. \tag{2.17}$$

For details see *Clément et al.* (1989) and *van Neerven* (1992) where mainly dual semigroups have been considered, but the proofs equally work in the more general case of a l.l.c integrated semigroup.

We now consider the mapping

$$\jmath : X \to X^{\odot *}, \\ \langle x^\odot, \jmath x\rangle = \langle x, x^\odot\rangle, \quad x \in X, x^\odot \in X^\odot. \tag{2.18}$$

The last expression is defined because $X^\odot \subseteq X^*$. $\jmath$ maps X one-to-one and continuously into $X^{\odot *}$. $\jmath X_\circ$ is closed in $X^{\odot *}$; we conjecture that the same does not hold for $\jmath X$ in general because we have the following equivalencies.

PROPOSITION 2.5. *The following statements are equivalent:*

(i) $\jmath X$ *is closed in* $X^{\odot *}$.

(ii) $\jmath : X \to \jmath X \subseteq X^{\odot *}$ *is an open mapping.*

(iii) $X^\odot$ *norms* X.

(iv) *There exists a subspace* Y *of* X^* *and some* $N > 0$ *such that* Y *norms* X *and*

$$|\langle x, x^*\rangle| \leq N \limsup_{\lambda\to\infty} |\langle \lambda(\lambda - A)^{-1}x, x^*\rangle| \quad \forall x \in X, x^* \in Y.$$

(v) *There exists some* $N > 0$ *such that*

$$\|x\| \leq N \limsup_{\lambda\to\infty} \|\lambda(\lambda - A)^{-1}x\| \quad \forall x \in X.$$

Proof. (ii) and (iii) are obviously equivalent. (ii) obviously implies (i) and (i) implies (ii) by the open mapping theorem. Further (iii) $\Rightarrow$(iv) with $Y = X^\odot$. We first show (iv) $\Rightarrow$ (v). Since Y norms X, then there exists some $c > 0$ such that, for any $x \in X$, there exist some $x^* \in Y$, $\|x^*\| = 1$, such that

$$\|x\| \leq c|\langle x, x^*\rangle|.$$

Since $x^* \in Y$,

$$\|x\| \leq cN \limsup_{\lambda\to\infty} |\langle x, \lambda(\lambda - A)^{-1*}x^*\rangle| \leq cN \limsup_{\lambda\to\infty} \|\lambda(\lambda - A)^{-1}x\|.$$

Finally we show (v) $\Rightarrow$ (iii). From (v),

$$\begin{aligned}\|x\| \leq & \limsup_{\lambda\to\infty} \sup\{|\langle \lambda(\lambda - A)^{-1}x, x^*\rangle|;\ x^* \in X^*, \|x^*\| = 1\} \\ \leq & \limsup_{\lambda\to\infty} \sup\{|\langle x, \lambda(\lambda - A)^{-1*}x^*\rangle|;\ x^* \in X^*, \|x^*\| = 1\}.\end{aligned}$$

By Theorem 2.3 (iii),

$$\begin{aligned}\|x\| \leq & \ \sup\{|\langle x, y^{\odot}\rangle|; y^{\odot} \in X^{\odot}, \|y^{\odot}\| \leq M + 1\} \\ \leq & \ (M+1)\sup\{|\langle x, x^{\odot}\rangle|; x^{\odot} \in X^{\odot}, \|x^{\odot}\| \leq 1\}.\end{aligned}$$

Definition 2.6. A Θ-operator A is called *norming* if one (and then all) of the five equivalent statements in Proposition 2.5 are satisfied. In particular A is norming if and only if the quasi-norm

$$\|x\|_{\sim} = \limsup_{\lambda\to\infty} \|\lambda(\lambda - A)^{-1}x\|$$

is a norm that is equivalent to the original norm $\|\cdot\|$.

2.4 Integrated semigroups of locally bounded semi-variation

Let $\Phi(t), t \geq 0$, be a family of bounded linear operators on X. Then the semi-variation of Φ on the interval $[0, t]$ is defined as

$$\mathbf{V}_t(\Phi) = \sup\left\{\left\|\sum_{j=1}^{n}(\Phi(t_j) - \Phi(t_{j-1})x_j\right\|\right\}$$

where the supremum is taken over all partitions $0 = t_0 < \cdots < t_n = t$ and all $x_j \in X$ with $\|x_j\| \leq 1$. Φ is called to be of locally bounded semi-variation if $\mathbf{V}_t(\Phi) < \infty$ for all $t > 0$. An increasing family of bounded linear operator on an abstract M space is automatically of locally bounded semi-variation (*Diekmann, Gyllenberg, Thieme*, 1995, Proposition 7.1). If f is a continuous function on $[0, \infty)$, then the Stieltjes integral $\int_0^t \Phi(ds)f(s)$ exists. See *Hönig* (1975) and *Diekmann, Gyllenberg, Thieme* (1993, 1995).

THEOREM 2.7. *Let $\Phi(t), t \geq 0$, be an integrated semigroup that is of locally bounded semi-variation. Let $f : [0, \infty) \to X$ be continuous. Then*

$$I = \int_0^t \Phi(s) f(s) ds$$

is in the domain of the generator A of Φ and

$$AI = \int_0^t \Phi(ds) f(s) - \int_0^t f(s) ds.$$

If the Φ is l.L.c., then it is sufficient to assume that f is locally Bochner integrable.

Proof. Let us first assume that f is continuously differentiable. Then

$$I = \int_0^t \Phi(s) ds f(t) - \int_0^t \left(\int_0^s \Phi(r) dr \right) f'(s) ds.$$

By Theorem 2.1, $I \in D(A)$ and

$$\begin{aligned} AI &= \Phi(t) f(t) - t f(t) - \int_0^t (\Phi(s) - s) f'(s) ds \\ &= \int_0^t \Phi(ds) f(s) - \int_0^t f(s) ds. \end{aligned}$$

Our statement now follows for continuous f by approximating f by continuously differentiable functions and using the closedness of A.

We conclude this section with the following result.

THEOREM 2.8. *Let Φ and f as in Theorem 2.7. Define*

$$u(t) = \int_0^t \Phi(s) f(t-s) ds.$$

Then u is continuously differentiable and has its values in $D(A)$ and

$$(d/dt) u(t) = Au(t) + \int_0^t f(s) ds.$$

Proof. By Theorem 2.7 we know that u has its values in $D(A)$ and

$$Au(t) = \int_0^t \Phi(ds)f(t-s) - \int_0^t f(t-s)ds.$$

Again we first assume that f is continuously differentiable. Then u is continuously differentiable and

$$(d/dt)u(t) = \Phi(t)f(0) + \int_0^t \Phi(s)f'(t-s)ds = \int_0^t \Phi(ds)f(t-s)$$

by using the integration by parts formula for Stieltjes integrals. Again this formula can be extended to continuous f by approximating f by continuously differentiable functions. Notice that

$$\int_0^t \Phi(ds)f(t-s)$$

is a continuous function of t. See *Diekmann, Gyllenberg, Thieme* (1993), Proposition 2.4.

Corollary 2.9. *Let Φ and f as in Theorem 2.7. Then*

$$v(t) = \int_0^t \Phi(dr)f(t-r)$$

satisfies

$$v(t) = x + A\int_0^t v(s)ds + \int_0^t f(s)ds.$$

Proof. Notice that, by the proof of Theorem 2.8, $v(t) = (d/dt)u(t)$ with u being defined as in Theorem 2.8.

3. Commutative sums of generators with one domain being dense

We start with A being the generator of a C_0 semigroup and B the generator of an integrated semigroup of locally bounded semivariation.

THEOREM 3.1. *Let $S(t), t \geq 0$, be a C_0-semigroup on the Banach space X with generator A and $\Psi(t), t \geq 0$, an integrated semigroup on X with generator B. Assume that Ψ is of locally bounded semi-variation and that Ψ and S commute, i.e., $\Psi(t)S(r) = S(r)\Psi(t)$ for all $t \geq 0, r \geq 0$. Then*

$$\Xi(t)x = \int_0^t \Psi(dr)S(r)x \tag{3.1}$$

defines an integrated semigroup Ξ that is of locally bounded semi-variation and whose generator extends $A + B$. Ξ commutes with both Ψ and S.

REMARK 3.2. Moreover we have the following relations:

a) $D(A)$ and $D(B)$ are invariant under $\Xi(t)$ and $\Xi(t)$ commutes with A and B.

b) $\Xi(t)$ maps $D(B)$ into $D(A)$ and

$$A\Xi(t)x = S(t)x - x + \Psi(t)S(t)Bx + \Xi(t)Bx, \quad x \in D(B).$$

Moreover, if $x \in D(B)$, $\Xi(t)x$ is continuously differentiable and takes values in $D(A) \cap D(B)$ and

$$(d/dt)\Xi(t)x = x + (A+B)\Xi(t)x = x + A\Xi(t)x + \Xi(t)Bx.$$

c) $\int_0^t \Xi(s)ds$ maps $D(A) + D(B)$ into $D(B) \cap D(A)$ and

$$B\int_0^t \Xi(s)xds = \Xi(t)x - tx - \int_0^t \Xi(s)Axds, \quad x \in D(A),$$

$$A\int_0^t \Xi(s)xds = \Xi(t)x - tx - \int_0^t \Xi(s)Bxds, \quad x \in D(B).$$

d) If Ξ happens to be exponentially bounded, i.e., if there exist numbers g$M, \xi > 0$ such that $\|\Xi(t)\| \leq Me^{\xi t}$ for all $t \geq 0$, then the resolvent set of its generator, C, contains the ray (ξ, ∞) and

$$(\lambda - C)^{-1}(D(A) + D(B)) \subseteq D(A) \cap D(B), \quad \lambda > \xi.$$

Proof. Since Ψ is of locally bounded semi-variation and $S(r)x$ is continuous in r, the Stieltjes integral (3.1) exists. Moreover, after some straightforward calculations, one sees that Ξ is of locally bounded semi-variation. Obviously Ξ commutes with Ψ and S.

Let us check that Ξ is an integrated semigroup. Since Ψ and S commute and S is a semigroup,

$$\Xi(t)\Xi(u) = \int_0^t \Psi(dr)\left(\int_0^u \Psi(ds)S(r+s)\right).$$

Let $x \in D(A)$. Then, by integration by parts,

$$\begin{aligned}\Xi(t)\Xi(u)x = & \int_0^t \Psi(dr)\left(\Psi(u)S(r+u) - \int_0^u \Psi(s)S'(r+s)xds\right)\\ = & \int_0^t \Psi(dr)\Psi(u)S(r+u)x - \\ & - \int_0^u \left(\int_0^t \Psi(dr)\Psi(s)S'(r+s)x\right)ds.\end{aligned}$$

Since Ψ is an integrated semigroup, by (2.7),

$$\begin{aligned}\Xi(t)\Xi(u)x = & \int_0^t (\Psi(r+u) - \Psi(r))S(r+u)xdr\\ & - \int_0^u \left(\int_0^t (\Psi(r+s) - \Psi(r))S'(r+s)xdr\right)ds\\ = & \int_0^t (\Psi(r+u) - \Psi(r))S(r+u)xdr\\ & - \int_0^t \left(\int_r^{u+r} (\Psi(s) - \Psi(r))S'(s)xds\right)dr.\end{aligned}$$

Integrating by parts again,

$$\begin{aligned}\Xi(t)\Xi(u)x = & \int_0^t \left(\int_r^{u+r} \Psi(ds)S(s)x\right)dr\\ = & \int_0^t (\Xi(u+r) - \Xi(r))xdr.\end{aligned}$$

Since $D(A)$ is dense in X, this relation holds for all $x \in D(A)$ and Ξ is an integrated semigroup by (2.7).

Before we show that the generator of Ξ extends $A+B$, we show Remark 3.2.

Since $(\lambda - A)^{-1}$ commutes with $S(t)$ and $\Psi(t)$, it also commutes with $\Xi(t)$. Hence $D(A)$ is invariant under $\Xi(t)$ and $A\Xi(t)x = \Xi(t)Ax$ for $t \geq 0, x \in D(A)$. Since Ξ commutes with Ψ, it is immediate from (2.8) that $D(B)$ is invariant under $\Xi(t)$ and $B\Xi(t)x = \Xi(t)Bx$ for $x \in D(B)$.

In order to show that $\Xi(t)$ maps $D(B)$ into $D(A)$ let $x \in D(B)$. Since S commutes with Ψ, by (2.8) applied to Ψ and B, $S(r)x \in D(B)$ and $BS(r)x = S(r)Bx$, and

$$\Xi(t)x = \int_0^t (S(r)x + \Psi(r)S(r)Bx)dr. \tag{3.2}$$

Integrating by parts,

$$\begin{aligned}\Xi(t)x = &\int_0^t S(r)xdr + \int_0^t \Psi(r)(-d/dr)\left(\int_r^t S(s)Bxds\right)dr \\ = &\int_0^t S(r)xdr + \int_0^t \Psi(dr)\left(\int_r^t S(s)Bxds\right).\end{aligned}$$

Since A is the generator of S and a closed operator, $\Xi(t)x \in D(A)$ and

$$\begin{aligned}A\Xi(t)x = &\ S(t)x - x + \int_0^t \Psi(dr)(S(t) - S(r)Bx \\ = &\ S(t)x - x + \Psi(t)S(t)Bx - \Xi(t)Bx.\end{aligned}$$

Further, by (3.2), $\Xi(t)x$ is differentiable in t and $(d/dt)\Xi(t)x = S(t)x + \Psi(t)S(t)Bx$. This finishes part (b) in Remark 3.2.

Turning to part (c), let $x \in D(A)$. Then, integrating by parts,

$$\Xi(t)x = \Psi(t)S(t)x - \int_0^t \Psi(s)S(s)Axds.$$

By Theorem 2.7, $\Xi(t)x - \Psi(t)S(t)x \in D(B)$ and

$$\begin{aligned} B(\Xi(t)x - \Psi(t)S(t)x) &= -\int_0^t \Psi(ds)S(s)Ax + \int_0^t S(s)Axds \\ &= -\Xi(t)Ax + S(t)x - x. \end{aligned} \tag{3.3}$$

Since B is closed,

$$\int_0^t (\Xi(s)x - \Psi(s)S(s)x)ds \in D(B).$$

Using Theorem 2.7 another time, we have

$$u(t) = \int_0^t \Xi(s)xds \in D(B)$$

and, by (3.3),

$$Bu(t) = \Xi(t)x - tx - \int_0^t \Xi(t)Axds.$$

The other statement in part (c) follows from part (b) by integration.

Part (d) follows from Remark 3.2 (c) because A and B are closed and

$$(\lambda - C)^{-1} = \lambda^2 \int_0^\infty e^{-\lambda t} \left(\int_0^t \Xi(s)ds \right) dt.$$

This finishes the proof of Remark 3.2.

Finally we show that the generator of Ξ, C, extends $A + B$. Let $x \in D(A) \cap D(B)$. Since B commutes with Ξ, by Remark 3.2 (c), we have

$$\int_0^t \Xi(s)(Ax + Bx)ds = \int_0^t \Xi(s)Axds + B\int_0^t \Xi(s)ds = \Xi(t)x - tx.$$

By (2.8), $x \in D(C)$ and $Cx = Ax + Bx$.

COROLLARY 3.3. *Let X be an abstract M space. Let $S(t), t \geq 0$, be a positive C_0-semigroup on X with generator A and $\Psi(t), t \geq 0$, an increasing integrated semigroup on X with generator B. Assume that Ψ and S commute, i.e., $\Psi(t)S(r) = S(r)\Psi(t)$ for all $t \geq 0$. Then*

$$\Xi(t)x = \int_0^t \Psi(dr)S(r)x$$

defines an increasing integrated semigroup Ξ *whose generator extends* $A + B$. Ξ *commutes with both* Ψ *and* S. *Sufficiently large* $\lambda > 0$ *are contained in the resolvent set of the generator of* Ξ, C, *and* $(\lambda - C)^{-1}(D(A) + D(B)) \subseteq D(A) \cap D(B)$.

We remark that a densely defined resolvent positive operator on an ordered Banach space generates an increasing integrated semigroup (*Arendt*, 1987b).

Proof. Increasing operator families on an abstract M space automatically have locally bounded semi-variation (*Diekmann, Gyllenberg, Thieme*, 1995, Proposition 7.1). Hence Theorem 3.1 applies and (3.1) implies that Ξ is an increasing family. An increasing integrated semigroup is automatically exponentially bounded (*Arendt*, 1987b), hence Remark 3.2 (d) applies.

COROLLARY 3.4. *Let* $S(t), t \geq 0$, *be a* C_0*-semigroup on* X *with generator* A *and* $\Psi(t), t \geq 0$, *a l.L.c. integrated semigroup on* X *with generator* B. *Assume that* Ψ *and* S *commute, i.e.,* $\Psi(t)S(r) = S(r)\Psi(t)$ *for all* $t \geq 0, r \geq 0$. *Then*

$$\Xi(t)x = \int_0^t \Psi(dr)S(r)x$$

defines a l.L.c. integrated semigroup Ξ *whose generator* C *extends* $A + B$. Ξ *commutes with both* Ψ *and* S. *Moreover, if*

$$\|S(t)\| \leq M_A e^{\omega_A t}$$

and

$$\|\Psi(t) - \Psi(r)\| \leq M_B \int_r^t e^{\omega_B s} ds,$$

then

$$\|\Xi(t) - \Xi(r)\| \leq M_A M_B \int_r^t e^{(\omega_B + \omega_A)s} ds.$$

COROLLARY 3.5 (*Da Prato, Grisvard*, 1975). *Let* A, B *be resolvent commutative linear operators whose resolvent sets contain rays* (ω_A, ∞) *and* (ω_B, ∞) *respectively and*

$$\begin{aligned} \|(\lambda - A)^{-n}\| &\leq M_A(\lambda - \omega_A)^{-n}, \quad \lambda > \omega_A, n \in \mathbf{N}, \\ \|(\lambda - B)^{-n}\| &\leq M_B(\lambda - \omega_B)^{-n}, \quad \lambda > \omega_B, n \in \mathbf{N}. \end{aligned}$$

Let A or B be densely defined. Then the operator $A+B$ with domain $D(A) \cap D(B)$ has a closure C whose resolvent set contains the ray (ω_C, ∞) with

$$\omega_C = \omega_A + \omega_B$$

and

$$\|(\lambda - C)^{-n}\| \le M_A M_B (\lambda - \omega_C)^{-n}, \quad \lambda > \omega_C, n \in \mathbf{N}.$$

Moreover $(\lambda - C)^{-1}(D(A) + D(B)) \subseteq D(A) \cap D(B)$ for $\lambda > \omega_C$ and C is resolvent commutative with both A and C.

Proof. By Theorem 2.3 and Corollary 3.4, the generator C of the l.L.c. semigroup Ξ satisfies the estimates

$$\|(\lambda - C)^{-n}\| \le M_A M_B (\lambda - \omega_C)^{-n}, \quad \lambda > \omega_C, n \in \mathbf{N}.$$

Since the integrated semigroup Ξ is exponentially bounded, the last statement in Corollary 3.5 follows from Remark 3.2 (d). The last statement in Corollary 3.5 implies that

$$(\lambda - A - B)(D(A) \cap D(B)) \supseteq D(A) + D(B).$$

Hence the range of $(\lambda - A - B)$ is dense. Since $A + B$ is extended by the closed operator C, it has a closure $\bar{L}$. Since C extends $\bar{L}$,

$$\|x\| \le M_A M_B \|(\lambda - \bar{L})x\|, \quad x \in D(\bar{L}), \lambda > \omega_C.$$

This implies that the range of $\lambda - \bar{L}$ is X. Hence $(\lambda - \bar{L})$ is invertible and the inverse coincides with $(\lambda - C)^{-1}$. Hence $C = \bar{L}$.

4. Favard class and Favard operators

Let A be an operator from a Banach space X to a Banach space Y with domain $D(A) \subseteq X$.

DEFINITION 4.1. The *Favard class* or *generalized domain* of A, denoted by $\mathrm{Fav}(A)$, consists of those elements $x \in X$ which are the limits of a sequence $x_n \in D(A)$ with $\sup_{n \in \mathbf{N}} \|Ax_n\| < \infty$.

Obviously $\mathrm{Fav}(A)$ is a linear subspace of X and

$$D(A) \subseteq \mathrm{Fav}(A) \subseteq \overline{D(A)}.$$

4.1 The Favard class of a Θ-operator

In this subsection we assume that A is a Θ-operator or, equivalently, the generator of a l.L.c. integrated semigroup. We recall that the *Yosida approximations* of A, A_λ, are defined by

$$A_\lambda = \lambda A(\lambda - A)^{-1} = \lambda^2(\lambda - A)^{-1} - \lambda I, \quad \lambda \in \rho(A). \tag{4.1}$$

It turns out that the Favard class of A consists of those elements for which the Yosida approximations are bounded.

LEMMA 4.2. *$x \in \mathrm{Fav}(A)$ if and only if* $\limsup_{\lambda\to\infty} \|A_\lambda x\| < \infty$.

Proof. Let $x \in X$ and assume that the Yosida approximations are bounded when applied to x. Define $x_n = n(n - A)^{-1}x$. Then it easily follows that $x_n \to x$, $n \to \infty$, and that the sequence (Ax_n) is bounded. Conversely assume that $D(A) \ni x_n \to x$ and (Ax_n) is bounded. Since A is a Θ-operator there exists a constant $c > 0$ such that

$$\|A_\lambda x_n\| \le c \quad \forall \lambda \ge \omega + 1, n \in \mathbf{N}.$$

As the A_λ are bounded linear operators we can take the limit for $n \to \infty$ and obtain

$$\|A_\lambda x\| \le c \quad \forall \lambda \ge \omega + 1.$$

Let $X_\circ = \overline{D(A)}$ and $A_\circ$ the part of A in $X_\circ$. One can easily check that $(\lambda - A_\circ)^{-1}$ is the restriction of $(\lambda - A)^{-1}$ to $X_\circ$. As a corollary of the previous lemma we obtain

LEMMA 4.3. $\mathrm{Fav}(A) = \mathrm{Fav}(A_\circ)$.

4.2 The Favard class and the integrated semigroup

As in the previous subsection we assume that A is the generator of a l.L.c. integrated semigroup Φ. Recall that the strong derivatives of $\Phi(t)$, $\Phi'(t) = S_\circ(t)$, exist on $X_\circ$, and form a C_0-semigroup on $X_\circ$ that is generated by $A_\circ$.

PROPOSITION 4.4. *For any $t \geq 0, x \in X$, $\Phi(t)x \in \mathrm{Fav}(A)$.*

Proof. Set $y_n = n \int_t^{t+1/n} \Phi(r)x dr$. Then $y_n \to \Phi(t)x$ for $n \to \infty$ and the elements

$$Ay_n = n(\Phi(t+1/n)x - \Phi(t)x) - tx$$

(see Theorem 2.1) form a bounded sequence because $\Phi(t)$ is locally Lipschitz in t. Hence $\Phi(t)x \in \mathrm{Fav}(A)$ by definition.

PROPOSITION 4.5. *The following statements are equivalent for $x_\circ \in X_\circ$:*

(i) $x_\circ \in \mathrm{Fav}(A)$.

(ii) $S_\circ(t)x_\circ$ *is locally Lipschitz in* $t \geq 0$.

(iii) $\liminf_{t \searrow 0}(1/t)\|S_\circ(t)x_\circ - x_\circ\| < \infty$

Proof. (i) $\Rightarrow$ (ii) : Let $D(A) \ni x_n \to x,\ n \to \infty$. Then, by (2.8),

$$S_\circ(t)x_n - S_\circ(r)x_n = (\Phi(t) - \Phi(r))Ax_n$$

and

$$\begin{aligned} \|S_\circ(t)x - S_\circ(r)x\| &= \lim_{n\to\infty} \|S_\circ(t)x_n - S_\circ(r)x_n\| \\ &\leq \|\Phi(t) - \Phi(r)\| \limsup_{n\to\infty} \|Ax_n\|. \end{aligned}$$

Hence (ii) follows from the fact that $\Phi(t)$ is locally Lipschitz in the operator norm. (ii) $\Rightarrow$ (iii) is obvious. (iii) $\Rightarrow$ (i): Choose a sequence $0 < t_n \to 0, n \to \infty$ such that

$$\liminf_{n\to\infty}(1/t_n)\|S_\circ(t_n)x_\circ - x_\circ\| < \infty.$$

Then $x_n = (1/t_n)\Phi(t_n)x_\circ \to x_\circ,\ n \to \infty$, as $x_\circ \in X_\circ$, and the elements

$$Ax_n = (1/t_n)S_\circ(t_n)x_\circ - x_\circ$$

form a bounded sequence.

COROLLARY 4.6. *The operators $S_\circ(t), t \geq 0$, map the Favard class of A into itself.*

DEFINITION 4.7. The generator A of a l.L.c. semigroup is called a *Favard operator* if its Favard class coincides with its domain, i.e., $D(A) = \mathrm{Fav}(A)$.

We close this section by an applicable condition for a Θ-operator to be a Favard operator.

THEOREM 4.8. *Let A be the generator of a l.L.c. integrated semigroup. Assume that there exists a total subspace Π of X^* and a subspace Γ of X^* such that $(\lambda - A)^{-1*}\Pi \subseteq \bar{\Gamma}$ for some $\lambda \in \rho(A)$ and that for all $z^* \in \Gamma^*$ there exists some $y \in X$ such that*

$$\langle x^*, z^* \rangle = \langle y, x^* \rangle \quad \forall x^* \in \Gamma. \tag{4.2}$$

Then A is a Favard operator.

Proof. Let $x_n \in D(A)$, $x_n \to x$ as $n \to \infty$, and Ax_n form a bounded sequence. Then Ax_n can be considered elements in Γ^*. By the Alaoglu-Bourbaki theorem there exists $z^* \in \Gamma^*$ such that z^* is the weak* closure of $\{Ax_n; n \geq m\}$ for all $m \in \mathbf{N}$. By assumption there exists some $y \in X$ satisfying (4.2). Since $(\lambda - A)^{-1*}\Pi \subseteq \bar{\Gamma}$, we have for all $x^* \in \Pi$ that

$$\begin{aligned}
\langle -x + \lambda(\lambda - A)^{-1}x, x^* \rangle &= \lim_{n\to\infty} \langle -x_n + \lambda(\lambda - A)^{-1}x_n, x^* \rangle \\
&= \lim_{n\to\infty} \langle (\lambda - A)^{-1}Ax_n, x^* \rangle \\
&= \lim_{n\to\infty} \langle Ax_n, (\lambda - A)^{-1*}x^* \rangle \\
&= \langle (\lambda - A)^{-1}x^*, z^* \rangle \\
&= \langle (\lambda - A)^{-1}y, x^* \rangle.
\end{aligned}$$

Since Π is total,

$$-x + \lambda(\lambda - A)^{-1}x = (\lambda - A)^{-1}y.$$

Hence $x \in D(A)$ and $Ax = y$.

COROLLARY 4.9. *Let X be a dual Banach space and A be the dual of a generator of a C_0-semigroup. Then A is a norming Favard operator.*

For related results and historical references see *van Neerven* (1992), Chapter 3.

5. Commutative sums of operators with both domains non-dense

We replace the density of the domain of A by A being a norming Favard operator. These concepts can be found in Section 2.3 and Section 4.2.

THEOREM 5.1. *Let A and B be resolvent commutative generators of l.L.c. integrated semigroups. Assume that A is a norming Favard operator. Then $A+B$ has a closure C that generates an l.L.c. integrated semigroup. Moreover $(\lambda - C)^{-1}(D(A)+D(B)) \subseteq D(A)\cap D(B)$ for $\lambda > \omega_C$.*

REMARK 5.2. Moreover we have the following relations for the integrated semigroup Ξ generated by C:

a) $D(A)$ and $D(B)$ are invariant under $\Xi(t)$ and $\Xi(t)$ commutes with A and B.

b) Let $x \in D(B)$. Then $\Xi(t)x \in D(A)$.

Moreover, if $x \in D(B)$ and $x \in X_\circ$, $Bx \in X_\circ = \overline{D(A)}$, then $\Xi(t)x$ is continuously differentiable and takes values in $D(A)\cap D(B)$ and

$$(d/dt)\Xi(t)x = x + (A+B)\Xi(t)x = x + A\Xi(t)x + \Xi(t)Bx.$$

c) $\int_0^t \Xi(s)ds$ maps $D(A)+D(B)$ into $D(B)\cap D(A)$ and

$$B\int_0^t \Xi(s)xds = \Xi(t)x - tx - \int_0^t \Xi(s)Axds, \quad x \in D(A),$$

$$A\int_0^t \Xi(s)xds = \Xi(t)x - tx - \int_0^t \Xi(s)Bxds, \quad x \in D(B).$$

Proof. Let A generate the l.L.c. integrated semigroup Φ and B the l.L.c. integrated semigroup Ψ. Let $X_\circ = \overline{D(A)}$. Then the part of A in $X_\circ$, $A_\circ$, generates a C_0-semigroup $S_\circ$ on $X_\circ$. Since Φ and Ψ commute, $X_\circ$ is invariant under Ψ. The restrictions of $\Psi(t)$ to $X_\circ$, $\Psi_\circ(t)$, form a l.L.c. integrated semigroup on $X_\circ$ which is generated by the part of B in $X_\circ$. By Corollary 3.4,

$$\Xi_\circ(t)x_\circ = \int_0^t \Psi_\circ(dr)S_\circ(r)x_\circ \tag{5.1}$$

defines a l.L.c integrated semigroup on $X_\circ$.

We now proceed as in *Clément et al.* (1989). By Lemma 4.2, $\Xi_\circ(t)$ leaves the generalized domain of A invariant because the Hille-Yosida approximations A_λ commute with $X_\circ$. Since A is a Favard operator, $D(A)$ is invariant under $\Xi_\circ(t)$. We define

$$\Xi(t)x = (\lambda - A)\Xi_\circ(t)(\lambda - A)^{-1}, \quad \lambda \in \rho(A). \tag{5.2}$$

Since $\Xi_\circ(t)$ and $(\lambda - A)^{-1}$ commute, it follows from the resolvent identity that this definition is independent of λ. Let $X^\odot = \overline{D(A^\odot)}$ and $S^\odot$ the C_0-semigroup on $X^\odot$ generated by $A^\odot$. Since $\Psi^*(t)$ and $(\lambda - A)^{-1*}$ commute, $X^\odot$ is invariant under $\Psi^*(t)$. Again by Corollary 3.4, the definition

$$\Xi^\odot(t)x^\odot = \int_0^t \Psi^*(dr)S^\odot(r)x^\odot \tag{5.3}$$

provides a l.L.c. integrated semigroup $\Xi^\odot(t)$ on $X^\odot$. Obviously

$$\langle \Xi_\circ(t)x_\circ, x^\odot\rangle = \langle x_\circ, \Xi^\odot(t)x^\odot\rangle, \quad x_\circ \in X_\circ, x^\odot \in X^\odot.$$

Let $x^\odot \in D(A^\odot)$. Then

$$\begin{aligned} \langle \Xi(t)x, x^\odot\rangle &= \langle \Xi_\circ(t)(\lambda - A)^{-1}x, (\lambda - A^\odot)x^\odot\rangle \\ &= \langle (\lambda - A)^{-1}x, \Xi^\odot(t)(\lambda - A^\odot)x^\odot\rangle \\ &= \langle x, (\lambda - A)^{-1*}\Xi^\odot(t)(\lambda - A^\odot)x^\odot\rangle \\ &= \langle x, (\lambda - A^\odot)^{-1}\Xi^\odot(t)(\lambda - A^\odot)x^\odot\rangle \\ &= \langle x, \Xi^\odot(t)x^\odot\rangle. \end{aligned}$$

Since $D(A^\odot)$ is dense in $X^\odot$, this relation holds for all $x \in X^\odot$. Now Ξ inherits (2.7) from $\Xi^\odot$. Since A is a norming operator, $X^\odot$ norms X by Proposition 2.5 and Definition 2.6, and so $\Xi(t)$ inherits the local Lipschitz continuity from $\Xi^\odot(t)$.

Let C be the generator of Ξ and $C_\circ$ the generator of $\Xi_\circ$. Since Ξ commutes with $(\mu - A)^{-1}$, it follows that C and A are resolvent commutative and $C_\circ$ is the part of C in $X_\circ$.

Let $x \in D(A) \cap D(B)$. Then $(\lambda - A)^{-1}x \in D(A_\circ) \cap D(B_\circ)$. Since $C_\circ$ extends $A_\circ + B_\circ$ by Corollary 3.4, by (2.8),

$$\int_0^t \Xi_\circ(s)(\lambda - A)^{-1}(Ax + Bx)ds = \int_0^t \Xi_\circ(s)(A_\circ + B_\circ)(\lambda - A)^{-1}x ds$$
$$= \Xi(t)(\lambda - A)^{-1}x - t(\lambda - A)^{-1}x.$$

Applying $(\lambda - A)$ yields

$$\int_0^t \Xi(s)(Ax + Bx)ds = \Xi(t)x - tx.$$

By (2.8), $x \in D(C)$ and $Cx = Ax + Bx$.

The claim that $(\lambda - C)^{-1}(D(A) + D(B)) \in D(A) \cap D(B))^{-1}$ (and thus C is the closure of $A + B$, see the end of the proof of Corollary 3.5) follows from Remark 5.2 (c) in the same way as in the proof of Remark 3.2 (d). Hence we prove Remark 5.2 now. (a) (5.1), (5.2) imply that $\Xi(t)$ commutes with the resolvents of A and B. Hence this statement holds.

(b) Let $x \in D(B)$. Then $(\lambda - A)^{-1}x \in D(B_\circ)$ and $B_\circ(\lambda - A)^{-1}x = (\lambda - A)^{-1}Bx$. It follows from (3.2) that

$$\Xi_\circ(t)(\lambda - A)^{-1}x = \int_0^t S_\circ(r)(\lambda - A)^{-1}x dr + \int_0^t \Psi(r)S(r)(\lambda - A)^{-1}Bx dr.$$

Hence, by (5.2),

$$\Xi(t)x = \Phi(t)x + \int_0^t \Psi(r)\Phi(dr)Bx, \tag{5.4}$$

where Φ is the integrated semigroup generated by A. Integrating by parts,

$$\Xi(t)x = \Phi(t)x + \Phi(t)\Psi(t)Bx - \int_0^t \Phi(r)\Psi(dr)Bx.$$

Since $\Phi(t)$ maps X into $X_\circ$ by Theorem 2.3, we can apply $S_\circ(s)$ to this equation and obtain

$$S_\circ(s)\Xi(t)x = (\Phi(t+s) - \Phi(s))(x + \Psi(t)Bx) - \int_0^t (\Phi(r+s) - \Phi(s))\Psi(dr)Bx.$$

Since $\Phi(t)$ is l.L.c. in t, so is $S_\circ(s)\Xi(t)x$ in s. By Proposition 4.5 (ii), $\Xi(t)x$ is in the generalized domain of A and so in the domain of A.

If x and Bx are in $X_\circ$, by (5.4),

$$\Xi(t)x = \int_0^t S_\circ(r)xdr + \int_0^t \Psi(r)S_\circ(r)Bxdr$$

and the statement follows exactly as in the proof of Remark 3.2 (b).

(c) Since Ξ commutes with Φ and Ψ, $D(A)$ and $D(B)$ are left invariant. We show that $\int_0^t \Xi(s)ds$ maps $D(A)$ into $D(B)$. The proof where A and B are interchanged, is similar. By Remark 3.2 (c), $\int_0^t \Xi_\circ(s)ds$ maps $D(A_\circ)$ into $D(B_\circ)$ and, for $x \in D(A)$,

$$\begin{aligned} B_\circ \int_0^t \Xi_\circ(s)(\lambda - A)^{-1}xds &= \Xi_\circ(t)(\lambda - A)^{-1}x - t(\lambda - A)^{-1}x \\ &\quad - \int_0^t \Xi_\circ(s)(\lambda - A)^{-1}Axds. \end{aligned}$$

Since $\Xi_\circ(t)$ leave $D(A)$ invariant we have, for

$$y = \int_0^t \Xi_\circ(s)(\lambda - A)^{-1}xds,$$

that $y \in D(A) \cap D(B)$ and $By \in D(A)$. Hence $Ay \in D(B)$ and $ABy = BAy$. Thus $\int_0^t \Xi(s)xds \in D(B)$ and

$$B\int_0^t \Xi(s)xds = \Xi(t)x - tx - \int_0^t \Xi(s)Axds$$

is a continuous function of t.

6. Integral solutions to Cauchy problems

Let the assumptions of any of the Theorems or Corollaries in Section 3 or Section 5 be satisfied. Let Ξ be the integrated semigroup generated by an extension C of $A+B$, $f : [0,\infty) \to X$ continuous Let Ξ be of locally bounded semi-variation. Let $x \in \overline{D(C)}$. By Theorem 2.8,

$$v(t) = (d/dt)\Xi(t)x + \int_0^t \Xi(dr)f(t-r)$$

solves

$$v(t) = x + C\int_0^t v(s)ds + \int_0^t f(r)dr.$$

Here we are interested in conditions on f and x under which we actually solve

$$v(t) = x + (A+B)\int_0^t v(s)ds + \int_0^t f(r)dr, \tag{6.1}$$

i.e., under which we have

$$\int_0^t v(s)ds \in D(A)\cap D(B) \quad \text{and} \quad x \in \overline{D(C)}.$$

As for x, Remark 5.2 (b) tells that $x \in D(B)$ and $x, Bx \in \overline{D(A)}$ guarantees that $\Xi(t)x$ is continuously differentiable, hence $x \in \overline{D(C)}$ by Theorem 2.3. From the proof of Theorem 2.8 we know that

$$\int_0^t v(s)ds = \Xi(t)x + \int_0^t \Xi(r)f(t-r)dr.$$

Let us assume that f is continuous and takes values in $D(A)$ and that $Af(t)$ is also a continuous function of t. We first assume that f is continuously differentiable and that the derivative of f also takes values in $D(A)$ and that $Af'(t)$ is also a continuous function of t and $(d/dt)Af(t) = Af'(t)$. Then

$$\int_0^t \Xi(r)f(t-r)dr = \int_0^t \left(\int_0^r \Xi(s)ds\right) f'(t-r)dr + \int_0^t \Xi(s)dsf(0).$$

By Remark 3.2 (c) or Remark 5.2 (c) we have that

$$\int_0^t \Xi(r)f(t-r)dr \in D(B)$$

and

$$
\begin{aligned}
&B\int_0^t \Xi(r)f(t-r)dr = \\
&= \int_0^t \left(\Xi(r)f'(t-r) - rf'(t-r) - \left(\int_0^r \Xi(s)ds\right)Af'(t-r)\right)dr \\
&\quad + \Xi(t)f(0) - tf(0) - \int_0^t \Xi(s)Af(0)ds \\
&= \int_0^t \Xi(dr)f(t-r) - \int_0^t f(t-r)dr - \int_0^t \Xi(dr)Af(t-r)dr.
\end{aligned}
\tag{6.2}
$$

Now assume that f is continuous, takes values in $D(A)$ and that $Af(t)$ is continuous. Set

$$f_n(t) = n\int_t^{t+1/n} f(s)ds.$$

Then f_n satisfies all the extra assumptions we made before and $f_n(t) \to f(t), Af_n(t) \to Af(t)$ as $n \to \infty$, locally uniformly on $[0,\infty)$. For

$$w_n = \int_0^t \Xi(r)f_n(t-r)dr, \quad w(t) = \int_0^t \Xi(r)f(t-r)dr$$

we see from (6.2) that $w_n(t) \in D(B)$, $w_n(t) \to w(t)$ and that $Bw_n(t)$ also converges as $n \to \infty$. Hence $w(t) \in D(B)$ because B is closed. An analogous statement holds if f takes values in $D(B)$ and $Bf(t)$ is continuous. So we have shown the following:

THEOREM 6.1. *Let the assumptions of any of the theorems or corollaries in Sections 3 or 5 hold. Assume that $x \in D(B)$ and that $x, Bx \in \overline{D(A)}$. Further assume that $f : [0,\infty) \to X$ is continuous and at least one of the following holds:*

(i) *f takes values in $D(A)$ and $Af(t)$ is a continuous function of t.*
(ii) *f takes values in $D(B)$ and $Bf(t)$ is a continuous function of t.*

Then there exists a continuous solution v of (6.1).

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