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Favard Classes and Hyperbolic Equations

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Summary. - *New regularity theorems are proved for the abstract Cauchy problem $u'(t) = Au(t) + f(t)$, $t \geq 0$, $u(0) = u_0$ where A is a Hille-Yosida operator, by using properties of the Favard classes, matrix operators and extrapolation spaces.*

0. Introduction

In a series of papers P. Grisvard (in collaboration with G. Da Prato) showed the importance of the interpolation spaces in the study of parabolic equations and even the necessity of their use to get the maximal regularity for the non homogeneous initial value problem:

$$\begin{cases} u'(t) = A_0 u(t) + f(t), & t \geq 0 \\ u(0) = u_0 \end{cases} \tag{0.1}$$

where $A_0 : D(A_0) \subset X_0 \to X_0$ is the generator of an analytic semigroup in a Banach space X_0 (see the bibliographical references of [A. Lunardi 1995]). In this paper we will suppose that A_0 is the generator of a strongly continuous semigroup (and even only a Hille-Yosida operator) so that (0.1) is the abstract version of hyperbolic initial value problems.

The existence of a (classical solution) in the hyperbolic case has been proved in 1953 by R. Phillips when f is differentiable and by T.

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Kato when f is continuous with values in $D(A_0)$ (see e.g. [A. PAZY 1993]).

We will show that these results can be improved by relaxing the conditions on A_0, u_0 and f: the methods used are based on a regularizing property of certain interpolation spaces (the Favard classes), on the theory of Hille-Yosida operators (see [G. DA PRATO, E. SINESTRARI 1987]), a method of homogeneization of (0.1) (see [R. NAGEL, E. SINESTRARI 1994]) and the theory of extrapolation spaces as introduced in [R. NAGEL 1983].

We will not consider applications, for which we refer to [R. NAGEL, E. SINESTRARI 1994, 1996; E. SINESTRARI 1996].

This paper is a part of a work in progress with R. Nagel and is dedicated to the memory of P. Grisvard who introduced me to the methods of interpolation spaces in the study of abstract evolution equations.

1. Favard classes and regularity

Let us recall the definition and the main properties of the Favard classes (see [P. BUTZER, H. BERENS 1967]).

In this section $(E, \|\cdot\|)$ will denote a Banach space and $\Lambda : D(\Lambda) \subset E \to E$ the generator of a strongly continuous semigroup $e^{\Lambda t}$. For the problems considered in this paper it is not a restriction to suppose that the semigroup is bounded i.e., there exists $M \geq 0$ such that

$$\|e^{\Lambda t}\|_{\mathcal{L}(E)} \leq M \quad , \qquad t \geq 0 \; . \tag{1.1}$$

The *Favard class* of $e^{\Lambda t}$ is the real interpolation space $(E, D(\Lambda))_{1,\infty}$ according to J.L. Lions i.e.

$$F := \text{Fav}\,(e^{\Lambda t}) := \left\{ x \in E \; ; \; [x]_F := \sup_{t>0} \frac{\|e^{\Lambda t}x - x\|}{t} < \infty \right\}$$

with norm

$$\|x\|_F := \|x\| + [x]_F \; .$$

We have

$$D(\Lambda) \hookrightarrow F \hookrightarrow E \tag{1.2}$$

where $D(\Lambda)$ is given the graph norm: this norm is equivalent to that induced by F hence $D(\Lambda)$ is closed in F.

We will set $L^1(]0,+\infty[;E) = L^1(E)$ and similarly for the Sobolev space $W^{1,1}(E)$, the space of continuous functions $C(E)$ and so on.

If $f \in L^1_{\text{loc}}(E)$ we set for each $t \geq 0$

$$(e^\Lambda * f)(t) := \int_0^t e^{\Lambda s} f(t-s)\, \mathrm{d}s \ .$$

If A is a linear operator in a Banach space, $\rho(A)$ will denote its resolvent set.

THEOREM 1.1. *Let $(F_*, \|\cdot\|_*)$ be a Banach space such that*

$$F_* \hookrightarrow F \ . \tag{1.3}$$

If $f \in L^1_{\text{loc}}(F_*)$ then for each $t > 0$ we have

$$(e^\Lambda * f)(t) \in D(\Lambda) \tag{1.4}$$

$$\|(e^\Lambda * f)(t)\|_{D(\Lambda)} \leq cM \|f\|_{L^1(]0,t[;F_*)} \tag{1.5}$$

$$e^\Lambda * f \in C(D(\Lambda)) \tag{1.6}$$

where c is independent on f and t.

Proof. As for $t > 0$

$$\begin{aligned}\left\|\frac{1}{t}\int_0^t e^{\Lambda s} x\, \mathrm{d}s\right\|_{D(\Lambda)} &= \left\|\frac{1}{t}\int_0^t e^{\Lambda s} x\, \mathrm{d}s\right\| + \left\|\frac{e^{\Lambda t}x - x}{t}\right\| \leq \\ &\leq M\|x\| + \sup_{t>0}\left\|\frac{e^{\Lambda t}x - x}{t}\right\|\end{aligned}$$

we see that $F_* \hookrightarrow E$ satisfies (1.3) if and only if there exists $c > 0$ such that

$$\left\|\frac{1}{t}\int_0^t e^{\Lambda s} x\, \mathrm{d}s\right\|_{D(\Lambda)} \leq c\|x\|_*, \quad t > 0,\ x \in F_* \ . \tag{1.7}$$

Set for $t \geq 0$, $x \in E$

$$V(t)x := \int_0^t e^{\Lambda s} x\, \mathrm{d}s \ .$$

If χ_A is the characteristic function of the set A, denote by Φ the set of the step functions $\varphi : \mathbb{R}_+ \to F_*$

$$\varphi := \sum_{i=0}^{n} x_i \chi_{[t_1, t_{i+1}[} \tag{1.8}$$

where $n \in \mathbb{N}$, $x_i \in F_*(i = 1, \dots, n)$ and $0 = t_0 < t_1 < \cdots < t_{n+1}$.

Let us prove (1.4) and (1.5) for $f = \varphi$ given by (1.8). For $t \geq t_{n+1}$ we have

$$\begin{aligned} (e^\Lambda * \varphi)(t) &= \sum_{i=0}^{n} e^{\Lambda(t-t_{i+1})} \int_{t_i}^{t_{i+1}} e^{\Lambda(t_{i+1}-s)} x_i \, \mathrm{d}s = \\ &= \sum_{i=0}^{n} e^{\Lambda(t-t_{i+1})} V(t_{i+1} - t_i) x_i \ . \end{aligned}$$

Hence from (1.7) we get $(e^\Lambda * \varphi)(t) \in D(\Lambda)$ and

$$\begin{aligned} \|(e^\Lambda * \varphi)(t)\|_{D(\Lambda)} &\leq \sum_{i=0}^{n} \|e^{\Lambda(t-t_{i+1})}\| \; \|V(t_{i+1} - t_i) x_i\|_{D(\Lambda)} \leq \\ &\leq cM \sum_{i=0}^{n} \|x_i\|_* (t_{i+1} - t_i) = cM \|\varphi\|_{L^1(]0,t[;F_*)} \ . \end{aligned}$$

If $t < t_{n+1}$, setting $\psi = \varphi \cdot \chi_{[0,t[}$ we have $\psi \in \Phi$: and so from the preceding result we see that each $f \in \Phi$ verifies (1.4) and (1.5): this can be proved also for $f \in L^1(F_*)$ because Φ is dense in $L^1(F_*)$.

To prove (1.6) let us choose $f \in L^1(F_*)$: given $T > 0$ there exists $\{f_n\}$ with $f_n \in C^1([0,T]; F_*)$ such that

$$\lim_{n \to \infty} \|f - f_n\|_{L^1(]0,T[;F_*)} = 0.$$

As $e^\Lambda * f_n \in C([0,T]; D(\Lambda))$ (see corollary 2.5 of [A. Pazy 1993]) and

$$\sup_{0 \leq t \leq T} \|(e^\Lambda * f)(t) - (e^\Lambda * f_n)(t)\|_{D(\Lambda)} \leq cM \|f - f_n\|_{L^1(]0,T[;F_*)}$$

we deduce for $n \to \infty$ that $e^\Lambda * f \in C([0,T]; D(\Lambda))$ and (1.6) is proved.

□

THEOREM 1.2. *Let F_* verify (1.3). Given $f \in L^1(F_*)$ and $u_0 \in D(\Lambda)$ there exists $u \in C(D(\Lambda))$, differentiable (in E) for $t \geq 0$ a.e. and solution of*

$$\begin{cases} u'(t) = \Lambda u(t) + f(t), & t \geq 0 \text{ a.e.} \\ u(0) = u_0 \end{cases} \tag{1.9}$$

Proof. We can assume $u_0 = 0$. Setting $u(t) = \int_0^t e^{\Lambda(t-s)} f(s)\, \mathrm{d}s$, $t \geq 0$ we have for $0 \leq t < t+h$

$$\frac{u(t+h) - u(t)}{h} = \frac{e^{\Lambda h} u(t) - u(t)}{h} + \\ + \frac{1}{h} \int_t^{t+h} e^{\Lambda(t+h-s)} [f(s) - f(t)]\, \mathrm{d}s + \frac{1}{h} \int_0^h e^{\Lambda s} f(t)\, \mathrm{d}s\,. \tag{1.10}$$

Now by virtue of (1.4)

$$\lim_{h \to 0^+} \frac{e^{\Lambda h} u(t) - u(t)}{h} = \Lambda u(t) \tag{1.11}$$

and $f \in L^1(E)$ implies

$$\lim_{h \to 0^+} \frac{1}{h} \int_t^{t+h} \|f(s) - f(t)\|\, \mathrm{d}s = 0$$

for $t \geq 0$ a.e.; letting $h \to 0^+$ in (1.10) we obtain

$$u'_+(t) = \Lambda u(t) + f(t), \quad t \geq 0 \text{ a.e.} \tag{1.12}$$

For $0 \leq t - h < t$ we have

$$\frac{u(t-h) - u(t)}{-h} = \frac{e^{\Lambda h} u(t-h) - u(t-h)}{h} + \\ + \frac{1}{h} \int_{t-h}^{t} e^{\Lambda(t+h-s)} [f(s) - f(t)]\, \mathrm{d}s + \frac{1}{h} \int_0^h e^{\Lambda s} f(t)\, \mathrm{d}s\,. \tag{1.13}$$

Now

$$\frac{e^{\Lambda h} u(t-h) - u(t-h)}{h} - \frac{e^{\Lambda h} u(t) - u(t)}{h} = \\ = \frac{1}{h} \int_0^h e^{\Lambda s} [u(t-h) - u(t)]\, \mathrm{d}s$$

and so by virtue of (1.6) and (1.11)

$$\lim_{h\to 0^+} \frac{e^{\Lambda h}u(t-h)-u(t-h)}{h} = \Lambda u(t)\ .$$

Proceding as above we deduce from (1.13) for $h \to 0^+$

$$u'_-(t) = \Lambda u(t) + f(t)\ , \quad t \geq 0 \text{ a.e.} \tag{1.14}$$

From (1.12) and (1.14) the conclusion follows.

□

THEOREM 1.3. *Let F_* verify* (1.3). *Given $f \in L^1(F_*) \cap C(E)$ and $u_0 \in D(\Lambda)$ there exists a unique solution $u \in C(D(\Lambda)) \cap C^1(E)$ of*

$$\begin{cases} u'(t) = \Lambda u(t) + f(t), & t \geq 0 \\ u(0) = u_0\ . \end{cases} \tag{1.15}$$

Proof. By repeating the proof of the preceding theorem we can deduce now that (1.9) holds for every $t \geq 0$: hence $u' \in C(E)$. The uniqueness follows from the fact that when $f = u_0 = 0$ the only solution $u \in C(D(\Lambda)) \cap C^1(E)$ of (1.15) is zero.

□

2. Hille-Yosida operators

Let $(X, \|\cdot\|)$ be a Banach space. $A : D(A) \subset X \to X$ is called *a Hille-Yosida operator* if there exist $\omega \in \mathbb{R}$ and $M \geq 1$ such that if $\lambda > \omega$ then $\lambda \in \rho(A)$ and

$$\|(\lambda-\omega)^n(\lambda-A)^{-n}\| \leq M$$

for each $n \in \mathbb{N}$ (see [E. SINESTRARI 1994]).

When $D(A)$ is dense then by virtue of Hille-Yosida theorem, A generates a strongly continuous semigroup: more generally we have the following result.

THEOREM 2.1. *If* $X_0 = (\overline{D(A)}, \|\cdot\|)$ *and* $A_0 : D(A_0) \subset X_0 \to X_0$ *with*

$$D(A_0) = \{x \in D(A)\,; Ax \in X_0\}$$

and $A_0 x = Ax$, *then* A_0 *is the generator of a strongly continuous semigroup in* X_0. *In addition*

$$D(A_0) \subset D(A) \hookrightarrow F\ . \tag{2.1}$$

For the proof of (2.1) see proposition 3.2 of [R. NAGEL, E. SINESTRARI 1994].

The Favard classes of a semigroup provide examples of Hille-Yosida operators as the following theorem shows.

THEOREM 2.2. *Let* $\Lambda : D(\Lambda) \subset E \to E$ *be the generator of a bounded semigroup and* F *its Favard class and set* $\Lambda_F : D(\Lambda_F) \subset F \to F$ *with* $D(\Lambda_F) = \{x \in D(\Lambda), \Lambda x \in F\}$: *then* Λ_F *is a Hille-Yosida operator and*

$$\overline{D(\Lambda_F)}^F = D(\Lambda) \tag{2.2}$$

Proof. By using (1.2), we see that if $\lambda \in \rho(\Lambda)$ then $\lambda \in \rho(\Lambda_F)$ and we have $(\lambda - \Lambda_F)^{-1} = (\lambda - \Lambda)^{-1}|_F$: and for each $n \in \mathbb{N}$, $\|(\lambda - \Lambda_F)^{-n}\|_{\mathcal{L}(F)} \leq \|(\lambda - \Lambda)^{-n}\|_{\mathcal{L}(E)}$; as Λ generates a semigroup the conclusion follows.

□

REMARK 2.3. The restriction of $e^{\Lambda t}$ to F is a semigroup which is not strongly continuous if $D(\Lambda) \neq F$: in fact we have

$$\lim_{t \to 0} \|e^{\Lambda t} x - x\|_F = 0$$

if and only if $x \in D(\Lambda)$ (see Corollary 3.1.8 of [P. BUTZER, H. BERENS 1967]).

Even if Λ_F is only a Hille-Yosida operator we can obtain regularity results for the initial value problem

$$u'(t) = \Lambda u(t) + f(t), \quad t \geq 0;\ u(0) = u_0\ . \tag{2.3}$$

THEOREM 2.4. *Let Λ be the generator of a strongly continuous semigroup and F its Favard class.*

(i) *If $f \in W^{1,1}(F)$, $u_0 \in D(\Lambda)$, $\Lambda u_0 \in F$ and $\Lambda u_0 + f(0) \in D(\Lambda)$ then* (2.3) *has a unique solution $u \in C^1(F)$ such that $\Lambda u \in C(F)$.*

(ii) *If $f(t) \in D(\Lambda)$, $t \geq 0$ a.e.; $f, \Lambda f \in L^1(F)$, $u_0 \in D(\Lambda^2)$ then there exists a unique $u \in W^{1,1}_{\text{loc}}(F)$ such that $\Lambda u \in C(F)$ and satisfying* (2.3) *for $t \geq 0$ a.e.*

(iii) *If $f(t) \in D(\Lambda)$, $t \geq 0$ a.e., $f \in C(F)$; $\Lambda f \in L^1(F)$, $u_0 \in D(\Lambda^2)$ then* (2.3) *has a unique solution $u \in C^1(F)$ such that $\Lambda u \in C(F)$.*

Proof. As Λ_F is a Hille-Yosida operator we can use theorems 8.1 and 8.3 of [G. DA PRATO, E. SINESTRARI 1987] taking into account also (2.2).

□

3. Homogeneization

In this section we exhibit a method for reducing the non homogeneous problem

$$\begin{cases} u'(t) = \Lambda u(t) + f(t) \\ u(0) = u_0 \end{cases} \tag{3.1}$$

with $\Lambda : D(\Lambda) \subset E \to E$ generator of a strongly continuous semigroup $e^{\Lambda t}$ to a homogeneous one in a suitable product space (see [R. NAGEL, E. SINESTRARI 1994]).

Let us consider in the Banach space $L^1(E) = L^1(]0, +\infty[; E)$ the left translations semigroup

$$(S(t)f)(s) := f(t+s);\ t, s \geq 0$$

for $f \in L^1(E)$: it is known that its generator is the a.e. derivative with domain $W^{1,1}(E)$.

In the next theorem we show the homogeneization procedure and a method to obtain regularity results for (3.1) (in particular the Phillips theorem):

THEOREM 3.1. *In the Banach space*

$$Z := E \oplus L^1(E) \tag{3.2}$$

the semigroup

$$G(t)\begin{pmatrix} x \\ f \end{pmatrix} := \begin{pmatrix} e^{\Lambda t}x + (e^{\Lambda} * f)(t) \\ S(t)f \end{pmatrix} \tag{3.3}$$

has the generator $\mathcal{A} : D(\mathcal{A}) \subset Z \to Z$ *defined by*

$$D(\mathcal{A}) := D(\Lambda) \oplus W^{1,1}(E) \tag{3.4}$$

$$\mathcal{A}\begin{pmatrix} x \\ f \end{pmatrix} := \begin{pmatrix} \Lambda x + f(0) \\ f' \end{pmatrix} . \tag{3.5}$$

Hence given $f \in W^{1,1}(E)$ *and* $u_0 \in D(\Lambda)$, *problem* (3.1) *has a unique solution* $u \in C^1(E) \cap C(D(\Lambda))$, *given by the first component of* $U(t) = G(t)\binom{u_0}{f}$.

Proof. The first part is a direct consequence of the definition of $G(t)$. To prove the last part let us observe that $\binom{u_0}{f} \in D(\mathcal{A})$: hence the homogeneous problem in Z

$$U'(t) = \mathcal{A}U(t), \quad t \geq 0 \; ; \quad U(0) = U_0 \tag{3.6}$$

has a unique solution $U \in C^1(Z) \cap C(D(\mathcal{A}))$ given by $U(t) = G(t)\binom{x_0}{f}$. Now setting $U(t) = \binom{u(t)}{v(t)}$, equation (3.6) is equivalent to the system

$$\begin{cases} u'(t) = \Lambda u(t) + v(t)(0), & t \geq 0 \\ v'(t) = D_s v(t), & t \geq 0 \, . \end{cases}$$

But $v(t) = S(t)f$ and so $v(t)(0) = f(t)$: hence u satisfies (3.1).

□

An advantage of the homogeneization is the possibility of using the semigroup theory's classical procedures (restrictions, perturbations and so on) applied to $G(t)$ to get regularity results for the solution u of (3.1)

Let us show a restriction procedure.

THEOREM 3.2. *Let* $D \hookrightarrow E$ *and* $\mathcal{F} \hookrightarrow L^1(E)$ *verify the following properties:*

(3.7) $e^{\Lambda t}$ *is a semigroup on* D

(3.8) $S(t)$ *is a semigroup on* $\mathcal{F}$

(3.9) *given* $t \geq 0$ *and* $f \in \mathcal{F}$, *we have* $(e^{\Lambda} * f)(t) \in D$

(3.10) *given* $t \geq 0$, *there is* $c(t) > 0$ *such that*
$$\|(e^{\Lambda} * f)(t)\|_D \leq c(t)\|f\|_{\mathcal{F}}, \ \forall f \in \mathcal{F}$$

(3.11) *given* $f \in \mathcal{F}$, *we have* $\lim_{t\to 0} \|(e^{\Lambda} * f)(t)\|_D = 0$

Then the restriction of $G(t)$ *to the space*

$$Z_* := D \oplus \mathcal{F} \tag{3.12}$$

is a strongly continuous semigroup with generator $\mathcal{A}_*$, *restriction of* $\mathcal{A}$ *to*

$$D(\mathcal{A}_*) \ := \ \left\{ \begin{pmatrix} x \\ f \end{pmatrix}; \ x \in D \cap D(\Lambda); \right.$$
$$\left. f \in \mathcal{F} \cap W^{1,1}(E), \ f' \in \mathcal{F}, \ \Lambda x + f(0) \in D \right\}. \tag{3.13}$$

(3.14)

Hence given $\binom{u_0}{f} \in D(\mathcal{A}_*)$, *problem* (3.1) *has a unique solution* $u \in C^1(D)$.

Proof. Properties (3.7)–(3.11) guarantee that $G(t) \in \mathcal{L}(Z_*)$ and that $\lim_{t\to 0} \|G(t) \binom{x}{f} - \binom{x}{f}\|_{Z_*} = 0$ for each $\binom{x}{f} \in Z_*$: hence the part of $\mathcal{A}$ in Z_* is the generator of the semigroup $G(t)|_{Z_*}$. The last part of the theorem is proved as the corresponding one in theorem 3.1.

□

We can give now a regularity result for (3.1) as an application of this theorem and theorem 1.1.

THEOREM 3.3. *Let* F_* *be a Banach space such that* $F_* \hookrightarrow F$. *Given* $f \in W^{1,1}(F_*)$ *and* $u_0 \in D(\Lambda)$ *such that* $\Lambda u_0 + f(0) \in D(\Lambda)$, *there exists a unique solution* $u \in C^1(D(\Lambda))$ *of problem* (3.1).

Proof. By virtue of theorem 1.1, conditions (3.9)–(3.11) are satisfied by setting $D = D(\Lambda)$ and $\mathcal{F} = L^1(F_*)$. Now

$$D(\mathcal{A}_*) = \left\{ \begin{pmatrix} x \\ f \end{pmatrix} \in D(\Lambda) \oplus W^{1,1}(F_*) \ ; \ \Lambda x + f(0) \in D(\Lambda) \right\}$$

and so the conclusion follows from the last part of theorem 3.2.

□

Examples of perturbation methods and their applications to Volterra integrodifferential equations are given in [R. NAGEL, E. SINESTRARI 1994].

4. Extrapolation spaces

In this section we will recall some definitions and results about extrapolation spaces: for details see [R. NAGEL, E. SINESTRARI 1994].

We will assume that $A_0 : D(A_0) \subset X_0 \to X_0$ is the generator of a bounded semigroup $T_0(t)$ in $(X_0, \|\cdot\|)$ such that $A_0^{-1} \in \mathcal{L}(X_0)$.

DEFINITION 4.1. *The extrapolation space of X_0 (associated with the operator A_0) is the completion of $Y_0 := (X_0, \|\cdot\|_{-1})$ where $\|x\|_{-1} := \|A_0^{-1}x\|$, $x \in X_0$. It will be denoted by X_{-1}.*

The extrapolated semigroup T_{-1} in X_{-1} is the (unique) continuous extension of $T_0(t) : Y_0 \subset X_{-1} \to X_{-1}$, $t \geq 0$.

The Favard class of T_{-1} will be denoted by F_{-1}.

The following results are proved in [R. NAGEL, E. SINESTRARI 1994]

THEOREM 4.2. *The semigroup T_{-1} is strongly continuous and*

$$\|T_{-1}(t)\|_{\mathcal{L}(X_{-1})} = \|T_0(t)\|_{L(X_0)} \ .$$

If $A_{-1} : D(A_{-1}) \subset X_{-1} \to X_{-1}$ is the generator of T_{-1}, then

$$D(A_{-1}) = X_0 \text{ with equivalent norms} \tag{4.1}$$

$$\|A_{-1}x\|_{-1} = \|x\|, \quad x \in X_0 \tag{4.2}$$

(4.3) A_{-1} *is an extension of* A_0

(4.4) *there exists* $A_{-1}^{-1} \in \mathcal{L}(X_{-1})$

(4.5) $A_{-1}(F) = F_{-1}$ *and* $\|x\|_F = \|A_{-1}x\|_{F_{-1}},\ x \in F$.

For the applications we will need the following

THEOREM 4.3. *Let* $A : D(A) \subset X \to X$ *be a Hille-Yosida operator and* $T_0(t)$ *the strongly continuous semigroup generated by* $A_0 : D(A_0) \subset X_0 \to X_0$ *with* $X_0 = \overline{D(A)}$, $A_0 = A|_{D(A_0)}$ *and* $D(A_0) := \{x \in D(A),\ Ax \in \overline{D(A)}\}$ *(see Theorem 2.1). We have*

$$D(A_0) \subset D(A) \hookrightarrow F \hookrightarrow X_0 \subset X \hookrightarrow F_{-1} \hookrightarrow X_{-1} \tag{4.6}$$

and A_{-1} *is an extension of* A.

The inclusion $X \hookrightarrow F_{-1}$ is important because it allows us to apply theorem 3.3 to find a new proof of a theorem of [G. DA PRATO, E. SINESTRARI 1985, 1987].

THEOREM 4.4. *Let* $A : D(A) \subset X \to X$ *be a Hille-Yosida operator. Given* $f \in W^{1,1}(X)$ *and* $u_0 \in D(A)$ *such that* $Au_0 + f(0) \in \overline{D(A)}$ *there exists a unique* $u \in C^1(X) \cap C(D(A))$, *solution of*

$$\begin{cases} u'(t) = Au(t) + f(t), & t \geq 0 \\ u(0) = u_0 \end{cases} \tag{4.7}$$

Proof. As $X \hookrightarrow F_{-1}$ we can use theorem 3.3 with $E = X_{-1}$, $\Lambda = A_{-1}$ and $X = F_*$. Hence, given $f \in W^{1,1}(X)$ there is a solution $u \in C^1(X_0)$ if $u_0 \in X_0$ and $A_{-1}u_0 + f(0) \in X_0$: as $f(0) \in X$ this implies $A_{-1}u_0 \in X$ and so $u_0 \in D(A)$.

□

As in many applications to differential operators A in non reflexive Banach spaces X, the restriction of A_{-1} to F, F and F_{-1} can be characterized, it is interesting to use the extrapolation space restricted to F_{-1} as in the following results.

THEOREM 4.5. (i) *If* $f \in W^{1,1}(F_{-1})$, $u_0 \in F$ *and* $A_{-1}u_0 + f(0) \in X_0$ *there exists a unique* $u \in C^1(F_{-1})$ *such that* $A_{-1}u \in C(F_{-1})$ *and solution of*

$$\begin{cases} u'(t) = A_{-1}u(t) + f(t), & t \geq 0 \\ u(0) = u_0 \end{cases} \tag{4.8}$$

(ii) *If* $f \in L^1(F)$ *and* $u_0 \in D(A_0)$ *there exists a unique solution* $u \in W^{1,1}_{\text{loc}}(F_{-1}) \cap C(F)$ *of* (4.8) *for* $t \geq 0$ *a.e.*

(iii) *If* $f \in L^1(F) \cap C(F_{-1})$ *and* $u_0 \in D(A_0)$ *there exists a unique solution* $u \in C^1(F_{-1}) \cap C(F)$ *of* (4.8).

Proof. We can use the fact that the part of A_{-1} in F_{-1} is a Hille-Yosida operator, and apply theorem 2.4 taking into account that $A_{-1}u_0 \in F_{-1}$ or $u_0 \in D(A_0^2)$ are equivalent to $u_0 \in F$ or $u_0 \in D(A_0)$ respectively and $A_{-1}f \in L^1(F_{-1})$ or $A_{-1}f \in C(F_{-1})$ are equivalent to $f \in L^1(F)$ or $f \in C(F)$.

□

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