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# Well-Posedness of Difference Elliptic Equation

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SUMMARY. - *It is considered the difference analog of Poisson equation on the plane and it is established the exact with respect of step h coercive inequality in the $C^h$-norm for its solutions.*

## 0 Introduction

### 0.1 Well-Posedness of difference elliptic equation

The partial differential equation

$$-(\partial^2 v/\partial x_1^2 + \partial^2 v/\partial x_2^2) + v = f \qquad (0.1)$$

on the plane $R^2$ of points $x = (x_1, x_2)$ is considered. It is naturally to call the function $v(x) = v(x_1, x_2)$ the (classical) solution of equation (0.1), if it has the continuous and bounded derivatives till the second order and if it satisfies the equation (0.1). We will consider the partial differential equation (0.1) as the operator equation in the Banach space $C = C(R^2)$ of the continuous and bounded (scalar) functions $\psi(x) = \psi(x_1, x_2)$ with the norm

$$\|\psi\|_C = \sup_{x\in R^2} |\psi(x)|. \qquad (0.2)$$

For the existence of such solution of equation (0.1), evidently, it is necessary, that

$$f \in C. \qquad (0.3)$$

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We will say, that equation (0.1) is well-posed in $C$, if the following two conditions are fulfilled:

(a$_1$) The unique solution $v(x) = v(x; f)$ of equation (0.1) in $C$ exists for any $f \in C$. It means, that formula

$$[v(f)](x) = v(x; f) \tag{0.4}$$

defines the homogeneous and additive operator, acting from $C$ in the Banach space $C^2 = C^2(R^2)$ of (scalar) functions $\psi(x) = \psi(x_1, x_2)$, having continuous and bounded partial derivatives till the second order, with the norm

$$\|\psi\|_{C^2} = \|\psi\|_C + \sum_{i=1}^{2} \|\partial\psi/\partial x_i\|_C + \sum_{i,j=1}^{2} \|\partial^2\psi/\partial x_i \partial x_j\|_C. \tag{0.5}$$

(a$_2$) The operator $v(f)$, as the operator in $C$, is continuous. This property is, evidently, equivalent to inequality

$$\|v(f)\|_C \leq M \cdot \|f\|_C \tag{0.6}$$

with some $1 \leq M < +\infty$, does not depending on $f \in C$.

It turns out, that property (a$_1$) leads to the essentially more stronger inequality. In fact, the acting in $C$ with domain $C^2$ operators

$$(A_i\psi)(x) = \partial^2\psi/\partial x_i^2 \tag{0.7}$$

evidently, are closed. Then from properties (a$_1$) and (a$_2$) it follows, that operators

$$[A_i v(f)](x) = -\partial^2 v(x; f)/\partial x_i^2 \tag{0.8}$$

are the closed operators defined on the whole Banach space. Therefore, in virtue of Banach's theorem, the operators (0.8) are bounded. This leads to the coercive inequality

$$\|v\|_{C^2} \leq M\|f\|_C \tag{0.9}$$

for the solution in $C$ of problem (0.1) with some $1 \leq M < +\infty$, does not depending on $f \in C$. However, it is well known, equation (0.1) is not well-posed in $C$. The corresponded counter-example can be given by formula

$$\begin{array}{ll} v_\alpha(x) = (x_1^2 - x_2^2) \cdot \ln^\alpha \frac{1}{x_1^2+x_2^2} & (0 < x_1^2 + x_2^2 \leq 1/9) \\ v_\alpha(x) = 0 & (4/9 \leq x_1^2 + x_2^2 \text{ and } x_1 = x_2 = 0) \\ v_\alpha \in C^2 & (1/10 \leq x_1^2 + x_2^2). \end{array} \tag{0.10}$$

It means, that $v_\alpha \in C^1(R^2)$, and for $0 \leq x_1^2 + x_2^2 \leq 1/9$

$$\begin{aligned} \partial^2 v_\alpha / \partial x_1^2 &= 2 \cdot \ln^\alpha \frac{1}{x_1^2 + x_2^2} + a_{\alpha,1}(x_1, x_2), \\ \partial^2 v_\alpha / \partial x_2^2 &= -2 \cdot \ln^2 \frac{1}{x_1^2 + x_2^2} + a_{\alpha,2}(x_1, x_2) \end{aligned} \tag{0.11}$$

for some continuous functions $a_{\alpha,i}(x_1, x_2), i = 1, 2$. Therefore, evidently, equation (0.1) is not well-posed in $C$, and the coercive inequality (0.9) is not true for the any solution in $C$ of this equation.

## 0.2 Well-Posedness of difference equation

We will consider now the difference analog of differential equation (0.1), namely difference equation, $(i, j = \overline{-\infty, +\infty})$,

$$\begin{array}{l} -[(v_{i+1,j} - 2v_{i,j} + v_{i-1,j}) \cdot h^{-2} + \\ +(v_{i,j+1} - 2v_{i,j} + v_{i,j-1}) \cdot h^{-2}] + v_{i,j} = f_{i,j} \end{array} \tag{0.12}$$

for some $0 < h \leq 1$. We will consider equation (0.12) as operator equation

$$-[D_1^{h,2} v^h + D_2^{h,2} v^h] + v^h = f^h \tag{0.13}$$

in the Banach space $C^h$ of bounded (scalar) grid functions

$$\psi^h = (\psi_{i,j}; i, j = \overline{-\infty, +\infty}) \tag{0.14}$$

with norm

$$\|\psi^h\|_{C^h} = \sup_{i,j=\overline{-\infty,+\infty}} |\psi_{i,j}|. \tag{0.15}$$

Here operators $D_k^{h,2} (k = 1, 2)$ are defined by formulas

$$\begin{array}{l} D_1^{h,2} \psi^h = [(\psi_{i+i,j} - 2\psi_{i,j} + \psi_{i-1,j}) \cdot h^{-2}; i, j = \overline{-\infty, +\infty}] \\ D_2^{h,2} \psi_h = [(\psi_{i,j+1} - 2\psi_{i,j} + \psi_{i,j-1}) \cdot h^{-2}; i, j = \overline{-\infty, +\infty}]. \end{array} \tag{0.16}$$

For any $f^h \in C^h$ equation (0.13) has the unique solution $v^h \in C^h$, and the difference coercive inequality

$$\|D_1^{h,2} v^h\|_{C^h} + \|D_2^{h,2} v^h\|_{C^h} + \|v^h\|_{C^h} \leq M_C(h) \cdot \|f^h\|_{C^h} \qquad (0.17)$$

takes place with some $1 \leq M_C(h) < +\infty$, does not depending on $f^h \in C^h$. In fact, let us consider more general, than (0.13), operator equation with parameter $\lambda > 0$

$$-(D_1^{h,2} v^h + D_2^{h,2} v^h) + \lambda v^h = f^h \qquad (0.18)$$

or (infinite) system of linear algebraic equations, $(i, j = \overline{-\infty, +\infty})$,

$$\begin{aligned} -[(v_{i+1,j} - 2v_{i,j} + v_{i-1,j}) \cdot h^{-2} + \\ + (v_{i,j+1} - 2v_{i,j} + v_{i,j-1}) \cdot h^{-2}] + \lambda v_{i,j} = f_{i,j}\,. \end{aligned} \qquad (0.19)$$

Since, evidently, operators (0.16) are bounded (for fixed $h$), then, in virtue of contraction mapping principle, equation (0.18) for any $f^h \in C^h$ has a unique solution $v^h \in C^h$, if $\lambda > 0$ is sufficiently large. Further we apply the maximum principle [to system (0.19)] and obtain estimate

$$\|v^h\|_{C^h} \leq \lambda^{-1} \cdot \|f^h\|_{C^h}. \qquad (0.20)$$

Therefore equation (0.18) has the unique solution $v^h \in C^h$ for any $f^h \in C^h$ and $\lambda > 0$, i.e. operator $\lambda I - (D_1^{h,2} + D_2^{h,2})$ has the bounded inverse for any $\lambda > 0$, and estimate

$$\|[\lambda I - (D_1^{h,2} + D_2^{h,2})]^{-1}\|_{C^h \to C^h} \leq \lambda^{-1} \qquad (0.21)$$

is true. Since $D_k^{h,2} (k = 1, 2)$ are bounded operators (for fixed $h$), then coercive inequality (0.17) takes place. The value $M_C(h)$ in this inequality must tend to $+\infty$, when $h \to +0$, since the differential coercive inequality (0.9) is not true. It is the consequence of ill-posedness in $C$ of differential equation (0.1). From inequality (0.21) and from formulas (0.16), evidently, it follows, that we can put in inequality (0.17)

$$M_C(h) = M \cdot h^{-2} \qquad (0.22)$$

with some $1 \leq M < +\infty$, does not depending on $f^h \in C^h$ and $0 < h \leq 1$. It turns out that essentially more exact result takes place. Namely for the solution $v^h$ of equation (0.13) in $C^h$ coercive inequality (0.17) is true with

$$M_C(h) = M_0 \cdot \ln 1/h\,, \qquad (0 < h \leq 1/2) \tag{0.23}$$

with some $1 \leq M_0 < +\infty$, does not depending on $h$. It is in particular the consequence of theory of difference equation, which is devoted this paper. Formula (0.23) means, that

$$\sup_{f^h \in C^h, f^h \neq 0} [\|D_1^{h,2} v^h\|_{C^h} + \|D_2^{h,2} v^h\|_{C^h}] \cdot \|f^h\|_{C^h}^{-1} \leq M_0 \cdot \ln 1/h. \tag{0.24}$$

It turns out, that (0.24) is the exact with respect to order of $h \to +0$ estimate. In fact, let [see formula (0.10)]

$$v_{i,j} = v_1(x_1, x_2)(x_1 = ih, x_2 = jh; i, j = -\infty + \infty). \tag{0.25}$$

Then from formulas (0.11) it follows, that $(0 < x_1^2 + x_2^2 \leq 1/9)$

$$\begin{aligned}
&(v_{i+1,j} - 2 \cdot v_{i,j} + v_{i-1,j}) \cdot h^{-2} = \\
&= \int_0^1 y\{\int_{-1}^1 [2 \cdot \ln \frac{1}{(x_1 + hyz)^2 + x_2^2} + a_{1,1}(x_1 + hyz, x_2)] dz\} dy, \\
&(v_{i,j+1} - 2 \cdot v_{i,j} + v_{i,j-1}) \cdot h^{-2} = \\
&= \int_0^1 y\{\int_{-1}^1 [-2 \cdot \ln \frac{1}{x_1^2 + (x_2 + hyz)^2} + a_{1,2}(x_1, x_2 + hyz)] dz\} dy.
\end{aligned} \tag{0.26}$$

Therefore for some $a_0 > 0$ and sufficiently small $h > 0$ estimates o from below

$$\begin{aligned}
|(v_{i+1,j} - 2v_{i,j} + v_{i+1,j}) \cdot h^{-2}|, |(v_{i,j+1} - 2v_{i,j} + v_{i,j-1}) \cdot h^{-2}| \geq \\
\geq 8 \cdot (1 - a_0 \cdot h) \cdot \ln 1/h
\end{aligned} \tag{0.27}$$

are true. Finally, from (0.10) and (0.26) it follows, that estimates

$$\begin{aligned}
|f_{i,j}| \leq m_0, f_{i,j} \quad = \quad & -[(v_{i+1,j} - v_{i,j} + v_{i-1,j}) \cdot h^{-2} + \\
& + (v_{i,j+1} - 2v_{i,j} + v_{i,j-1}) \cdot h^{-2}] + v_{i,j}
\end{aligned} \tag{0.28}$$

take place for some $0 < m_0 < +\infty$, does not depending on $h$. Therefore from (0.27) and (0.28) it follows, that estimate from below

$$\sup_{f^h \in C^h, f^h \neq 0} [\|D_1^{h,2} v^h\|_{C^h} + \|D_2^{h,2} v^h\|_{C^h} + \|v^h\|_{C^h}] \cdot \|f^h\|_{C^h}^{-1} \geq \\ \geq \frac{8 - a_0 h}{m_0} \cdot \ln 1/h \tag{0.29}$$

holds for sufficiently small $h > 0$.

Let $v$ be the solution in $C$ of equation (0.1), having the continuous and bounded partial derivatives till the fourth order. Let further $v_{i,j} (i, j = \overline{-\infty, +\infty})$ be the solution of system (0.12) for

$$f_{i,j} = f(ih, jh). \tag{0.30}$$

Then, evidently, values

$$z_{i,j} = v(ih, jh) - v_{i,j} \tag{0.31}$$

are the solution of system

$$-[(z_{i+1,j} - 2z_{i,j} + z_{i-1,j}) \cdot h^{-2} + (z_{i,j+1} - 2z_{i,j} + z_{i,j-1}) \cdot h^{-2}] + \\ + z_{i,j} = \Gamma_{i,j}, \tag{0.32}$$

and for values $\Gamma_{i,j}$ estimates

$$|\Gamma_{i,j}| \leq M \cdot h^2 \tag{0.33}$$

take place for some $1 \leq M < +\infty$. Then from (0.24) estimate from above

$$\|D_1^{h,2} z^h\|_{C^h} + \|D_2^{h,2} z^h\|_{C^h} \leq M_1 \cdot h^2 \cdot \ln 1/h \tag{0.34}$$

follows for some $1 \leq M_1 < +\infty$, does not depending on $h$.

Finally, let $f(x_1, x_2) \not\equiv 0$ be the smooth function, which partial derivatives till the second order sufficiently quickly tend to zero, when $x_1^2 + x_2^2 \to +\infty$. Then, evidently

$$\sup_{x \in R^2} |\partial^4 v / \partial x_1^4 + \partial^4 v / \partial x_2^4| > 0, \tag{0.35}$$

and therefore estimate

$$\sup_{i,j = \overline{-\infty, +\infty}} |\Gamma_{i,j}| \geq m \cdot h^2 \tag{0.36}$$

is true for some $0 < m < +\infty$. Estimate (0.36) and triangle inequality lead us to estimate from below

$$\|D_1^{h,2} z^h\|_{C^h} + \|D_2^{h,2} z^h\|_{C^h} \geq m_1 \cdot h^2 \tag{0.37}$$

for some $0 < m_1 < +\infty$, does not depending on $h$. Estimates (0.34) and (0.37) give the almost exact estimate of convergence rate of difference method (0.12) of approximate solution of differential equation (0.1) in the difference coercive norm.

### 0.3 The content of paper

This paper is devoted to the investigation of well-posedness of differential equation

$$d^2v/dt^2 + Av = f(t)\,, \qquad (-\infty < t < +\infty) \tag{0.38}$$

and its difference analog

$$-(v_{i+1} - 2v_i + v_{i-1}) \cdot h^{-2} + Av_i = f_i\,, \qquad (i = \overline{-\infty, +\infty}) \tag{0.39}$$

in arbitrary Banach space $E$. Here $A$ is the (unbounded) linear closed operator in $E$ with dense in $E$ domain $D(A)$.

About the notion of well-posedness of differential and difference equations see monograph [1].

The investigation of well-posedness is based on the property of positivity of linear operators. About this notion see monograph [2].

Equation (0.38) is considered as operator equation in the functional (abstract) Hölder space $C^\alpha(E)(0 < \alpha < 1)$ and for any positive in $E$ operator $A$ coercive inequality

$$\|Av\|_{C^\alpha(E)} \leq M \cdot \alpha^{-1} \cdot (1-\alpha)^{-1} \cdot \|f\|_{C^\alpha(E)} \tag{0.40}$$

is established for its solution $v$ in $C^\alpha(E)$ with some $1 \leq M < +\infty$, does not depending on $f \in C^\alpha(E)$ and $0 < \alpha < 1$.

To the differential equation (0.38) the P. Grisvard's theory [3], is applicable but it leads us to the coercive inequality

$$\|Av\|_{C^\alpha(E)} \leq M \cdot \alpha^{-2} \cdot (1-\alpha)^{-1} \cdot \|f\|_{C^\alpha(E)}. \tag{0.41}$$

The difference equation (0.39) is considered as operator equation in the Hölder space $C^{h,\alpha}(E)(0 < \alpha < 1)$ of (abstract) grid function, and for any strongly positive in $E$ operator $A$ it is established coercive inequality

$$\|Av^h\|_{C^{h,\alpha}(E)} \leq M \cdot \alpha^{-1} \cdot (1-\alpha)^{-1} \cdot \|f^h\|_{C^{h,\alpha}(E)} \qquad (0.42)$$

for its solution $v^h$ in $C^{h,\alpha}(E)$ with some $1 \leq M < +\infty$, does not depending not only on $f^h \in C^{h,\alpha}(E)$ and $0 < \alpha < 1$, but on $h$. From inequality (0.42) it follows, that

$$\|Av^h\|_{C^h(E)} \leq M \cdot \ln 1/h \cdot \|f^h\|_{C^h(E)}, \qquad (0.43)$$

where $C^h(E)$ is the Banach space of uniformly bounded grid function $\psi^h = (\psi_i \in E; i = \overline{-\infty, +\infty})$. Inequality (0.43) leads us to the formula (0.23) of exact value $M_C(h)$ in the difference coercive inequality (0.17)

To difference equation (0.39) also the P. Grisvard's theory is applicable even in more general case, when $A$ is only positive operator in $E$, but it which leads us to inequality

$$\|Av^h\|_{C^{h,\alpha}(E)} \leq M \cdot \alpha^{-2} \cdot (1-\alpha)^{-1} \|f^h\|_{C^{h,\alpha}(E)}. \qquad (0.44)$$

From (1.44) only estimate

$$\|Av^h\|_{C^h(E)} \leq M \cdot \overset{2}{\ln} 1/h \cdot \|f^h\|_{C^h(E)} \qquad (0.45)$$

follows.

## 1. Differential equation of the second order in the Banach space

### 1.1 Well-Posedness in the space $C(E)$

In the arbitrary Banach space $E$ differential equation

$$-v''(t) + Av(t) = f(t), \qquad (-\infty < t < +\infty) \qquad (1.1)$$

is considered. Here $v(t)$ and $f(t)$ are unknown and given (abstract) functions, defined on $(-\infty, +\infty)$ with values in $E$; $A$ is acting in $E$

linear (unbounded) closed operator with dense in $E$ domain $D(A)$. We will consider differential equation (1.1) as operator equation in functional Banach space $C(E) = C[(-\infty, +\infty), E]$ of all defined on $(-\infty, +\infty)$ (abstract) continuous and bounded functions $\psi(t)$ with norm

$$\|\psi\|_{C(E)} = \sup_{-\infty<t<+\infty} \|\psi(t)\|_E. \tag{1.2}$$

We will call function $v(t) \in C(E)$ the solution in $C(E)$ of equation (1.1), if $v''(t), Av(t) \in C(E)$ and equation (1.1) is satisfied. For existence of such solution $v(t)$ of equation (1.1), evidently, it is necessary, that

$$f(t) \in C(E). \tag{1.3}$$

We will say, that equation (1.1) is well-posed in $C(E)$, if two conditions are satisfy:

($a_1$) For any function $f(t) \in C(E)$ equation (1.1) has unique in $C(E)$ solution $v(t) \equiv v(t; f)$. It, in particular, means, that $v(t; f)$ is acting in $C(E)$ additive and homogeneous operator, which is defined on whole $C(E)$, and operators

$$d^2[v(t; f)]/dt^2 \quad \text{and } Av(t; f). \tag{1.4}$$

have these properties also.

($a_2$) The operator $v(t; f)$ is continued in $C(E)$. It means, that inequality

$$\|v(t; f)\|_{C(E)} \leq M_S \cdot \|f(t)\|_{C(E)} \tag{1.5}$$

holds with some $1 \leq M_S < +\infty$, does not depending on $f(t) \in C(E)$. From this it follows, that operators (1.4) are closed in Banach space $C(E)$ and, in virtue of Banach's theorem, are bounded. This leads us to coercive inequality

$$\|v''(t)\|_{C(E)} + \|Av(t)\|_{C(E)} \leq M_C \cdot \|f(t)\|_{C(E)} \tag{1.6}$$

for solution of well-posed in $C(E)$ equation (1.1) with some $1 \leq M_C < +\infty$, does not depending on $f(t) \in C(E)$. Inequality (1.6) permits to investigate spectral properties of operator $A$. For any $u \in D(A)$ and $\lambda \geq 0$ we will put

$$\lambda u + Au = \psi. \tag{1.7}$$

Then, evidently, function $v(t) = \exp\{i\sqrt{\lambda}t\} \cdot u(i = \sqrt{-1})$ is solution in $C(E)$ of equation (1.1) for function $f(t) = \exp\{i\sqrt{\lambda}t\} \cdot \psi$. Therefore from (1.6) it follows, that

$$\lambda\|u\|_E + \|Au\|_E \leq M_C \cdot \|\psi\|_E. \tag{1.8}$$

We will suppose, that operator $A$ has bounded in $E$ inverse. Then, evidently, from inequality (1.8) it follows, that operator $\lambda I + A$ has bounded inverse for any $\lambda \geq 0$, and estimate

$$\|(\lambda I + A)^{-1}\|_{E \to E} \leq M \cdot (\lambda + 1)^{-1} \tag{1.9}$$

is true for some $1 \leq M < +\infty$. Such operator is called (see [2]) positive operator. So, if equation (1.1) is well-posed in functional Banach space $C(E)$, then $A$ is positive operator in Banach space $E$ (under condition, that operator $A$ has bounded inverse). Whether the positivity of operator $A$ in $E$ is the sufficient condition of the well-posedness of equation (1.1) in $C(E)$?

For arbitrary Banach space $F$ let us consider the acting in $C(F) = C[(-\infty, +\infty), F]$ operator $A$, defending by formula

$$A\psi(x) = -\psi''(x) + \psi(x)\,, \qquad (-\infty < x < +\infty) \tag{1.10}$$

on functions $\psi(x) \in C(F)$, such that $\psi''(x) \in C(F)$. Evidently, that operator $\lambda I + A$ has bounded inverse for any $\lambda \geq 0$, and formula

$$(\lambda I + A)^{-1}\psi(x) = \frac{1}{2\sqrt{\lambda + 1}} \int_{-\infty}^{+\infty} \exp\{-\sqrt{1 + \lambda}|x - y|\} \cdot \psi(y)dy \tag{1.11}$$

holds. From (1.11) estimate (1.9) (for $M = 1$) follows, i.e. $A$ is positive operator in Banach space $E = C(F)$. However the contraexample from §0 introduction shows, that equation (1.1) is ill-posed in $C(F)$.

### 1.2 Formula for solution in $C(E)$ of equation (1.1)

From estimate (1.9) it follows, that operator $\lambda I + A$ has bounded inverse for complex numbers $\lambda = \sigma + i\tau \in \mathcal{G}_\varepsilon^+ = \mathcal{G}_\varepsilon^+(M)(0 < \varepsilon < 1)$, such that, $(\sigma \leq 0)$,

$$|\tau| \leq \frac{1 - \varepsilon}{M} \cdot (1 + \sigma)(\sigma \geq 0) \quad \text{or } (\sigma^2 + \tau^2)^{1/2} \leq \frac{1 - \varepsilon}{M}\,, \tag{1.12}$$

and estimate

$$\|(\lambda I + A)^{-1}\|_{E\to E} \le M_1 \cdot \varepsilon^{-1} \cdot (1+|\lambda|)^{-1} \qquad (1.13)$$

holds for some $1 \le M_1 < +\infty$, does not depending on $0 < \varepsilon < 1$. It means, that spector $\sigma$ (A) of positive operator $A$ is outside of set $\mathcal{G}_\varepsilon^- = -\mathcal{G}_\varepsilon^+$, and inside of $\mathcal{G}_\varepsilon^-$ and on its boundary $\partial\mathcal{G}_\varepsilon^-$ estimate

$$\|(\lambda I - A)^{-1}\|_{E\to E} \le M_1 \cdot \varepsilon^{-1} \cdot (1+|\lambda|)^{-1} \qquad (1.14)$$

holds. Therefore for any analytic in the neighborhood of $\sigma$ (A) (scalar) function $\psi(z)$, such that estimate

$$(1+|z|)^\alpha \cdot |\psi(z)| \le M_2 \qquad (1.15)$$

takes place for some $0 < \alpha < +\infty, 1 \le M_2 < +\infty$, Cauchy-Riesz's formula defines bounded operator

$$\psi(A) = \frac{1}{2\pi i} \int\limits_{\partial\mathcal{G}_\varepsilon^-} \psi(z) \cdot (zI - A)^{-1} dz\,, \qquad (i = \sqrt{-1}). \qquad (1.16)$$

In particular (see [2]), negative fractional powers $A^{-\alpha}(\alpha > 0)$ of positive operator $A$ are defined, $A^{-\alpha} = (A^{-1})^\alpha$ for integer $\alpha > 0$, and semigroup identity

$$A^{-(\alpha+\beta)} = A^{-\alpha} \cdot A^{-\beta}\,, \quad (0 < \alpha, \beta < +\infty) \qquad (1.17)$$

is true. From this it follows, that positive fractional proves $A^\alpha$ can be defined by formula

$$A^\alpha = (A^{-\alpha})^{-1}\,, \qquad (\alpha > 0). \qquad (1.18)$$

Operators $A^\alpha, (\alpha > 0)$, already are unbounded, and their domains $D(A^\alpha)$ are dense in $E$. Following moment inequality

$$\|A^\alpha u\|_E \le M(\alpha,\beta) \cdot \|A^\beta u\|_E^{\alpha/\beta} \cdot \|u\|_E^{1-\alpha/\beta} \qquad (1.19)$$

$[0 \le \alpha \le \beta < +\infty, u \in D(A^\beta)]$ takes place with some $1 \le M(\alpha,\beta) < +\infty$, does not depending on $u \in D(A^\beta)$. Operators $A^\alpha$ for $\alpha \in (0,1)$

have better spectral properties, than operator A. In particular, from identity [see (1.17)]

$$\lambda I + A = (\sqrt{\lambda} I - \sqrt{A}) \cdot (\sqrt{\lambda} I + \sqrt{A}), \qquad (1.20)$$

inequality (1.19) (for $\alpha = 1/2, \beta = 1$) and estimate (1.14) it follows, that operator $\sqrt{\lambda} I - \sqrt{A}$ has the bounded inverse for $\lambda \in \mathcal{G}_\varepsilon^-$, and estimate

$$\|(\sqrt{\lambda} I - \sqrt{A})^{-1}\|_{E\to E} \leq M_3 \cdot \varepsilon^{-1} \cdot (1 + |\lambda|^{1/2})^{-1} \qquad (1.21)$$

holds. In particular, this means, that operator $\lambda I + \sqrt{A}$ has bounded inverse for any complex number $\lambda$, such that $Re\lambda \geq 0$, and estimate

$$\|(\lambda I + \sqrt{A})^{-1}\|_{E\to E} \leq M \cdot (1 + |\lambda|)^{-1} \qquad (1.22)$$

holds. Acting in Banach space $E$ linear operator $B$ with dense in $E$ domain $D(B)$ is called strongly positive (see [2]), if operator $\lambda I + B$ has bounded inverse for any complex number $\lambda$ with $Re\lambda \geq 0$, and estimate

$$\|(\lambda I + B)^{-1}\|_{E\to E} \leq M \cdot (1 + |\lambda|)^{-1} \qquad (1.23)$$

is true for some $1 \leq M < +\infty$. Operator B is strongly positive, iff $-B$ is generator of analytic semigroup $\exp\{-\tau B\} (\tau \geq 0)$ of linear bounded operator in $E$ with exponentially decreasing norm, when $t \to +\infty$, i.e. estimates

$$\|\exp\{-tB\}\|_{E\to E}, \|t \cdot B \cdot \exp\{-tB\}\|_{E\to E} \leq$$
$$\leq M(B) \cdot \exp\{-a(B)t\}, \quad (t > 0) \qquad (1.24)$$

are true for some $1 \leq M(B) < +\infty, 0 < a(B) < +\infty$. Thus $\sqrt{A}$ is strongly positive operator, and estimates

$$\|\exp\{-t\sqrt{A}\}\|_{E\to E}, \|t \cdot \sqrt{A} \cdot \exp\{-t\sqrt{A}\}\|_{E\to E} \leq$$
$$\leq M(\sqrt{A}) \cdot \exp\{-a(\sqrt{A}) \cdot t\}, \quad (t > 0) \qquad (1.25)$$

take place. The consideration of operator $\sqrt{A}$ permits to reduce differential equation (1.1) of the second order to equivalent system

$$v'(t) + \sqrt{A} v(t) = z(t), -z'(t) + \sqrt{A} z(t) = f(t), \quad (-\infty < t < +\infty) \qquad (1.26)$$

of differential equations of the first order. This prompt [see formula (1.11)] that for solution $v(t)$ in $C(E)$ of equation (1.1) formula

$$v(t) = \frac{1}{2\sqrt{A}} \int_{-\infty}^{+\infty} \exp\{-\sqrt{A}|t-s|\} \cdot f(s)ds \tag{1.27}$$

must be true. In fact, with help of integration by parts and estimates (1.25) identity

$$\frac{1}{2\sqrt{A}} \int_{-\infty}^{+\infty} \exp\{-\sqrt{A}|t-s|\} \cdot v''(s)ds =$$

$$= -v(s) + \frac{1}{2\sqrt{A}} \int_{-\infty}^{+\infty} \exp\{-\sqrt{A}|t-s|\} \cdot Av(s)ds \tag{1.28}$$

is established. From (1.28) and closedness in $E$ of operator $A$, in virtue of (1.1), formula (1.27) follows. This, in particular, means, that equation (1.1) cannot have more that one solution in $C(E)$. Finally, it is easy to see, that formula (1.27) defines solution in $C(E)$ of equation (1.1), if, for example,

$$Af(t) \quad \text{or } f''(t) \quad \text{belong to } C(E). \tag{1.29}$$

It turns out, that formula (1.27) defines solution in $C(E)$ of equation (1.1) under essentially less restriction on smoothness of function $f(t)$.

### 1.3 Well-posedness in the space $C^\alpha(E)$

We will consider differential equation (1.1) as operator equation in functional Banach space of Hölder $C^\alpha(E) = C[(-\infty,+\infty),E)$ $(0 < \alpha < 1)$ of all defined on $(-\infty,+\infty)$ with value in $E$ (abstract) functions $\psi(t)$ with norm

$$\|\psi\|_{C^\alpha(E)} = \sup_{-\infty<t<+\infty} \|\psi(t)\|_E + \sup_{-\infty<t<t+s<+\infty} \|\psi(t+s)-\psi(t)\|_E \cdot s^{-\alpha}. \tag{1.30}$$

Analogously to the case of space $C(E)$ the notion of solution $v(t)$ in $C^\alpha(E)$ of equation (1.1) in the space $C^\alpha(E)$ are defined. The well-posedness in $C^\alpha(E)$ of equation (1.1) means, that coercive inequality

$$\|v''\|_{C^\alpha(E)} + \|Av\|_{C^\alpha} \leq M(\alpha) \cdot \|f\|_{C^\alpha(E)} \tag{1.31}$$

is true for its solutions $v(t)$ in $C^\alpha(E)$ with some $1 \leq M(\alpha) < +\infty$, does not depending on $f(t) \in C^\alpha(E)$. As in the case of space $C(E)$ it is established, that from (1.31) the positivity of operator A in $E$ follows. It turns out, that this property of operator in $E$ is not only the necessary, but also the sufficient condition of well-posedness of equation (1.1) in the space $C^\alpha(E)$ for all $\alpha \in (0,1)$. Since, evidently, the set of smooth (abstract) functions [for example, functions, satisfied condition (1.29)] is dense in $C^\alpha(E)$, then it is sufficiently to establish the inequality

$$\|Av(t)\|_{C^\alpha(E)} \leq M(\alpha) \cdot \|f(t)\|_{C^\alpha(E)} \tag{1.32}$$

for function $v(t)$, defined by formula (1.27) for smooth function $f(t)$ with some $1 \leq M(\alpha) < +\infty$, does not depending on $f(t)$. From (1.27), evidently, it follows, that

$$Av(t) = \frac{\sqrt{A}}{2} \int_0^{+\infty} \exp\{-\sqrt{A}|t-s|\} \cdot [f(s) - f(t)]ds + f(t). \tag{1.33}$$

The application of estimates (1.25) leads us to estimate

$$\|Av(t)\|_E \leq \frac{1}{2} \cdot M(\sqrt{A}) \cdot \int_{-\infty}^{+\infty} \exp\{-a(\sqrt{A})|t-s|\} \cdot |t-s|^{\alpha-1} ds \cdot$$
$$\cdot H^\alpha(f) + \|f\|_{C(E)}. \tag{1.34}$$

Here and in what follows

$$H^\alpha(f) = \sup_{-\infty<t+s<+\infty} \|f(t+s) - f(t)\|_E \cdot s^{-\alpha}. \tag{1.35}$$

It means, that estimate

$$\|Av\|_{C(E)} \leq M(\sqrt{A}) \cdot [a(\sqrt{A})]^{-\alpha} \cdot \Gamma(\alpha) \cdot H^\alpha(f) + \|f\|_{C(E)} \tag{1.36}$$

is true. Here and in what follows $\Gamma(\alpha)$ is the Euler's Gamma-function. Further from (1.33) for $h > 0$ identity

$$
\begin{aligned}
&Av(t+h) - Av(t) = \\
&= [f(t+h) - f(t)] + \frac{\sqrt{A}}{2} \int_{-\infty}^{+\infty} \langle \exp\{-\sqrt{A}|z+h-s|\}\cdot \\
&\cdot [f(s) - f(t+h)] - \exp\{-\sqrt{A}|t-s|\} \cdot [f(s) - f(t)]\rangle ds \\
&= \frac{\sqrt{A}}{2} \int_{-2h}^{2h} \langle \exp\{-\sqrt{A}|z+h|\} \cdot [f(t-z) - f(t+h)] - \\
&- \exp\{-\sqrt{A}|z|\} \cdot [f(t-z) - -f(t)]\rangle dz + \\
&+ \frac{\sqrt{A}}{2} \int_{2h}^{+\infty} [\exp\{-\sqrt{A}|z+h|\} - \exp\{-\sqrt{A}|z|\}]\cdot \\
&\cdot [f(t-z) - f(t)]dz + \frac{\sqrt{A}}{2} \int_{-\infty}^{-2h} \exp\{-\sqrt{A}|z+h|\} - \\
&- \exp\{-\sqrt{A}|z|\}] \cdot [f(t-z) - f(t+h)]dz + \\
&+ [1 + \tfrac{1}{2}\exp\{-2h\sqrt{A}\} - \tfrac{1}{2}\cdot \exp\{-3h\sqrt{A}\}] \cdot [f(t+h) - f(t)] \\
&= J_1(h) + J_2(h) + J_3(h) + J_4(h)
\end{aligned}
\tag{1.37}
$$

follows. The application of estimates (1.25) gives

$$
\begin{aligned}
\|J_1(h)\|_E &\le \frac{1}{2} \cdot M(\sqrt{A}) \cdot H^\alpha(f) \cdot \int_{-2h}^{2h} [|z+h|^{\alpha-1} + |z|^{\alpha-1}]dz \\
&= (2\alpha)^{-1} \cdot M(\sqrt{A}) \cdot H^\alpha(f)(3^\alpha + 2^{\alpha+1} + 1) \cdot h^\alpha.
\end{aligned}
\tag{1.38}
$$

Further for $0 < z < z+h$ we have

$$
\exp\{-\sqrt{A}|z+h|\} - \exp\{-\sqrt{A}|z|\} = -\int_{z}^{z+h} \sqrt{A} \cdot \exp\{-\sqrt{A}\cdot s\} ds,
$$

and therefore, in virtue of estimate (1.25), we obtain

$$
\|J_2(h)\|_E \le \frac{4 \cdot M^2(\sqrt{A}) \cdot H^\alpha(f)}{2} \cdot \int_{2h}^{+\infty} [\int_{z}^{z+h} \frac{ds}{s^2}] z^\alpha dz
$$

$$= \frac{2^\alpha \cdot M^2(\sqrt{A}) \cdot H^\alpha(f)}{1-\alpha} \cdot h^\alpha. \tag{1.39}$$

Analogously

$$\begin{aligned} \|J_3(h)\|_E &\leq \frac{4 \cdot M^2(\sqrt{A}) \cdot H^\alpha(f)}{2} \cdot \int_{2h}^{+\infty} [\int_{z-h}^{z} \frac{ds}{s^2}] z^\alpha dz \\ &= \frac{2^{\alpha+1} \cdot M^2(\sqrt{A}) \cdot H^\alpha(f)}{1-\alpha} \cdot h^\alpha. \end{aligned} \tag{1.40}$$

Finally, evidently, that

$$\|J_4(h)\|_E \leq [1 + M(\sqrt{A})] \cdot H^\alpha(f) \cdot h^\alpha. \tag{1.41}$$

Identity (1.37) and estimates (1.38)–(1.41) mean that estimate

$$\begin{aligned} H^\alpha(Av) \leq\ & [M(\sqrt{A}) \tfrac{3^\alpha + 2^{\alpha+1} + 1}{2\alpha} + \\ & + M^2(\sqrt{A}) \tfrac{2^\alpha + 2^{\alpha+1}}{1-\alpha} + 1 + M(\sqrt{A})] \cdot H^\alpha(f) \end{aligned} \tag{1.42}$$

is true. Estimates (1.36) and (1.42) lead us to following result:

THEOREM 1.1. *Equation (1.1) is well-posed in functional Banach space $C^\alpha(E)(0 < \alpha < 1)$, iff $A$ is positive operator in Banach space $E$. For solution $v$ in $C^\alpha(E)$ of equation (1.1) coercive inequality*

$$\|Av\|_{C^\alpha(E)} \leq M \cdot \alpha^{-1} \cdot (1-\alpha)^{-1} \cdot \|f\|_{C^\alpha(E)} \tag{1.43}$$

*takes place with some $1 \leq M < +\infty$, does not depending on $f \in C^\alpha(E)$ and $\alpha \in (0,1)$.*

## 1.4 The application of P. Grisvard's theory

From definition (1.9), evidently, it follows, that $A_1 = A - a_1 I$ for sufficiently small $a_1 > 0$ is also positive operator, i.e., estimate

$$\|(\lambda + A_1)^{-1}\|_{E \to E} \leq M_1 \cdot (\lambda + 1)^{-1} \tag{1.44}$$

holds for any $\lambda \geq 0$ and some $1 \leq M_1 < +\infty$. It means, that spector $\sigma(A_1)$ lies outside the correspondent set $\mathcal{G}_\varepsilon^-(M_1), (0 < \varepsilon <$

1). Therefore there exist numbers $\varphi \in (0, \pi)$, such that $\sigma(A_1)$ lies inside the angle of complex plane

$$|\arg \lambda| \leq \varphi, \tag{1.45}$$

and outside of this set and on its boundary estimate

$$\|(\lambda I - A_1)^{-1}\|_{E \to E} \leq M(\varphi) \cdot (|\lambda| + 1)^{-1} \tag{1.46}$$

holds with some $1 \leq M(\varphi) < +\infty$. The infimum of such numbers $\varphi$ is called the spectral angle $\varphi(A_1) = \varphi(A_1, E)$ of positive operator $A_1$ in Banach space $E$. It is evident, that

$$0 \leq \varphi(A_1) < \pi. \tag{1.47}$$

We will continue operator $A_1$ to acting in $C(E)$ operator $\tilde{A}_1$ by formula

$$(\tilde{A}_1 \psi)(t) = A_1 \psi(t), \qquad (-\infty < t < +\infty) \tag{1.48}$$

on functions $\psi(t) \in C(E)$, such that $A_1 \psi(t) \in C(E)$. It is evident, that $\tilde{A}_1$ is positive operator in $C(E)$, inequality

$$\|(\lambda + A_1)^{-1}\|_{C(E) \to C(E)} \leq M_1 \cdot (\lambda + 1)^{-1} \tag{1.49}$$

holds with the same numbers $M_1$ and $\lambda$, that in the inequality (1.44), and

$$\varphi[\tilde{A}_1, C(E)] = \varphi(A_1, E). \tag{1.50}$$

Further we will define acting in $C(E)$ operator $B_1$ by formula

$$(\tilde{B}_1 \psi)(t) = -\psi^{''}(t) + a_1 \psi(t) \tag{1.51}$$

on functions $\psi(t) \in C(E)$, such that $\psi^{''}(t) \in C(E)$. It is evident, that for any complex numbers $\lambda$ with $|\arg \lambda| < \pi$ and $f(t) \in C(E)$ differential equation

$$\lambda v(t) - v^{''}(t) + a_1 v(t) = f(t) \tag{1.52}$$

has unique solution $v(t) = v(t; \lambda)$ in $C(E)$, defining [see formula (1.11)] by formula

$$\frac{1}{2\sqrt{\lambda + a_1}} \int\limits_{-\infty}^{+\infty} \exp\{-\sqrt{\lambda + a_1}|t - s|\} \cdot f(s) ds, \tag{1.53}$$

which leads us to estimate

$$\|v\|_{C(E)} \leq [|\lambda|^2 + a_1^2 - 2|\lambda| \cdot a_1 \cdot \cos(\pi - |\arg\lambda|)]^{-1} \cdot \|f\|_{C(E)}. \quad (1.54)$$

It means, that $\tilde{B}_1$ is positive operator in $C(E)$, and

$$\varphi[\tilde{B}_1, C(E)] = 0. \quad (1.55)$$

In particular, this means, that $-\tilde{B}_1$ is generator of analytic semigroup $\exp\{-t\tilde{B}_1\}(t \geq 0)$ with exponentially decreasing norm, i.e. [see estimates (1.24)] estimates

$$\|\exp\{-t\tilde{B}_1\}\|_{C(E)\to C(E)},\ \|t\tilde{B}_1 \cdot \exp\{-t\tilde{B}_1\}\|_{C(E)\to C(E)} \leq$$
$$\leq M(\tilde{B}_1) \cdot \exp\{-ta(\tilde{B}_1)\} \quad (1.56)$$

$(t > 0)$ hold. Now we will consider differential equation (1.1) as operator equation

$$\tilde{B}_1 v + \tilde{A}_1 v = f \quad (1.57)$$

in functional Banach space $C(E)$. Operator $\tilde{B}_1$ and $\tilde{A}_1$, evidently, commute, and, in virtue of (1.47) and (1.55),

$$\varphi(\tilde{B}_1) + \varphi(\tilde{A}_1) < \pi. \quad (1.58)$$

Therefore to operator equation (1.57) the P. Grisvard's theory [3] is applicable.

Let $C_\alpha(\tilde{B}_1) = C_\alpha[C(E), \tilde{B}_1](0 < \alpha < 1)$ be functional Banach space with norm

$$\|\psi\|_{C_\alpha(\tilde{B}_1)} = \sup_{\lambda>0} \|\lambda^\alpha \tilde{B}_1(\lambda I + \tilde{B}_1)^{-1}\psi\|_{C(E)}. \quad (1.59)$$

It turns out (Grisvard's theorem) that for any $f \in C_\alpha(\tilde{B}_1)$ there exists unique solution $v = v(t) = v(t; f)$ in $C_\alpha(\tilde{B}_1)$ of equation (1.57), and coercive inequality

$$\|\tilde{A}_1 v\|_{C_\alpha(\tilde{B}_1)} \leq \tilde{M}_1 \cdot \alpha^{-1} \cdot (1-\alpha)^{-1} \cdot \|f\|_{C_\alpha(B_1)}, \quad (1.60)$$

takes place for some $1 \leq \tilde{M}_1 < +\infty$, does not depending on $f \in C_\alpha(\tilde{B}_1)$ and $\alpha \in (0,1)$. Further, in interpolation theory of linear operators (see, for examples, references in [1]) is proved, that spaces

$C_{\alpha/2}(B_1)$ coincide with Hölder spaces $C^\alpha(E)$ for $\alpha \in (0,1)$. In fact, from (1.53) for any $\lambda > 0$ and $f(t) \in C(E)$, evidently, it follows, that

$$\begin{aligned} &\tfrac{1}{\lambda} f(t) - (\lambda I + \tilde{B}_1)^{-1} f(t) = \tfrac{a_1}{\lambda(\lambda+a_1)} \cdot f(t) + \\ &+ \tfrac{1}{2\sqrt{\lambda+a_1}} \int\limits_{-\infty}^{+\infty} \exp\{-\sqrt{\lambda+a_1}|t-s|\} \cdot [f(t)-f(s)]ds, \end{aligned} \tag{1.61}$$

and therefore identity

$$\begin{aligned} &\lambda^{\alpha/2} \tilde{B}_1 (\lambda I + \tilde{B}_1)^{-1} f(t) = \tfrac{a_1 \lambda^{\alpha/2}}{\lambda+a_1} \cdot f(t) + \tfrac{1}{2}(\tfrac{\lambda}{\lambda+a_1})^{1+\alpha/2} \cdot \\ &\cdot (\sqrt{\lambda+a_1})^{1+\alpha} \int\limits_{-\infty}^{+\infty} \exp\{-\sqrt{\lambda+a_1}|t-s|\} \cdot [f(t)-f(s)]ds \end{aligned} \tag{1.62}$$

is true. Let $f(t) \in C^\alpha(E)$. Then from (1.62) it follows, that $f(t) \in C_{\alpha/2}(\tilde{B}_1)$, and estimate

$$\|f\|_{C_{\alpha/2}(\tilde{B}_1)} \leq a_1^{\alpha/2} \cdot (\alpha/2)^{\alpha/2} \cdot (1-\alpha/2)^{1-\alpha/2} \cdot \|f\|_{C(E)} + \Gamma(1+\alpha) \cdot H^\alpha(f) \tag{1.63}$$

holds. Further, since $\tilde{B}_1$ is positive operator in $C(E)$, then

$$\begin{aligned} B_1^{-1} f(t) &= \int\limits_0^{+\infty} (\lambda + \tilde{B}_1)^{-2} f(t) dt \\ &= \int\limits_0^{+\infty} \lambda^{-\alpha/2} (\lambda + \tilde{B}_1)^{-1} \cdot \lambda^{\alpha/2} (\lambda + \tilde{B}_1)^{-1} f(t) d\lambda \end{aligned} \tag{1.64}$$

for any $f \in C(E)$. If $f \in C_\alpha(\tilde{B}_1)$, then from definition (1.59) it follows, that function $\psi_\alpha(\lambda,t) = \lambda^{\alpha/2} \tilde{B}_1 (\lambda + \tilde{B}_1)^{-1} f(t) \in C(E)$ for any $\lambda \geq 0$, and estimate

$$\|\psi_\alpha(\lambda,t)\|_{C(E)} \leq \|f(t)\|_{C_{\alpha/2}(\tilde{B}_1)} \tag{1.65}$$

is true. In virtue of closeness in $C(E)$ of operator $\tilde{B}_1$, identity

$$f(t) = \int\limits_0^{+\infty} \lambda^{-\alpha/2} (\lambda + \tilde{B}_1)^{-1} \psi_\alpha(\lambda,t) d\lambda \tag{1.66}$$

takes place. Applying formula (1.53), we will obtain, that

$$f(t) = \int\limits_0^{+\infty} \lambda^{-\alpha/2} \cdot [\frac{1}{2\sqrt{\lambda+a_1}} \int\limits_{-\infty}^{+\infty} \exp\{-\sqrt{\lambda+a_1}|t-s|\} \cdot \psi_\alpha(\lambda,s) ds] d\lambda. \tag{1.67}$$

From formula (1.67) and estimate (1.65) it follows, that

$$\begin{aligned}\|f(t)\|_{C(E)} &\leq \int_0^{+\infty} \lambda^{-\alpha/2} \cdot (\lambda + a_1)^{-1} d\lambda \cdot \|f(t)\|_{C_{\alpha/2}(\tilde{B}_1)} \\ &= a_1^{-\alpha/2} \cdot \frac{\pi}{\sin \pi\alpha/2} \cdot \|f(t)\|_{C_{\alpha/2}(\tilde{B}_1)}. \qquad (1.68)\end{aligned}$$

Further formula (1.67) leads us to identity

$$f(t+h) - f(t) = \int_0^{+\infty} \lambda^{-\alpha/2} \frac{1}{2\sqrt{\lambda + a_1}} \langle \int_{-\infty}^{\infty} [\exp\{-\sqrt{\lambda + a_1}|z+h|\} - \\ - \exp\{-\sqrt{\lambda + a_1}|z|\}] \psi_\alpha(\lambda, t - z) dz \rangle d\lambda. \qquad (1.69)$$

From identity (1.69) and estimate (1.65) it follows, that

$$\begin{aligned}&\|f(t+h) - f(t)\|_E \leq \\ &\leq \int_0^{+\infty} \lambda^{-\alpha/2} \cdot \frac{1}{2\sqrt{\lambda + a_1}} \langle \int_{-\infty}^{+\infty} |\exp\{-\sqrt{\lambda + a_1}|z+h|\} - \\ &- \exp\{-\sqrt{\lambda + a_1}|z|\} dz \rangle d\lambda \cdot \|f(t)\|_{C_{\alpha/2}(\tilde{B}_1)}. \qquad (1.70)\end{aligned}$$

The substitution $y = z \cdot \sqrt{\lambda + a_1}, s = |h| \cdot \sqrt{\lambda + a_1}$ leads us to inequality

$$\|f(t+h) - f(t)\|_E \leq |h|^\alpha \cdot \int_{|h|\sqrt{a_1}}^{+\infty} \frac{4 \cdot (1 - \exp\{-\frac{s}{2}\})}{(s^2 - h^2 a)^{\alpha/2}} \cdot \frac{ds}{s} \cdot \|f(t)\|_{C_{\alpha/2}(\tilde{B}_1)}. \qquad (1.71)$$

From (1.71) estimate

$$H^\alpha(f) \leq 4 \cdot \alpha^{-1} \cdot (1 - \alpha)^{-1} \cdot \|f\|_{C_{\alpha/2}(\tilde{B}_1)} \qquad (1.72)$$

follows. Inequality (1.63), (1.68) and (1.72) mean, that following inequalities of equivalence of norms of spaces $C^\alpha(E)$ and $C_{\alpha/2}(\tilde{B}_1)$

$$\alpha \cdot (1 - \alpha) \cdot M^- \cdot \|f\|_{C^\alpha(E)} \leq \|f\|_{C_{\alpha/2}(\tilde{B}_1)} \leq M^+ \cdot \|f\|_{C^\alpha(E)} \qquad (1.73)$$

are true for some $0 < M^{-1} \leq M^{+} < +\infty$, do not depending on $f \in C^{\alpha}(E) = C_{\alpha/2}(\tilde{B}_1)$ and $\alpha \in (0,1)$. Further from coercive inequality (1.60) under substitution $\alpha$ by $\alpha/2$ for $0 < \alpha < 1$ it follows, that

$$\|A_1 v\|_{C_{\alpha/2}(\tilde{B}_1)} \leq \frac{2\tilde{M}_1}{\alpha} \cdot \|f\|_{C_{\alpha/2}(\tilde{B}_1)}. \tag{1.74}$$

Inequalities (1.73) and (1.74) lead us to coercive inequality

$$\|A_1 v\|_{C^{\alpha}(E)} \leq 2\tilde{M}_1 \cdot M^{+} \cdot (M^{-1})^{-1} \cdot \alpha^{-2} \cdot (1-\alpha)^{-1} \cdot \|f\|_{C^{\alpha}(E)} \tag{1.75}$$

for solution of equation (1.1).

REMARK 1.1. Coercive inequality (1.43), evidently, is stronger, that coercive inequality (1.75) with respect to order $\alpha \to +0$. It will be important under consideration of difference analogue of differential equation (1.1).

## 2. Difference equation of the second order in the Banach space

### 2.1 The estimates of resolvent povers of operator $-B(h^2 A)$, and stability in the space $C^h(E)$

We will consider the difference analog of differential equation (1.1), namely difference equation

$$-(v_{i+1} - 2v_i + v_{i-1}) \cdot h^{-2} + Av_i = f_i\,, \qquad (i = \overline{-\infty, +\infty}). \tag{2.1}$$

Here $v_i \in D(A)$ and $f_i \in E$ are unknown and given elements, $h \in (0,1]$ is some given number. Difference equation of the second order (2.1) is equivalent to system of difference equations of the first order

$$(v_i - v_{i-1}) \cdot h^{-1} + \tilde{B} v_i = z_i, -(z_{i+1} - z_i) \cdot h^{-1} + \tilde{B} z_i = (1 + h\tilde{B}) f_i \tag{2.2}$$

$(i = \overline{-\infty, +\infty})$, which is analogous to system of differential equations (1.26). Here operator $B = B(h^2 A) = h\tilde{B}$ is defined by formula

$$B = h^2 A/2 + [(h^2 A/2)^2 + h^2 A]^{1/2} \tag{2.3}$$

i.e. $B$ is the solution of operator quadratic equation

$$B^2 \cdot (I + B)^{-1} = h^2 A. \tag{2.4}$$

We will consider the Banach space $C^h(E)$ of grid functions

$$\psi^h = (\psi_i \in E, i = \overline{-\infty, +\infty}) \tag{2.5}$$

with norm

$$\|\psi^h\|_{C^h(E)} = \sup_{i=\overline{-\infty,+\infty}} \|\psi_i\|_E. \tag{2.6}$$

System (2.2) permits to show, then for any $f^h \in C^h(E)$ there exists unique solution $v^h, Av^h = [Av_i, i = \overline{-\infty, +\infty}] \in C^h(E)$, of equation (2.1), defining by formula

$$Av_i = B \cdot (2+B)^{-1} \cdot \sum_{k=-\infty}^{+\infty} (1+B)^{-|i-k|} f_k\,, \qquad (i = \overline{-\infty, +\infty}) \tag{2.7}$$

which is analogous to formula (1.27). The basis of these statements will be given under supposition, that $A$ is positive operator in $E$, and estimate (1.9) will be comfortable to write in form

$$\|(\lambda I + A)^{-1}\|_{E \to E} \leq M(A) \cdot [\lambda + a(A)]^{-1} \tag{2.8}$$

for any $\lambda \geq 0$ and some $1 \leq M(A) < +\infty, 0 < a(A) < +\infty$. For the investigation of spectral properties of unbounded operator $B(h^2 A)$ we will construct the bounded operator $[\lambda I + B(h^2 A)]^{-1}$ for $\lambda \geq 0$. Since (scalar) function

$$B(z) = z/2 + (z^2/4 + z)^{1/2} \tag{2.9}$$

is analytic on whole complex plane, except points $z = 0, -4$ and $B(z) \cdot z^{-1} \to 1$, when $|z| \to +\infty$, then, in virtue to estimate (2.8), the Cauchy-Riesz's formula gives

$$[\lambda I + B(h^2 A)]^{-1} = \frac{1}{2\pi i} \int_{h^2 \partial G_\varepsilon^-} [\lambda + B(z)]^{-1} \cdot (zI - h^2 A)^{-1} dz\,. \tag{2.10}$$

Finally, since $z = 0, -4$ are the bifurcation points of function $B(z)$, then the deformation of integration contour, in virtue of Cauchy's theorem, leads to formula

$$[\lambda I + B(h^2 A)]^{-1} = \frac{1}{2\pi} \int_0^4 (\lambda^2 - \lambda\rho + \rho)^{-1} \cdot \sqrt{\rho(4-\rho)} \cdot (\rho I + h^2 A)^{-1} d\rho. \tag{2.11}$$

Since, evidently, function $(0 \leq \rho \leq 4,\ \lambda \geq 0)$

$$M_1(\lambda,\rho) = (2\pi)^{-1} \cdot (\lambda^2 - \lambda\rho + \rho)^{-1} \cdot \sqrt{\rho(4-\rho)} \geq 0, \tag{2.12}$$

then, in virtue of estimate (2.8), estimate

$$\|[\lambda I + B(h^2A)]^{-1}\|_{E\to E} \leq M(A) \cdot \int_0^4 M_1(\lambda,\rho) \cdot [\rho + h^2 a(A)]^{-1} d\rho \tag{2.13}$$

is true. The application now of formulas (2.11) and (2.12) in the case, when operator $h^2A$ is replaced by the number $h^2a(A)$, gives estimates

$$\begin{aligned}\|[\lambda I + B(h^2A)]^{-1}\|_{E\to E} &\leq M(A) \cdot \{\lambda + B[h^2a(A)]\}^{-1} \\ &\leq M(A) \cdot [\lambda + h\sqrt{a(A)}]^{-1}. \end{aligned} \tag{2.14}$$

Analogously to formula (2.11) for any $m = \overline{1,+\infty}$ formula

$$[\lambda I + B(h^2A)]^{-m} = \int_0^4 M_m(\lambda,\rho) \cdot (\rho I + h^2A)^{-1} d\rho \tag{2.15}$$

is established. However, function $M_m(\lambda,\rho)$ for $m \geq 2$ changes the sing on the segment $0 \leq \rho \leq 4$. Therefore the method, which was applied in the case $m = 1$, does not work in the case $m \geq 2$.

We will suppose supplemently, that $-A$ is generator of strongly continuous semigroup $\exp\{-tA\}(t \geq 0)$ with exponentially decreasing norm, i.e. estimates

$$\|\exp\{-tA\}\|_{E\to E} \leq M(A) \cdot \exp\{-ta(A)\}\ (t \geq 0) \tag{2.16}$$

take place for some $1 \leq M(A) < +\infty, 0 < a(A) < +\infty$. Then, if it is well-known (see, for example [2]), there exists the bounded inverse $(\lambda I + A)^{-1}$ for any complex number $\lambda$ with $Re\lambda > -a(A)$, and formula

$$(\lambda I + A)^{-1} = \int_0^{+\infty} \exp\{-t\lambda\} \cdot \exp\{-tA\} dt \tag{2.17}$$

holds. Formula (2.17) means, that the resolvent of operator $-A$ is the Laplase transform of the semigroup $\exp\{-tA\}$. From (2.17), in particular, it follows, that $A$ is positive operator in $E$ and, in virtue of estimate (2.16), estimate (2.8) is true. Further from (2.15) it follows, that

$$[\lambda I + B(h^2A)]^{-m} = \int_0^{+\infty} \mathcal{L}_m(\lambda, t) \cdot \exp\{-th^2A\}dt, \tag{2.18}$$

$$\mathcal{L}_m(\lambda, t) = \int_0^4 M_m(\lambda, \rho) \cdot \exp\{-\rho t\}d\rho. \tag{2.19}$$

In the case, when $h^2A$ is the positive numbers, formula (2.18) means, that function $[\lambda I + B(h^2A]^{-m}$ is (for fixed $\lambda \geq 0$) the Laplase transform of function $\mathcal{L}_m(\lambda, t)$. Then from properties of Laplase transform it follows, that $\mathcal{L}_m(\lambda, t)$ is the convolution of $m$ copies of function $\mathcal{L}_1(\lambda, t)$, and this convolution is defined by recurrent correlation

$$\mathcal{L}_{m+1}(\lambda, t) = \int_0^t \mathcal{L}_m(\lambda, s) \cdot \mathcal{L}_1(\lambda, t-s)ds\,, \qquad (m = \overline{1, +\infty})\,. \tag{2.20}$$

Since [see (2.12)] $M_1(\lambda, \rho) \geq 0$, then from (2.19) it follows, that $\mathcal{L}_1(\lambda, t) \geq 0$. Therefore, in virtue of (2.20),

$$\mathcal{L}_m(\lambda, t) \geq 0\,, \qquad (m = \overline{1, +\infty}). \tag{2.21}$$

Inequality (2.21) permits to apply by estimate of norm of operator $[\lambda I + B(h^2A)]^{-m}$, defining by formula (2.18), the same approach, as in the case $m = 1$ for formula (2.11). Namely estimates ($\lambda \geq 0$)

$$\begin{aligned} \|[\lambda I + B(h^2A)]^{-m}\|_{E\to E} &\leq M(A) \cdot \{\lambda + B[h^2a(A)]\}^{-m} \\ &\leq M(A) \cdot [\lambda + h\sqrt{a(A)}]^{-m} \end{aligned} \tag{2.22}$$

are true. In particular,

$$\begin{aligned} \|[1 + B(h^2A)]^{-m}\|_{E\to E} &\leq M(A) \cdot \{1 + B[h^2a(A)]\}^{-m} \\ &\leq M(A) \cdot [1 + h\sqrt{a(A)}]^{-m}. \end{aligned} \tag{2.23}$$

Estimate (2.14) for $\lambda = 2$, identity $B \cdot (2+B)^{-1} = I - 2 \cdot (2+B)^{-1}$ and estimates (2.23) show, that, in virtue of formula (2.7), the grid functions $v^h, Av^h = (Av_i; i = \overline{-\infty, +\infty}) \in C^h(E)$ for any grid function $f^h \in C^h(E)$. Further from formula (2.7) it follows, that

$$\begin{aligned}
(v_{i+1} - 2v_i + v_{i-1}) &= A^{-1} \cdot B \cdot (2+B)^{-1} \cdot \sum_{r=-\infty}^{+\infty} (1+B)^{-|r|} \\
&\quad \cdot (f_{i+1-r} - 2f_{i-r} + f_{i-1-r}) \\
&= A^{-1} \cdot B \cdot (2+B)^{-1} \cdot \sum_{\ell=-\infty}^{+\infty} [(1+B)^{-|\ell+1|} - \\
&\quad - 2 \cdot (1+B)^{-|\ell|} + (1+B)^{-|\ell-1|}] \cdot f_{i-\ell} \\
&= -A^{-1} \cdot B \cdot (2+B)^{-1} \cdot (2B) \cdot \\
&\quad \cdot (1+B)^{-1} \cdot f_i + A^{-1} \cdot B \cdot (2+B)^{-1} \cdot B^2 \cdot \\
&\quad \cdot (1+B)^{-1} \cdot \sum_{\ell \neq 0} (1+B)^{-|\ell|} \cdot f_{i-\ell}. \qquad (2.24)
\end{aligned}$$

Finally, formula (2.4) leads us to identity

$$(v_{i+1} - 2v_i + v_{i-1}) = h^2 \cdot B \cdot (2+B)^{-1} \cdot \sum_{\ell=-\infty}^{+\infty} (1+B)^{-|\ell|} \cdot f_{i-\ell} - h^2 \cdot f_i, \tag{2.25}$$

which, in virtue of (2.3), means, that defined by formula (2.7) grid function $v^h$ is the solution of difference equation (2.1). It is showed analogously, that formula (2.7) define unique solution on of equation (2.1). So, the well-posedness of equation (2.1) in the Banach space $C^h(E)$ is established. This statement, as in the case of differential equation (1.1), is equivalent to two inequalities

$$\|v^h\|_{C^h(E)} \leq M_s(h) \cdot \|f^h\|_{C^h(E)}, \tag{2.26}$$

$$\|Av^h\|_{C^h(E)} \leq M_C(h) \cdot \|f^h\|_{C^h(E)}. \tag{2.27}$$

Inequality (2.26) is, evidently, the corollary of inequality (2.27), since $A^{-1}$ is the bounded operator in $E$.

However under investigation of convergence of difference method it is necessary to establish the well-posedness of equation (2.1) in

Banach space $C^h(E)$ not for some fixed $h \in (0,1]$, but in the aggregate of such spaces for all $h \in (0,1]$. To this aim we must establish inequalities

$$\|v^h\|_{C^h(E)} \leq M_s \cdot \|f^h\|_{C^h(E)}, \tag{2.28}$$

$$\|Av^h\|_{C^h(E)} \leq M_C \cdot \|f^h\|_{C^h(E)} \tag{2.29}$$

with some $1 \leq M_s, M_C < +\infty$ do not depending on $f^h \in C^h(E)$ and $h \in (0,1]$. Inequality (2.29) is (see §1), generally speaking, not true for any Banach space $E$ and generator $-A$ of strongly continuous semigroup $\exp\{-tA\}(t \geq 0)$ with exponentially decreasing norm. It turns out, that more weaker inequality (2.28) is true. In fact, from formulas (2.7) (2.4) and identity $(1+B)\cdot(2+B)^{-1} = I - (2+B)^{-1}$ it follows, that

$$v_i = (h \cdot B^{-1}) \cdot [I - (2+B)^{-1}] \cdot [f_i + \sum_{r \neq 0} (1+B)^{-|r|} \cdot f_{i-r}] \cdot h. \tag{2.30}$$

Further we use estimate (2.14) for $\lambda = 0$ and estimate (2.23) and obtain

$$\begin{aligned} \|v^h\|_{C^h(E)} \quad \leq \quad & M(A) \cdot [a(A)]^{-1/2} \cdot [1 + M(A)]^{1/2} \\ & \cdot \{1 + 2 \cdot [a(A)]^{-1/2}\} \cdot \|f^h\|_{C^h(E)}. \end{aligned} \tag{2.31}$$

This property is called the stability of difference equation (2.1) in the Banach space $C^h(E)$.

REMARK 2.1. If $\exp\{-tA\}(t \geq 0)$ is the strongly continuous semigroup with exponentially decreasing norm, then $A$ is called the normal positive operator. I do not know, if the normal positivity of operator $A$ in Banach space $E$ is the necessary condition of the stability of difference equation (2.1) in the functional Banach space $C^h(E)$.

## 2.2 Stability in the space $C^{h,\alpha}(E)$

We will consider difference equation (2.1) as operator equation in the Banach space $C^{h,\alpha}(E)_{(0<\alpha<1)}$ of grid functions $\psi^h = (\psi_i \in E; i = \overline{-\infty, +\infty})$ with norm

$$\|\psi^h\|_{C^{h,\alpha}} = \sup_{i=-\infty,+\infty} \|\psi_i\|_E + \sup_{-\infty<i<i+k<+\infty} \|\psi_{i+k} - \psi_i\|_E \cdot (kh)^{-\alpha}. \tag{2.32}$$

The well-posedness of difference equation (2.1) in the aggregate of such spaces for all $h \in (0,1]$ means that for solutions $v^h$ of equation (2.1) stability inequality

$$\|v^h\|_{C^{h,\alpha}(E)} \leq M_S(\alpha) \cdot \|f^h\|_{C^{h,\alpha}(E)} \tag{2.33}$$

and coercive inequality

$$\|Av^h\|_{C^{h,\alpha}(E)} \leq M_C(\alpha) \cdot \|f^h\|_{C^{h,\alpha}(E)} \tag{2.34}$$

are true for some $1 \leq M_S(\alpha), M_C(\alpha) < +\infty$, do not depending on $f^h \in C^{h,\alpha}(E)$ and $h \in (0,1]$. Let us establish inequality (2.33). To this aim we will mark, that from estimate (2.33) the estimate

$$\begin{aligned} \|[I + B(h^2 A)]^{-m}\|_{E \to E} &\leq M(A) \cdot \exp\{-\delta(A) \cdot mh\}, \delta(A) \\ &= \sqrt{a(A)} \cdot [1 + \sqrt{a(A)}]^{-1} \end{aligned} \tag{2.35}$$

follows. Further, from formulas (2.7) and (2.4), evidently, it follows, that ($r = \overline{1, +\infty}$)

$$\begin{aligned} v_{i+r} - v_i &= h^2 \cdot B^{-1} \cdot (1+B) \cdot (2+B)^{-1} \cdot \sum_{\ell=-\infty}^{+\infty} [(1+B)^{-|\ell+r|} \\ &\quad \cdot (f_{i-\ell} - f_{i+r}) - (1+B)^{-|\ell|}(f_{i-\ell} - f_i)] + \\ &\quad + h^2 (B^{-2} + B^{-1}) \cdot (f_{i+r} - f_i) \,. \end{aligned} \tag{2.36}$$

If $r$ is odd, then

$$\begin{aligned} & v_{i+r} - v_i = \\ & = [h^2 \cdot B^{-2} - h \cdot (hB^{-1})] \cdot (f_{i+r} - f_i) - \\ & - h \cdot (hB^{-1}) \cdot (2+B)^{-1} \cdot (1+B)^{-(r-1)} \cdot (f_{i+r} - f_i) - \\ & - (hB^{-1}) \cdot \sum_{|\ell| < |\ell+r|, \ell \neq 0} (2+B)^{-1} \cdot \\ & \cdot (1+B)^{-(|\ell+r|-1)} \cdot h \cdot (f_{i+r} - f_i) - \\ & - (hB^{-1}) \cdot \sum_{|\ell| > |\ell+r|} (2+B)^{-1} \cdot (1+B)^{-(|\ell|-1)} \cdot h \cdot (f_{i+r} - f_i) \\ & + h^2 \cdot \sum_{\substack{|\ell| < |\ell+r| \\ \ell \neq 0}} \sum_{m=|\ell|+1}^{|\ell+r|} (2+B)^{-1} \cdot (1+B)^{-(m-1)} \cdot (f_i - f_{i-\ell}) \end{aligned}$$

$$+ h^2 \cdot \sum_{|\ell|>|\ell+r|} \sum_{m=|\ell+r|+1}^{|\ell|} (2+B)^{-1} \cdot (1+B)^{-(m-1)} \cdot (f_{i+r} - f_{i-\ell})$$
$$= J_1(h) - J_2(h) - J_3(h) - J_4(h) + J_5(h) - J_6(h). \tag{2.37}$$

From estimate (2.14) for $\lambda = 0$ it follows, that

$$\|J_1(h)\|_E \leq [M^2(A) \cdot a^{-1}(A) + h \cdot M(A) \cdot a^{-1/2}(A)] \cdot H^{h,\alpha}(f^h) \cdot |rh|^\alpha. \tag{2.38}$$

Here and in what follows

$$H^{h,\alpha}(f^h) = \sup_{-\infty<i<i+k<+\infty} \|f_{i+k} - f_i\|_E \cdot (kh)^{-\alpha}. \tag{2.39}$$

Further, analogously to formula (2.18), it is established, that

$$[2+B(h^2A)]^{-1}[1+B(h^2A)]^{-m} = \int_0^{+\infty} \mathcal{L}_1(2,t)*\mathcal{L}_m(1,t)\exp\{-th^2A\}dt. \tag{2.40}$$

Here $\mathcal{L}_1(2,t) * \mathcal{L}_m(1,t)$ is the convolution [see formula (2.20)] of functions $\mathcal{L}_1(2,t)$ and $\mathcal{L}_m(1,t)$. Therefore, analogously to estimates (2.23) and (2.35), the estimates, $(m = \overline{1,+\infty})$,

$$\|(2+B)^{-1} \cdot (1+B)^{-m}\|_{E\to E} \leq M(A) \cdot \{1+h[a(A)]^{1/2}\}^{-(m+1)}, \tag{2.41}$$

$$\|(2+B)^{-1} \cdot (1+B)^{-m}\|_{E\to E} \leq M(A) \cdot \exp\{-\delta(A) \cdot (m+1)h\}, \tag{2.42}$$

are true. Therefore

$$\begin{aligned} \|J_2(h)\|_E \quad \leq \quad & H^{h,\alpha}(f^h) \cdot |rh|^\alpha \cdot h \cdot M(A) \cdot [a(A)]^{-1/2} \\ & \cdot M(A) \cdot \{1 + h \cdot [a(A)]^{1/2}\}^{-r}. \end{aligned} \tag{2.43}$$

Further analogously

$$\begin{aligned} \|J_3(h)\|_E \quad \leq \quad & H^{h,\alpha}(f^h) \cdot |rh|^\alpha \cdot \\ & \cdot \sum_{|\ell|<|\ell+r|,\ell\neq 0} M(A) \cdot \exp\{-\delta(A) \cdot |\ell+r| \cdot h\} \cdot h, \end{aligned} \tag{2.44}$$

$$\begin{aligned} \|J_4(h)\|_E \quad \leq \quad & H^{h,\alpha}(f^h) \cdot |rh|^\alpha \cdot \\ & \cdot \sum_{|\ell|>|\ell+r|} M(A) \cdot \exp\{-\delta(A) \cdot |\ell| \cdot h\} \cdot h. \end{aligned} \tag{2.45}$$

Therefore, evidently, that

$$\begin{aligned}\|J_3(h)\|_E + \|J_4(h)\|_E &\le \\ &\le |rh|^\alpha \cdot H^{h,\alpha}(f^h)\cdot M(A)\cdot \int_{-\infty}^{+\infty} \exp\{-\delta(A)\cdot |x|\}dx \\ &= |rh|^\alpha \cdot H^{h,\alpha}(f^h)\cdot M(A)\cdot 2\cdot \delta^{-1}(A). \qquad (2.46)\end{aligned}$$

Further

$$\begin{aligned}\|J_5(h)\|_E &\le \sum_{|\ell|<|\ell+r|,\ell\neq 0} \sum_{m=|\ell|+1}^{|\ell+r|} M(A)\cdot \\ &\quad \cdot \exp\{-\delta(A)\cdot |m|\cdot h\}\cdot |\ell h|^\alpha \cdot h^2 \cdot H^{h,\alpha}(f^h) \\ &\le H^{h,\alpha}(f^h)\cdot M(A)\cdot \\ &\quad \cdot \int_{|x|<|x+rh|} |y|^\alpha [\int_{|x|}^{|x+rh|} \exp\{-\delta(A)y\}dy]dx. \qquad (2.47)\end{aligned}$$

Since, it is evident, that

$$\int_{|x|}^{|x+rh|} \exp\{-\delta(A)y\}dy \le \exp\{-\delta(A)\cdot |x|\}\cdot \min\{|rh|, \delta^{-1}(A)\}, \quad (2.48)$$

then from (2.47) it follows, that

$$\begin{aligned}\|J_5(h)\|_E &\le |rh|^\alpha \cdot H^{h,\alpha}(f^h)\cdot M(A)\cdot \delta^{\alpha-1}(A)\cdot \\ &\quad \int_{|x|<|x+rh|} \exp\{-\delta(A)\cdot |x|\}\cdot |x|^\alpha dx. \qquad (2.49)\end{aligned}$$

It is established analogously, that

$$\begin{aligned}\|J_6(h)\|_E &\le |rh|^\alpha \cdot H^{h,\alpha}(f^h)\cdot M(A)\cdot \delta^{\alpha-1}(A)\cdot \\ &\quad \int_{|x|>|x+rh|} \exp\{-\delta(A)\cdot |x|\}\cdot |x|^\alpha dx. \qquad (2.50)\end{aligned}$$

Therefore the following estimate is true:

$$\begin{aligned}\|J_5(h)\|_E + \|J_6(h)\|_E &\leq |rh|^\alpha H^{h,\alpha}(f^h)\cdot M(A)\cdot \delta^{\alpha-1}(A)\cdot \\ &\quad \cdot \int_{-\infty}^{+\infty} \exp\{-\delta(A)|x|\}\cdot |x|^\alpha \\ &= |rh|^\alpha H^{h,\alpha}(f^h)\cdot M(A)\cdot \delta^{-2}(A)\Gamma(1+\alpha).\end{aligned} \tag{2.51}$$

If $r$ is even, then to the right part of (2.37) must be added member

$$-J_7(h) = -h\cdot(hB^{-1})\cdot(2+B)^{-1}\cdot(1+B)^{-(\frac{r}{2}-1)}\cdot(f_{i+r}-2f_{i+r/2}+f_i). \tag{2.52}$$

Evidently, the following estimate is true:

$$\|J_7(h)\|_E \leq |rh|^\alpha\cdot H^{h,\alpha}(f^h)\cdot h\cdot a^{-1/2}(A)\cdot\{1+h[a(A)]^{1/2}\}^{-\frac{r}{2}}. \tag{2.53}$$

Estimates (2.38), (2.43), (2.46), (2.51) and (2.53) together with estimate (2.31) mean, that for solution $v^h$ of difference equation (2.1) inequality

$$\|v^h\|_{C^{h,\alpha}(E)} \leq M_S\cdot\|f^h\|_{C^{h,\alpha}(E)} \tag{2.54}$$

is true for some $1 \leq M_S < +\infty$, does not depending on $f^h \in C^{h,\alpha}(E), h \in (0,1]$ and $\alpha \in (0.1)$. So, it is established, that difference equation (2.1) is stable in the Banach space $C^{h,\alpha}(E)$ [uniformly with respect to $\alpha \in (0,1)$].

REMARK 2.2. From the stability of difference equation (2.1) in the space $C^h(E)$, evidently, its stability in the space $C^{h,1(E)}$ follows. Therefore the stability in the space $C^{h,\alpha}(E)(0 < \alpha < 1)$ can be also established with the help of interpolation theory of linear operators.

## 2.3 The additional estimates of the resolvent povers of operator $-B(h^2A)$

We will suppose now, that $-A$ is generator of analytic semigroup $\exp\{-tA\}$ with exponentially decreasing norm, i.e. estimates [see estimates (1.24)]

$$\|\exp\{-tA\}\|_{E\to E}, \|tA\cdot\exp\{-tA\}\|_{E\to E} \leq M(A)\cdot\exp\{-ta(A)\} \tag{2.55}$$

are true for some $1 \leq M(A) < +\infty, 0 < a(A) < +\infty$.

The proof of well-posedness of difference equation (2.1) in the spaces $C^{h,\alpha}(E)$, i.e. the proof of inequality (2.34), is based on estimates of norms of operators

$$B\cdot(2+B)^{-1}\cdot(1+B)^{-m}, B^2\cdot(2+B)^{-1}\cdot(1+B)^{-m}(m=\overline{1,+\infty}). \quad (2.56)$$

In the case $m=1$ from evident identities

$$\begin{aligned} B\cdot(2+B)^{-1}\cdot(1+B)^{-1} &= (2+B)^{-1}-(1+B)^{-1}\cdot(2+B)^{-1}, \\ B^2\cdot(2+B)^{-1}\cdot(1+B)^{-1} &= 1-(1+B)^{-1}-2\cdot(2+B)^{-1}+ \\ &\quad +2\cdot(2+B)^{-1}\cdot(1+B)^{-1} \end{aligned} \quad (2.57)$$

and estimate (2.41) the uniform with respect to $h\in(0,1]$ estimates

$$\begin{aligned} &\|B\cdot(2+B)^{-1}\cdot(1+B)^{-1}\|_{E\to E}\leq M(A), \\ &\|B^2(2+B)^{-1}\cdot(1+B)^{-1}\|_{E\to E}\leq 1+3\cdot M(A) \end{aligned} \quad (2.58)$$

follow. Let further $m\geq 2$. In virtue of formulas (2.4) and (2.40), we have

$$B^2\cdot(2+B)^{-1}\cdot(1+B)^{-m}=\int\limits_0^{+\infty}\mathcal{L}_1(2,t)*\mathcal{L}_{m-1}(1,t)\cdot h^2A\cdot\exp\{-th^2A\}dt. \quad (2.59)$$

We apply estimate (2.55) and obtain

$$\begin{aligned} &\|B^2\cdot(2+B)^{-1}\cdot(1+B)^{-m}\|_{E\to E}\leq M(A)\cdot \\ &ds\int\limits_0^{+\infty}\mathcal{L}_1(2,t)*\mathcal{L}_{m-1}(1,t)\cdot t^{-1}\cdot\exp\{-th^2\cdot a(A)\}dt. \end{aligned} \quad (2.60)$$

Further from evident formula

$$t^{-1}=\int\limits_0^{+\infty}\exp\{-ts\}ds(t>0) \quad (2.61)$$

it follows, that

$$\begin{aligned} &\|B^2\cdot(2+B)^{-1}\cdot(1+B)^{-m}\|_{E\to E}\leq \\ &M(A)\cdot\int\limits_0^{+\infty}\langle\int\limits_0^{+\infty}\mathcal{L}_1(2,t)*\mathcal{L}_{m-1}(1,t)\cdot \\ &\cdot\exp\{-t[s+h^2\cdot a(A)]\}dt\rangle ds. \end{aligned} \quad (2.62)$$

Finally the application of formula (2.40) for the case, when operator $h^2A$ is replaced by number $h^2a(A)$, gives estimate

$$\|B^2 \cdot (2+B)^{-1} \cdot (1+B)^{-m}\|_{E\to E} \leq M(A) \cdot \int_0^{+\infty} \{2 + B[s+h^2a(A)]\}^{-1} \cdot \cdot \{1+B[s+h^2a(A)]\}^{-(m-1)} ds. \tag{2.63}$$

We will remind, that here

$$B(z) = z/2 + \sqrt{z^2/4 + z}, \tag{2.64}$$

and therefore the following estimate is true:

$$\|B^2(h^2A)\cdot[2+B(h^2A)]^{-1}\cdot[1+B(h^2A)]^{-m}\|_{E\to E} \leq 2\cdot M(A)\cdot(m^2-1)^{-1}. \tag{2.65}$$

For the estimate of norm of the first from operators (2.56) we will use the moment inequality [see inequality (1.19)] for powers of positive operators in Banach space. Estimate (2.14) means, that $B = B(h^2a)$ is positive operator for any fixed $h \in (0,1]$. It turns out, that moment inequality

$$\|B\psi\|_E \leq M_B \cdot \|B^2\psi\|_E^{1/2} \cdot \|\psi\|_E^{1/2} \tag{2.66}$$

takes place for any element $\psi$ from domain $D(B^2)$ of operator $B^2$ with some $1 \leq M_B < +\infty$, does not depending on $\psi \in D(B^2)$ and $h \in (0,1]$. In fact, from estimate (2.14) it follows, that

$$B\psi = \int_0^{+\infty} (\lambda + B)^{-2} \cdot B^2\psi d\lambda. \tag{2.67}$$

Further we use identity

$$B^2(\lambda+B)^{-2} = I - 2\lambda \cdot (\lambda+B)^{-1} + \lambda^2 \cdot (\lambda+B)^{-2}. \tag{2.68}$$

Therefore from (2.22) it follows, that

$$\|(\lambda+B)^{-2}\|_{E\to E} \leq M(A)\cdot\lambda^{-2}, \|B^2(\lambda+B)^{-2}\|_{E\to E} \leq 1 + 3M(A). \tag{2.69}$$

Then formula (2.67) leads us to inequality $[N \in (0, +\infty)]$

$$\|B\psi\|_E \leq [1 + 3 \cdot M(A)] \cdot N \cdot \|\psi\|_E + M(A) \cdot N^{-1} \cdot \|B^2\psi\|_E. \quad (2.70)$$

Let $\psi \neq 0$. Then $B^2\psi \neq 0$, since there exists the bounded $B^{-2}$. Therefore we can put in (2.70)

$$N = [M(A)]^{1/2} \cdot [1 + 3 \cdot M(A)]^{-1/2} \cdot \|B^2\psi\|_E^{1/2} \cdot \|\psi\|_E^{-1/2} \in (0, +\infty). \quad (2.71)$$

This value $N$ leads us to inequality (2.66) with

$$M_B = 2 \cdot [M(A)]^{1/2} \cdot [1 + 3 \cdot M(A)]^{1/2}. \quad (2.72)$$

From (2.66), evidently, it follows, that

$$\|B \cdot (2 + B)^{-1} \cdot (1 + B)^{-m}\|_{E\to E} \leq$$
$$\leq M_B \cdot \|B^2 \cdot (2 + B)^{-1} \cdot (1 + B)^{-m}\|_{E\to E}^{1/2} \cdot$$
$$\cdot \|(2 + B)^{-1} \cdot (1 + B)^{-m}\|_{E\to E}^{1/2}. \quad (2.73)$$

We use estimates (2.65) and (2.41) and obtain from (2.73) estimate

$$\|B(h^2A) \cdot [2 + B(h^2A)]^{-1} \cdot [1 + B(h^2A)]^{-m}\|_{E\to E} \leq$$
$$\leq M_B \cdot M(A) \cdot \sqrt{2} \cdot (m^2 - 1)^{-1/2}. \quad (2.74)$$

Estimates (2.58), (2.65) and (2.74) show, that estimates

$$\|B \cdot (2 + B)^{-1} \cdot (1 + B)^{-m}\|_{E\to E} \leq \tilde{M}_1 \cdot m^{-1} \; (m = \overline{1, +\infty}), \quad (2.75)$$

$$\|B^2 \cdot (2 + B)^{-1} \cdot (1 + B)^{-m}\|_{E\to E} \leq \tilde{M}_2 \cdot m^{-2} \; (m = \overline{1, +\infty}) \quad (2.76)$$

are true for some $1 \leq \tilde{M}_1, \tilde{M}_2 < +\infty$, do not depending on $h \in (0, 1]$. Estimate (2.35) permits to obtain more precise estimates. Namely for $m > 1$ we use evident identity

$$(2+B)^{-1} \cdot (1+B)^{-m} = \{(2+B)^{-1} \cdot (1+B)^{-[m/2]}\} \cdot \{(1+B)^{-(m-[m/2]}\}. \quad (2.77)$$

Here $[m/2]$ is the integer part of number $m/2$. Then from (2.75), (2.76), evidently, it follows, that

$$\|B(h^2A) \cdot [2 + B(h^2A)]^{-1} \cdot [1 + B(h^2A)]^{-m}\|_{E\to E} \leq$$
$$\leq M_1 \cdot m^{-1} \cdot \exp\{-\delta(A) \cdot mh\}, \quad (2.78)$$

$$\|B^2(h^2A)\cdot[2+B(h^2A)]^{-1}\cdot[1+B(h^2a)]^{-m}\|_{E\to E}\leq \\ \leq M_2\cdot m^{-2}\cdot\exp\{-\delta(A)\cdot mh\} \qquad (2.79)$$

for some $1\leq M_1, M_2<+\infty$, do not depending on $h\in(0,1]$ and $m=\overline{1,+\infty}$.

### 2.4 Well-posedness in the space $C^{h,\alpha}(E)$

From formula (2.7), evidently, it follows, that, $(i=\overline{-\infty,+\infty})$,

$$Av_i = B\cdot(2+B)^{-1}\cdot\sum_{k=-\infty}^{+\infty}(1+B)^{-|i-k|}(f_k-f_i)+f_i\,, \qquad (2.80)$$

The application of estimate (2.78) leads us to estimate

$$\|Av_i\|_E \leq M_1\cdot H^{h,\alpha}(f^h)\cdot\sum_{k=-\infty,k\neq i}^{+\infty}|i-k|^{-1}\cdot \\ \cdot\exp\{-\delta(A)\cdot|i-k|h\}\cdot|(i-k)h|^{\alpha}+\|f_i\|_E. \qquad (2.81)$$

Therefore

$$\|Av_i\|_E\leq M_1\cdot H^{h,\alpha}(f^h)\cdot\int_{-\infty}^{+\infty}|z|^{\alpha-1}\cdot\exp\{-\delta(A)|z|\}dz+\|f_i\|_E, \quad (2.82)$$

i.e. estimate

$$\|Av^h\|_{C^h(E)}\leq M_1\cdot\delta^{-\alpha}(A)\cdot\Gamma(\alpha)\cdot H^{h,\alpha}(f^h)+\|f^h\|_{C^h(E)} \qquad (2.83)$$

is true. Further for any integer $r>0$ from (2.80), evidently, identity

$$Av_{i+r}-Av_i = \sum_{m=-\infty}^{+\infty}[B\cdot(2+B)^{-1}\cdot(1+B)^{-|m+r|}(f_{i-m}-f_{i+r}) \\ -B\cdot(2+B)^{-1}\cdot(1+B)^{-|m|}\cdot(f_{i-m}-f_i)]+ \\ +(f_{i+r}-f_i) \qquad (2.84)$$

follows, which we will to transform to following form

$$\begin{aligned} Av_{i+r} - Av_i \ = \\ &= [1 - (2+B)^{-1} \cdot (1+B)^{-3r} - \\ &- (2+B)^{-1} \cdot (1+B)^{-2r}] \cdot (f_{i+r} - f_i) + \\ &+ \sum_{m=-2r}^{2r} [B \cdot (2+B)^{-1} \cdot (1+B)^{-|m+r|} \cdot (f_{i-m} - f_{i+r}) - \\ &- B \cdot (2+B)^{-1} \cdot (1+B)^{-|m|} \cdot (f_{i-m} - f_i) + \\ &+ \sum_{m=2r+1}^{+\infty} \sum_{n=1}^{r} B^2 \cdot (2+B)^{-1} \cdot (1+B)^{-|m+n|} \cdot (f_i - f_{i-m}) + \\ &+ \sum_{m=2r+1}^{+\infty} \sum_{n=0}^{r-1} B^2 \ \cdot (1+B)^{-|m-n|} \cdot (f_{i+m} - f_{i+r}) \\ &= J_1(h) + J_2(h) + J_3(h) + J_4(h) \end{aligned} \tag{2.85}$$

From (2.41), evidently, it follows, that

$$\|J_1(h)\|_E \le |rh|^\alpha \cdot H^{h,\alpha}(f^h) \cdot [1 + 2 \cdot M(A)]. \tag{2.86}$$

Further we use estimate (2.78) and obtain

$$\begin{aligned} \|J_2(h)\|_E \ \le \ & \sum_{m=-2r}^{2r} M_1 \cdot H^{h,\alpha}(f^h) \cdot |(m+r)h|^{\alpha-1} \cdot h + \\ & + \sum_{m=-2r}^{2r} M_1 \cdot H^{h,\alpha}(f^h) \cdot |mh|^{\alpha-1} \cdot h. \end{aligned} \tag{2.87}$$

Therefore

$$\|J_2(h)\|_E \le M_1 \cdot H^{h,\alpha}(f^h) \cdot \int_{-2rh}^{2rh} [|x + rh|^{\alpha-1} + |x|^{\alpha-1}] dx. \tag{2.88}$$

It means, that estimate

$$\|J_2(h)\|_E \le |rh|^\alpha \cdot H^{h,\alpha}(f^h) \cdot M_1 \cdot \alpha^{-1} \cdot (3^\alpha + 1 + 2^{\alpha+1}) \tag{2.89}$$

is true. Finally, in virtue of estimate (2.79), we have

$$\|J_3(h)\|_E \le M_2 \cdot H^{h,\alpha}(f^h) \cdot \sum_{m=2r+1}^{+\infty} \sum_{n=1}^{r} (m+n)^{-2} \cdot |mh|^\alpha. \tag{2.90}$$

Therefore

$$\|J_3(h)\|_E \leq M_2 \cdot H^{h,\alpha}(f^h) \cdot \int\limits_{(2r+1)h}^{+\infty} x^\alpha [\int\limits_0^{rh} (x+y)^{-2} dy] dx. \quad (2.91)$$

It means that estimate

$$\|J_3(h)\|_E \leq |rh|^\alpha \cdot H^{h,\alpha}(f^h) \cdot M_2 \cdot (1-\alpha)^{-1} \cdot 2^{\alpha-1} \quad (2.92)$$

holds. Analogously

$$\|J_4(h)\|_E \leq M_2 \cdot H^{h,\alpha}(f^h) \cdot \sum_{m=2r+1}^{+\infty} \sum_{n=0}^{r-1} |m-n|^{-2} \cdot |(m-r)h|^\alpha. \quad (2.93)$$

Therefore

$$\|J_4(h)\|_E \leq M_2 \cdot H^{h,\alpha}(f^h) \cdot \int\limits_{(2r+1)h}^{+\infty} |x-rh|^\alpha [\int\limits_0^{rh} (x-y)^{-2} dy] dx. \quad (2.94)$$

It means, that estimate

$$\|J_4(h)\|_E \leq |rh|^\alpha \cdot H^{h,\alpha}(f^h) \cdot M_2 \cdot (1-\alpha)^{-1} \cdot 2^{\alpha-1} \quad (2.95)$$

takes place. Identity (2.84) and estimates (2.86), (2.89), (2.92), (2.95) lead us to estimate

$$H^{h,\alpha}(v^h) \leq M \cdot H^{h,\alpha}(f^h) \cdot \alpha^{-1} \cdot (1-\alpha)^{-1} \quad (2.96)$$

with some $1 \leq M < +\infty$, does not depending on $f^h \in C^{h,\alpha}(E), h \in (0,1]$ and $\alpha \in (0,1)$. From estimates (2.83) and (2.96) we obtain the following result:

THEOREM 2.1. *Let $A$ be strongly positive operator in Banach space $E$, i.e. $-A$ be generator of analytic semigroup $\exp\{-tA\}, (t \geq 0)$, with exponentially decreasing norm. Then difference equation (2.1) is well-posed in the functional Banach space $C^{h,\alpha}(E)$, $(0 < \alpha < 1)$, and coercive inequality*

$$\|Av^h\|_{C^{h,\alpha}(E)} \leq M \cdot \alpha^{-1}(1-\alpha)^{-1} \cdot \|f^h\|_{C^{h,\alpha}(E)} \quad (2.97)$$

*holds for its solutions $v^h$ with some $1 \leq M < +\infty$, does not depending on $f^h \in C^{h,\alpha}(E)$, $h \in (0,1]$ and $\alpha \in (0,1)$.*

REMARK 2.3. I do not know if the strong positivity of operator $A$ in Banach space $E$ is the necessary condition of well-posedness of difference equation (2.1) in the functional Banach space $C^{h,\alpha}(E)$.

### 2.5 Almost well-posedness in the space $C^h(E)$

The value $M_C(h)$ in coercive inequality (2.27) for solutions in $C^h(E)$ of difference equation (2.1) must, generally speaking, tend to $+\infty$, when $h$ tends to $+0$. It is the consequence of differential equation (1.1) theory (see contraexample in §0 introduction). The theory of difference equation (2.1) permits to obtain the estimate of convergence rate to $+\infty$ such value $M_C(h)$. Namely in virtue of definition (2.39), from estimate (2.83) it follows, that

$$\|Av^h\|_{C^h(E)} \leq M \cdot \alpha^{-1} \cdot h^{-\alpha} \cdot \|f^h\|_{C^h(E)} \tag{2.98}$$

for any $\alpha \in (0, 1/2]$, $h \in (0, e^{-2}]$ and some $1 \leq M < +\infty$, does not depending on $f^h \in C^h(E), h$ and $\alpha$. We will put here

$$\alpha = (\ln 1/h)^{-1} \tag{2.99}$$

and obtain

$$\|Av^h\|_{C^h(E)} \leq M \cdot e \cdot \ln 1/n \cdot \|f^h\|_{C^h(E)}. \tag{2.100}$$

It means, that we can put

$$M_C(h) = M \cdot e \cdot \ln 1/n \tag{2.101}$$

in inequality (2.27). It is naturally to say, that inequality (2.100) means the almost well-posedness of difference equation (2.1) in the space $C^h(E)$. In order to apply this result to two-dimensional difference equation (0.12) and to prove formula (0.23), we must prove that difference operator of the second order with respect of one variable in the space $C^h$ of grid functions of two variables, is strongly positive operator, uniformly with respect to $h \in (0, 1]$. We will mark that analogous fact for ordinary differential operator $A$, acting in the space $C(F)$ and defined by formula (1.10), evidently, from formula (1.11) follows.

In space $C^h(F)$ of grid function $\psi^h = (\psi_i; i = \overline{-\infty, +\infty})$ with values in any Banach space $F$ we will define difference operator $A^h$ by formula, $(i = \overline{-\infty, +\infty})$,

$$(A^h v^h)_i = -(v_{i+1} - 2v_i + v_{i-1}) \cdot h^{-2} + a \cdot v_i \tag{2.102}$$

for some $0 < h \leq 1$ and $a > 0$. The resolvent of operator $A^h$ is defined by solution of difference equation, $(i = \overline{-\infty, +\infty})$,

$$-(v_{i+1} - 2v_i + v_{i-1}) \cdot h^{-2} + av_i + \lambda v_i = f_i\,. \tag{2.103}$$

For any complex number $\lambda$ with $Re\lambda > -a$ and $f^h \in C^h(F)$ equation (2.103) has unique solution $v^h$, defining [see formula (2.7)] by formulas

$$(\lambda + a) \cdot v_i = b \cdot (2 + b)^{-1} \cdot \sum_{k=-\infty}^{+\infty} (1 + b)^{-|i-k|} \cdot f_k, \tag{2.104}$$

$$b = b[h^2(\lambda + a)], b(z) = z/2 + \sqrt{z^2/4 + z}. \tag{2.105}$$

In fact, evidently, $|1 + b| > 1$, when $Re\lambda > -a$.

For some integer $N > 0$ we will put

$$(\lambda + a) \cdot v_i^N = b \cdot (2 + b)^{-1} \cdot \sum_{|k| \leq N} (1 + b)^{-|i-k|} f_k. \tag{2.106}$$

Evidently ($N$ is the finite number), that function $(\lambda + a)v_i^N$ (for any fixed $i = \overline{-\infty, +\infty}$) is analytic and bounded in the complex halfplane $Re\lambda \geq -a, \lambda \neq 0$, and $|(\lambda + a)v_i^N| \to 0$, when $|\lambda| \to +\infty$.

Therefore, in virtue of maximum principle, the estimate of function $(\lambda + a) \cdot v_i^N$ is defined by its estimate on the line $\lambda + a = S \cdot \sqrt{-1}, -\infty < s < +\infty$. For $0 \leq s < +\infty$ and $t = h^2 s$, evidently, we have

$$\begin{aligned} |b|^2 &= (t^4/16 + t^2)^{1/2} + t^2/4 + \\ &\quad + t \cdot [\frac{1}{2}(t^4/16 + t^2)^{1/2} - t^2/8]^{1/2}, \\ |1 + b|^2 &= 1 + (t^4/16 + t^2)^{1/2} + t^2/4 + \\ &\quad + 2 \cdot [\frac{1}{2}(t^4/16 + t^2)^{1/2} + t^2/8]^{1/2} + \\ &\quad + t \cdot [\frac{1}{2}(t^4/16 + t^2)^{1/2} - t^2/8]^{1/2}, \\ |2 + b| &\geq |1 + b|. \end{aligned} \tag{2.107}$$

Further from (2.106), evidently, it follows, that estimate

$$\|(\lambda+a)v_i^N\|_F \leq 2\cdot|b|\cdot[|1+b|+1]\cdot[|1+b|^2-1]^{-1}\cdot\|f^{h,N}\|_{C^h(F)} \quad (2.108)$$

is true. Here we put

$$\|f^{h,N}\|_{C^h(F)} = \max_{|i|\leq N}\|f_i\|_F. \quad (2.109)$$

From (2.107) it follows, that

$$\|(\lambda + a)v_i^N\|_F \leq \gamma(t)\cdot\|f^{h,N}\|_{C^h(F)}, \quad (2.110)$$

$$\begin{aligned}
\gamma(t) &= \alpha(t)/\beta(t),\\
\alpha(t) &= 2\cdot\{(t^4/16+t^2)^{1/2}+t^2/4+\\
&\quad +t\cdot[\frac{1}{2}(t^4/16+t^2)^{\frac{1}{2}}-t^2/8]^{1/2}\}^{1/2}\cdot\\
&\quad \cdot\langle\{1+(t^4/16+t^2)^{1/2}+\\
&\quad +2[\frac{1}{2}(t^4/16+t^2)^{1/2}+t^2/8]^{1/2}+t^2/4+\\
&\quad +t\cdot[\frac{1}{2}(t^4/16+t^2)^{1/2}-t^2/8]^{1/2}\}^{1/2}+1\rangle,\\
\beta(t) &= (t^4/16+t^2)^{1/2}+\\
&\quad +2\cdot[\frac{1}{2}(t^4/16+t^2)^{1/2}+t^2/8]^{1/2}+t^2/4+\\
&\quad +t\cdot[\frac{1}{2}(t^4/16+t^2)^{1/2}-t^2/8]^{1/2}. \quad (2.111)
\end{aligned}$$

Finally, from (2.111) it follows, that

$$\sup_{0<t<+\infty}\gamma(t) = M < +\infty, \quad (2.112)$$

i.e. estimate

$$\|(\lambda + a)v_i^N\|_E \leq M\cdot\|f^{h,N}\|_{C^h(F)} \quad (2.113)$$

holds with some $1 \leq M < +\infty$ does not depending on $f^h, N$ and $h$. The same estimate is true also for $-\infty < s \leq 0$. Therefore, in virtue of maximum principle, estimate (2.113) takes place for all complex numbers $\lambda$ with $Re\lambda \geq -a$. Further from definition (2.109), evidently, it follows, that there exists the sequence $N_m \nearrow +\infty$ for $m \nearrow +\infty$, such that

$$\|f^{h,N_m}\|_{C^h(F)} \to \|f^h\|_{C^h(F)}. \tag{2.114}$$

Finally, if $Re\lambda > -a$, then, evidently,

$$\|v_i^{N_m} - v_i\|_F \to 0. \tag{2.115}$$

Therefore from estimate (2.113) and limit correlations (2.114) and (2.115) it follows, that

$$\|(\lambda + a)v_i\|_F \leq M \cdot \|f^h\|_F. \tag{2.116}$$

It means, that estimate

$$\|(\lambda I + A^h)^{-1}\|_{C^h(F) \to C^h(F)} \leq M \tag{2.117}$$

is true for $Re\lambda > -a$. Estimate (2.117) means, that $A^h$ is uniformly with respect to $h$ strongly positive operator in Banach space $C^h(F)$, i.e. estimates $(t > 0)$

$$\begin{aligned} &\| \exp\{-tA^h\}\|_{C^h(F)\to C^h(F)}, \|tA^h \cdot \exp\{-tA^h\}\|_{C^h(F)\to C^h(F)} \leq \\ &\leq M \cdot \exp\{-\delta t\} \end{aligned} \tag{2.118}$$

take place for some $1 \leq M < +\infty, 0 < \delta < +\infty$, do not depending on $h$. This property permits to apply formula (2.101) in coercive inequality (2.27) for solution of (abstract) difference equation (2.1) in the case, when operator $A = A^h$. Therefore formula (0.23) is true.

## 2.6 Application of P. Grisvard's theory

We will consider difference equation (2.1) as operator equation

$$A^h v^h + B^h v^h = f^h \tag{2.119}$$

in the Banach space $C^h(E)$. Here we will put, $[v_i \in E; i = \overline{-\infty, +\infty}]$,

$$(A^h v^h)_i = -(v_{i+1} - 2v_i + v_{i-1}) \cdot h^{-2} + av_i\,, \tag{2.120}$$

$$(B^h v^h)_i = [(A - aI)v_i], \ [v_i \in D(A); i = \overline{-\infty, +\infty}] \qquad (2.121)$$

for sufficiently small $a > 0$. From the strong positivity of operator $A$ in the Banach space $E$, evidently the strong positivity of operator $B^h$ in the functional Banach space $C^h(E)$, uniform with respect to $h$, follows. In point 2.5 is proved, that $A^h$ is uniformly with respect to $h$ strongly positive operator in $C^h(E)$. These facts, in particular, mean, that inequality

$$\varphi(A^h) + \varphi(B^h) \le \pi - \delta \qquad (2.122)$$

for spectral angles of operator $A^h$ and $B^h$ accordingly is true for some $\delta \in (0, \pi)$, does not depending on $h$. Finally, evidently, operators $A^h$ nd $B^h$ commute. Therefore to operator equation (2.119) the P. Grisvard's theory is appliable. Namely, let us define the Banach space $C^h_\alpha(A^h)(0 < \alpha < 1)$ of grid function $\psi^h \in C^h$ with norm

$$\|\psi^h\|_{C^h_\alpha(A^h)} = \sup_{\lambda \ge 0}(\lambda + a)^\alpha \cdot \|A^h \cdot (\lambda I + A^h)^{-1}\psi^h\|_{C^h(E)}. \qquad (2.123)$$

Then for any $f^h \in C^h_\alpha(A^h)$ there exists unique solution $v^h$ of equation (2.119), and coercive inequality

$$\|A^h v^h\|_{C^h_\alpha(A^h)} + \|B^h v^h\|_{C^h_\alpha(A^h)} \le M \cdot \alpha^{-1} \cdot (1 - \alpha)^{-1} \cdot \|f^h\|_{C^h_\alpha(A^h)} \qquad (2.124)$$

is true with some $1 \le M < +\infty$, does not depending on $f^h, h$ and $\alpha$. It turns out that spaces $C^h_{\alpha/2}(A^h)$ for $0 < \alpha < 1$ coincide with difference Hölder spaces $C^{h,\alpha}(E)$, and norms (2.32) and (2.123) are equivalent, uniformly with respect to $h$. Let us prove this statement. From formula (2.104) for resolvent of operator $-A^h$, evidently, it follows, that $(\lambda > 0)$

$$\begin{aligned}
[(\lambda + a)^{\alpha/2} \cdot A^h \cdot (\lambda I + A^h)^{-1} \cdot f^h]_i &= \\
&= \lambda \cdot (\lambda + a)^{\alpha/2 - 1} \cdot \frac{b}{2 + b} \cdot \sum_{k=-\infty}^{+\infty} (1 + b)^{-|i-k|} \cdot (f_i - f_k) \\
&\quad + a \cdot (\lambda + a)^{\alpha/2 - 1} \cdot f_i, \qquad (2.125)
\end{aligned}$$

$$b = b[h^2(\lambda + a)], \ b(z) = z/2 + \sqrt{z^2/4 + z} \ge \sqrt{z} \ (z \ge 0).$$

Therefore estimate

$$\begin{aligned}
&\|[(\lambda+a)^{\alpha/2}\cdot A^h\cdot(\lambda I+A^h)^{-1}f^h]_i\|_E \;\le \\
&\quad\le \;\frac{\lambda}{\lambda+a}\cdot\frac{2b}{2+b}\cdot\sum_{m=1}^{\infty}(1+b)^{-m}\cdot m^\alpha\cdot H^{h,\alpha}(f^h)+ \\
&\qquad + a\cdot(\lambda+a)^{\alpha/2-1}\cdot\|f^h\|_{C^h(E)}
\end{aligned}\tag{2.126}$$

holds. Further from identity, $(C>0)$,

$$\frac{1}{\Gamma(1-\alpha)}\cdot\int_0^C (C-t)^{-\alpha}t^\alpha\cdot t^{m-1}dt = C^m\cdot\frac{\Gamma(m+\alpha)}{\Gamma(m+1)}\tag{2.127}$$

and Stirling's formula it follows, that

$$C^m\cdot m^\alpha\le\frac{M}{\Gamma(1-\alpha)}\cdot\int_0^C (C-t)^{-\alpha}\cdot t^\alpha\cdot(m\cdot t^{m-1})dt\tag{2.128}$$

with some $1\le M<+\infty$, does not depending on $C>0, m=\overline{1,+\infty}$ and $\alpha\in(0,1)$. Therefore for $C\in(0,1)$ inequality

$$\sum_{m=1}^{+\infty}C^m\cdot m^\alpha\le\frac{M}{\Gamma(1-\alpha)}\cdot\int_0^C(C-t)^{-\alpha}\cdot t^\alpha\cdot(1-t)^{-2}dt\tag{2.129}$$

takes place. From (2.129), evidently, it follows, that

$$\sum_{m=1}^{+\infty}C^m\cdot m^\alpha\le\frac{M}{\Gamma(2-\alpha)}\cdot[C+\frac{2C^\alpha}{(1-C)^{1+\alpha}}\cdot\int_0^{C/1-C}\frac{s^{1-\alpha}}{(1+s)^3}ds].\tag{2.130}$$

We will apply (2.130) for $C=(1+b)^{-1}$ in inequality (2.126) and obtain

$$\begin{aligned}
&\|[(\lambda+a)^{\alpha/2}\cdot A^h\cdot(\lambda I+A^h)^{-1}f^h]_i\|_E \;\le \\
&\quad\le\;\frac{2M}{\Gamma(2-\alpha)}\cdot\Big[\frac{b^{1+\alpha}}{(1+b)(2+b)}+\frac{2(1+b)}{2+b}\int_0^{1/b}\frac{s^{1-\alpha}}{(1+s)^3}ds\Big] \\
&\qquad\cdot H^{h,\alpha}(f^h)+a^{\alpha/2}\cdot\|f^h\|_{C^h(E)}.
\end{aligned}\tag{2.131}$$

It means, that inequality

$$\|f^h\|_{C^h_{\alpha/2}(A^h)} \leq M_1 \cdot \|f^h\|_{C^{h,\alpha}(E)} \tag{2.132}$$

takes place for some $1 \leq M_1 < +\infty$, does not depending on $f^h, h$ and $\alpha$. Since $A^h$ is positive operator in $C^h(E)$, then formula

$$f^h = \int_0^{+\infty} A^h \cdot (\lambda I + A^h)^{-2} f^h d\lambda, f^h \in C^k_{\alpha/2}(A^h), \tag{2.133}$$

is true. It is evident, that (2.133) can be transform to formula

$$f^h = \int_0^{+\infty} (\lambda + a)^{-(1+\alpha/2)} \cdot (\lambda + a) \cdot (\lambda I + A^h)^{-1} \cdot \psi^h(\lambda, a) d\lambda, \tag{2.134}$$

where

$$\psi^h(\lambda, a) = (\lambda + a)^{\alpha/2} \cdot A^h \cdot (\lambda I + A^h)^{-1} f^h. \tag{2.135}$$

It means, in virtue of formula (2.104), that

$$\begin{aligned} f_i &= \int_0^{+\infty} (\lambda + a)^{-(1+\alpha/2)} \cdot \frac{b_a}{2 + b_a} \cdot \\ &\cdot \sum_{k=-\infty}^{+\infty} (1 + b_a)^{-|i-k|} \cdot \psi_k(\lambda, a) d\lambda, b_a \\ &= b[h^2(\lambda + a)]. \end{aligned} \tag{2.136}$$

From (2.136) and definitions (2.135) and (2.123), evidently, it follows, that

$$\|f^h\|_{C^h(E)} \leq \frac{2}{\alpha} \cdot a^{-\alpha/2} \cdot \|f^h\|_{C^h_{\alpha/2}(A^h)}. \tag{2.137}$$

Further for any integer $r > 0$ from (2.136) it follows, that

$$\begin{aligned} f_{i+r} - f_i &= \int_0^{+\infty} (\lambda + a)^{-(1+\alpha/2)} \cdot \tfrac{b_a}{2+b_a} \cdot \\ &\cdot \sum_{\ell=-\infty}^{+\infty} [(1 + b_a)^{|\ell+r|} - (1 + b_a)^{-|\ell|}] \cdot \psi_{i-\ell}(\lambda, a) d\lambda. \end{aligned} \tag{2.138}$$

From definition (2.135) and (2.123) it follows that

$$\|f_{i+r} - f_i\|_E \leq \|f^h\|_{C^h_{\alpha/2}(A^h)} \cdot \int\limits_0^{+\infty} (\lambda + a)^{-(1+\alpha/2)} \cdot \frac{b_a}{2+b_a} \cdot \cdot \sum_{e-\infty}^{+\infty} |(1+b_a)^{-|\ell+r|} - (1+b_a)^{-|\ell|}| d\lambda. \quad (2.139)$$

Substitution

$$t = h^2(\lambda + a) \quad (2.140)$$

leads us to inequality

$$\|f_{i+r} - f_i\|_E \cdot h^{-\alpha} \leq \|f^h\|_{C^h_{\alpha/2}(A^h)} \int\limits_{h^2 a}^{+\infty} t^{-(1+\alpha/2)} \cdot \cdot \frac{b}{2+b} \sum_{\ell=-\infty}^{+\infty} |(1+b)^{|\ell+r|} - (1+b)^{-|\ell|}| dt,$$
$$( b = b(t) = t/2 + \sqrt{t^2/4 + t}. ) \quad (2.141)$$

If $r$ is even, then

$$\|f_{i+r} - f_i\|_E \cdot h^{-\alpha} \leq \|f^h\|_{C^h_{\alpha/2}(A^h)} \cdot \int\limits_{h^2 a}^{+\infty} t^{-(1+\alpha/2)} \cdot \frac{1+b}{2+b} \cdot \cdot 2[1 - (1+b)^{-[\frac{r}{2}]}] \cdot [1 + (1+b)^{-1}]\, dt. \quad (2.142)$$

If $r$ is odd, then

$$\|f_{i+r} - f_i\|_E \cdot h^{-\alpha} \leq \|f^h\|_{C^h_{\alpha/2}(A^h)} \cdot \int\limits_{h^2 a}^{+\infty} t^{-(1+\alpha/2)} \cdot \frac{1+b}{2+b} \cdot \cdot 2\{[1 - (1+b)^{-r}] + [1 - (1+b)^{-[\frac{r}{2}]}] \cdot \cdot [1 - (1+b)^{-(r-[\frac{r}{2}]-1)}]\} dt. \quad (2.143)$$

Here $[\frac{r}{2}]$ is the integer part of number $r/2$. Let $r$ be even.
Then substitution

$$s = b(t), t = s^2 \cdot (1+s)^{-1}, dt = s \cdot (2+s) \cdot (1+s)^{-2} ds \quad (2.144)$$

leads us to inequality

$$\|f_{i+r} - f_i\|_E \cdot h^{-\alpha} \le \|f^h\|_{C^h_{\alpha/2}(A^h)} \int\limits_{b(h^2 a)}^{+\infty} (1+s)^{\alpha/2}(1+s)^{-(1+\alpha)} \cdot 2 \cdot [1-(1+s)^{-[\frac{r}{2}]}] \cdot [1+(1+s)^{-1}]ds \le 4 \cdot \|f^h\|_{C^h_{\alpha/2}(A^h)} \cdot \int\limits_0^{+\infty} [s^{-(1+\alpha)} + s^{-(1+\alpha/2)}] \cdot [1-(1+s)^{-[\frac{r}{2}]}]ds. \quad (2.145)$$

Since for any $N > 0$

$$\int\limits_0^{+\infty} [s^{-(1+\alpha)} + s^{-(1+\alpha/2)}] \cdot [1-(1+s)^{-[\frac{r}{2}]}]ds \le \quad (2.146)$$

$$\le \int\limits_0^{N} [\frac{r}{2}] \cdot (s^{-\alpha} + s^{-\alpha/2})ds + \int\limits_N^{+\infty} [s^{-(1+\alpha)} + s^{-(1+\alpha/2)}]ds,$$

then for

$$N = [\frac{r}{2}]^{-1} \quad (2.147)$$

we will obtain estimate

$$\|f_{i+r} - f_i\|_E \cdot h^{-\alpha} \le 12 \cdot 2^{-\alpha} \cdot \alpha^{-1} \cdot (1-\alpha)^{-1} \cdot r^{-\alpha} \cdot \|f^h\|_{C^h_{\alpha/2}(A^h)}. \quad (2.148)$$

The same estimate is true, when $r$ is odd. Estimates (2.137) and (2.148) mean, that following estimate

$$\|f^h\|_{C^{h,\alpha}(E)} \le M_2 \cdot \alpha^{-1} \cdot (1-\alpha)^{-1} \cdot \|f^h\|_{C^h_{\alpha/2}(A^h)} \quad (2.149)$$

holds with some $1 \le M_2 < +\infty$, does not depending on $f^h, h$ and $\alpha$. Estimates (2.132) and (2.149) mean, that following inequalities of equivalence of norms of spaces $C^{h,\alpha}(E)$ and $C^h_{\alpha/2}(A^h)$

$$\alpha \cdot (1-\alpha) \cdot M^- \cdot \|f^h\|_{C^{h,\alpha}(E)} \le \|f^h\|_{C^h_{\alpha/2}(A^h)} \le M^+ \cdot \|f^h\|_{C^{h,\alpha}(E)} \quad (2.150)$$

are true for some $0 < M^- \leq M^+ < +\infty$, do not depending on $f^h, h$ and $\alpha$. From (2.150) and (2.124)it follows, that inequality

$$\|Av^h\|_{C^{h,\alpha}(E)} \leq M \cdot \alpha^{-2} \cdot (1-\alpha)^{-1} \cdot \|f^h\|_{C^{h,\alpha}(E)} \tag{2.151}$$

is true for solution $v^h$ of difference equation (2.1) with some $1 \leq M < +\infty$, does not depending on $f^h, h$ and $\alpha$. It is evident, that for $\alpha \searrow +0$ inequality (2.151) is less exact, then inequality (2.97). From inequality (2.151), evidently, it follows, that

$$\|Av^h\|_{C^h(E)} \leq \tilde{M} \cdot (\ln 1/h)^2 \cdot \|f^h\|_{C^h(E)} \tag{2.152}$$

for $h \in (0, 1/2]$ and some $1 \leq \tilde{M} < +\infty$, does not depending on $f^h$ and$h$. Evidently, for $h \searrow +0$ inequality (2.152) is less exact, then inequality (2.100).

## References

[1] Allaberen Ashyralyev and Sobolevskii Pavel E., *Well-posedness of Parabolic Difference Equations*, Birkhäuser, 1994.

[2] Krasnoselski M. A. et al., *Integral operators in spaces of summable functions*, Noordhoff International Publishing, 1976.

[3] Grisvard P., *Equations differentielles abstraites*, Ann. Sci. École Norm. Sup., **3** (1970), 68–125.