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Elementary Operator-Theoretic Proof of Wiener's Tauberian Theorem

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To the memory of Professor P. Grisvard

Summary. - *We present a short and elementary proof of Wiener's general Tauberian theorem based on the theory of one-parameter groups of operators.*

In this paper we present a short and elementary proof of Wiener's Tauberian theorem based on methods from the theory of C_0-groups.

Let $\mathbf{T} = \{\mathbf{T}(t)\}_{t\in\mathbb{R}}$ be a C_0-group on a Banach space X, i.e. a strongly continuous one-parameter group of bounded linear operators on X. Then $\mathbf{T}$ defines a Banach algebra homomorphism $\mathbf{T} : L^1(\mathbb{R}) \to \mathcal{L}(X)$ by

$$\mathbf{T}(f)x := \int_{-\infty}^{\infty} f(t)T(t)x\,dt, \quad f \in L^1(\mathbb{R}),\ x \in X.$$

The *kernel* of $\mathbf{T}$, notation $I_{\mathbf{T}}$, is the ideal

$$I_{\mathbf{T}} := \{f \in L^1(\mathbb{R}) :\ \mathbf{T}(f) = 0\}.$$

The *Arveson spectrum* of $\mathbf{T}$, notation $\mathrm{Sp}(\mathbf{T})$, is the hull of $I_{\mathbf{T}}$, i.e. the set of all $\omega \in \mathbb{R}$ such that $\hat{f}(\omega) = 0$ for all $f \in I_{\mathbf{T}}$. Here, as usual,

$$\hat{f}(\omega) := \int_{-\infty}^{\infty} e^{-i\omega t} f(t)\,dt$$

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is the Fourier transform of $f \in L^1(\mathbb{R})$ at ω.

Our proof of Wiener's Tauberian theorem is based on the fact that $\mathrm{Sp}(\mathbf{T})$ is non-empty provided $\mathbf{T}$ is bounded and $X \neq \{0\}$. This is true in the more general setting of bounded strongly continuous Banach representations of LCA groups G [Ar] and is usually *derived* from Wiener's Tauberian theorem. The essential point about our proof of Wiener's Tauberian theorem is that in the case $G = \mathbb{R}$ the non-emptyness of the Arveson spectrum admits a direct and elementary operator-theoretic proof. For reasons of completeness, we shall give the complete proof below.

Assuming for the moment that $\mathrm{Sp}(\mathbf{T}) \neq \emptyset$ if $X \neq \{0\}$, Wiener's Tauberian theorem can be proved in a few lines as follows. The *right translation group* is the C_0-group U on $L^1(\mathbb{R})$ defined by

$$U(t)f(s) := f(s-t), \qquad t \in \mathbb{R}, \text{ a.a. } s \in \mathbb{R}.$$

Note that $U(f)g = f * g$ for all $f, g \in L^1(\mathbb{R})$; here $*$ denotes convolution.

THEOREM 1. (Wiener's Tauberian theorem) *If the Fourier transform of a function $f \in L^1(\mathbb{R})$ vanishes nowhere, then the linear span of the set of all translates of f is dense in $L^1(\mathbb{R})$.*

Proof. Let $X := \overline{\mathrm{span}\{U(t)f : t \in \mathbb{R}\}}$. We have to prove that $X = L^1(\mathbb{R})$. Consider the quotient space $Y := L^1(\mathbb{R})/X$ and let U_Y denote the associated quotient translation group on Y. Then U_Y is strongly continuous and bounded, and for all $g \in L^1(\mathbb{R})$ we have $U(f)g = f * g = g * f = U(g)f$. By the translation invariance of X, $U(g)f \in X$. Hence $U(f)g \in X$, so $U_Y(f)(g + X) = 0$ for all $g \in L^1(\mathbb{R})$. It follows that $U_Y(f) = 0$. On the other hand, by assumption $\hat{f}(\omega) \neq 0$ for all $\omega \in \mathbb{R}$. Therefore, $\mathrm{Sp}(U_Y) = \emptyset$. We conclude that $Y = \{0\}$ and $X = L^1(\mathbb{R})$. □

Although the above proof seems to be new, the idea to apply the theory of C_0-groups, and more generally, of strongly continuous Banach representations of LCA groups, to quotients of translation groups to derive results in Harmonic Analysis is not; it has been used by Huang [Hu] to study spectral synthesis in Beurling algebras and subsequently in [HNR] to identify a class of Banach subalgebras of $L^1(G)$ which have the Ditkin property.

Even for $G = \mathbb{R}$, the usual proofs of Theorem 1 are quite involved; cf. [Ka], [Lo], [Ru], [Yo].

It remains to prove that $\mathrm{Sp}(\mathbf{T}) \neq \emptyset$ if $X \neq \{0\}$. This is accomplished in two propositions. The first is a well-known result of Evans [Ev]. As usual, for $\lambda \in \varrho(A)$, the resolvent set of an operator A, we write $R(\lambda, A) := (\lambda - A)^{-1}$. We assume that the reader is familiar with the elementary theory of C_0-(semi)groups as presented in the first chapter of [Pa] or [Na].

Proposition 2. *Let $\mathbf{T}$ be a bounded C_0-group on a Banach space X, with infinitesimal generator A.*

(i) For all $f \in L^1(\mathbb{R})$ whose Fourier transform belongs to $L^1(\mathbb{R})$ we have

$$\hat{f}(\mathbf{T})x = \frac{1}{2\pi} \lim_{\delta \downarrow 0} \int_{-\infty}^{\infty} \hat{f}(-t)\,(R(\delta + it, A) - R(-\delta + it, A))\,x\,dt, \qquad x \in X.$$

(ii) If $\hat{f}$ is compactly supported and vanishes in a neighbourhood of $i\sigma(A)$, then $\hat{f}(\mathbf{T}) = 0$.

(iii) If $X \neq \{0\}$, then $\sigma(A) \neq \emptyset$.

Proof. For all $\delta > 0$ we have $\pm\delta - it \in \varrho(A)$, and for all $x \in X$ we have the identities

$$R(\delta - it, A)x = \int_0^{\infty} e^{-(\delta - it)s}\mathbf{T}(s)x\,ds$$

and

$$R(-\delta - it, A)x = -R(\delta + it, -A) = -\int_0^{\infty} e^{-(\delta + it)s}\mathbf{T}(-s)x\,ds.$$

Since $\hat{f} \in L^1(\mathbb{R})$, by the formula for the inverse Fourier transform we have

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(s)e^{its}\,ds, \quad \text{a.a. } t \in \mathbb{R}.$$

Hence by the dominated convergence theorem and Fubini's theorem,

$$\hat{f}(\mathbf{T})x = \lim_{\delta \downarrow 0} \int_{-\infty}^{\infty} e^{-\delta|t|} f(t)T(t)x\,dt$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\lim_{\delta\downarrow 0}\int_{-\infty}^{\infty} e^{-\delta|t|}\left(\int_{-\infty}^{\infty} e^{ist}\hat{f}(s)\,ds\right) T(t)x\,dt \\
&= \frac{1}{2\pi}\lim_{\delta\downarrow 0}\int_{-\infty}^{\infty} \hat{f}(s)\left(\int_{-\infty}^{\infty} e^{-\delta|t|}e^{ist}T(t)x\,dt\right) ds \\
&= \frac{1}{2\pi}\lim_{\delta\downarrow 0}\int_{-\infty}^{\infty} \hat{f}(s)\left(R(\delta - is, A) - R(-\delta - is, A)\right) x\,ds.
\end{aligned}
$$

This proves (i).

If $\hat{f}$ is compactly supported and vanishes on a neighbourhood of $i\sigma(A)$, then $\hat{f}(\mathbf{T})x = 0$ for all $x \in X$ by (i) and the dominated convergence theorem. This proves (ii).

Finally, assume $\sigma(A) = \emptyset$. Then (ii) implies that $\hat{f}(\mathbf{T}) = 0$ for all $f \in L^1(\mathbb{R})$ whose Fourier transform $\hat{f}$ has compact support. These functions are dense in $L^1(\mathbb{R})$; this can be seen in an elementary way by noting that $\lim_{\lambda\to\infty} K_\lambda * f = f$, where K_λ is the Fejér kernel, and recalling that $\hat{K}_\lambda$ is compactly supported. Thus $\hat{f}(\mathbf{T}) = 0$ for all $f \in L_\omega(\mathbb{R})$. In particular, by defining $f_0(t) := e^{-t}$ for $t \geq 0$ and $f_0(t) := 0$ for $t < 0$ we have $f_0 \in L^1(\mathbb{R})$ and $R(1, A) = \hat{f}_0(\mathbf{T}) = 0$. This implies $X = \overline{R(1, A)X} = \{0\}$. □

The second proposition is a special case of a result of Jorgensen [Jo]. For the real line, it admits the following simple proof.

PROPOSITION 3. *Let $\mathbf{T}$ be a bounded C_0-group with infinitesimal generator A on a Banach space X. Then $Sp(\mathbf{T}) = i\sigma(A)$.*

Proof. First let $\omega \notin i\sigma(A)$. Noting that $\sigma(A) \subset i\mathbb{R}$, we choose a function $f \in L^1(\mathbb{R})$ whose Fourier transform is compactly supported and vanishes in a neighbourhood of $i\sigma(A)$ but not on ω. By Proposition 2 (ii), $\hat{f}(\mathbf{T}) = 0$. But then $\hat{f}(\omega) \neq 0$ implies that $\omega \notin \mathrm{Sp}(\mathbf{T})$.

Conversely, let $\omega \in i\sigma(A)$. Since $\sigma(A) \subset i\mathbb{R}$ and since the topological boundary of $\sigma(A)$ is always contained in the approximate point spectrum (cf. [Na, Ch. A-III]), we see that $-i\omega$ is contained in the approximate point spectrum of A. Hence we may choose a sequence (x_n) of norm one vectors in X, $x_n \in D(A)$ for all n, such that $\lim_{n\to\infty} \|Ax_n + i\omega x_n\| \to 0$. In view of

$$
\mathbf{T}(t)x_n - e^{-i\omega t}x_n = \int_0^t e^{i\omega s}\mathbf{T}(s)(A + i\omega)x_n\,ds = 0,
$$

(x_n) is an approximate eigenvector of $T(t)$ with approximate eigenvalue $e^{-i\omega t}$.

Let $f \in L^1(\mathbb{R})$. By the dominated convergence theorem,

$$\lim_{n\to\infty} \left\| \int_{-\infty}^{\infty} f(t)(T(t)x_n - e^{-i\omega t}x_n)\,dt \right\| = 0.$$

Thus, using that $\|x_n\| = 1$,

$$\begin{aligned} \|\hat{f}(\mathbf{T})\| \geq \lim_{n\to\infty} \|\hat{f}(\mathbf{T})x_n\| &= \lim_{n\to\infty} \left\| \int_{-\infty}^{\infty} f(t)T(t)x_n\,dt \right\| \\ &= \left| \int_{-\infty}^{\infty} e^{-i\omega t} f(t)\,dt \right| \\ &= |\hat{f}(\omega)|. \end{aligned}$$

This inequality shows that $\hat{f}(\omega) = 0$ for all $f \in I_{\mathbf{T}}$. Therefore, $\omega \in \mathrm{Sp}(\mathbf{T})$. □

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