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Gaussian Estimates and Invariance of the L^p-Spectrum for Elliptic Operators of Higher Order

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In memoriam Pierre Grisvard

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open set and let $T_p = (T_p(t))_{t\geq 0}$ be consistent semigroups on $L^p(\Omega)$, $1 \leq p < \infty$, with generators A_p. It is natural to ask whether interesting properties of the semigroup, the generator or the solution of the associated inhomogeneous initial value problem on $L^p(\Omega)$ depend on p. Upper Gaussian estimates play an important role in this context; indeed, they are essential in questions concerning for example L^1-holomorphy, maximal L^p-regularity, bounded H^∞-calculus or characterization of certain interpolation spaces (see [Ou], [Hi2], [H-P], [D-R], [H-K-M]).

In this note we prove an upper Gaussian estimate of order $m\alpha$ for the semigroup generated by $-e^{i\pi\alpha}A^\alpha$, $\alpha \geq 1$, provided the holomorphic semigroup generated by A satisfies an upper Gaussian estimate of order m. Besides the application cited above, estimates of this type are in particular important for the question whether the spectrum $\sigma(A_p)$ of A_p is independent of p. Notice that this is not the case in general (see [H-V], [Da1, 4.3], [Ar, Sec. 3], [D-S-T]). However, it was shown in 1994 by Arendt [Ar] and Davies [Da2] that $\sigma(A_p)$ is independent of $p \in [1,\infty)$ provided that A_2 is self-adjoint and T_2

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satisfies an upper Gaussian estimate of order 2. Their result applies in particular to Schrödinger operators [Si] and to second order uniformly elliptic operators in divergence form with L^∞ coefficients acting on $L^2(\mathbb{R}^n)$ (see [Da1], [Au]) or on $L^2(\Omega)$ subject to certain boundary conditions (see [Da1], [A-tE]).

Less information is known for general elliptic operators of higher order. We refer to [Da3] for spectral properties of self-adjoint uniformly elliptic operators of order $2m$ satisfying certain quadratic form estimates. In the following, we generalize the result given by Arendt [Ar] to Gaussian estimates of higher order, i.e. we show that the connected component of the resolvent set containing a right half-plane of large class of elliptic operators of higher order is independent of p. In particular, we show that for $\alpha \geq 1$, $\sigma(A_p^\alpha)$ is independent of p provided that T_2 satisfies an upper Gaussian estimate of order m and A_2 is self-adjoint or that Ω is bounded.

We finally mention that the spectra of the $L^p(\mathbb{R}^n)$ realization of certain classes of hypoelliptic (pseudo)differential operators are independent of p only in an interval around $p = 2$ (see [Hi1], [L-S]).

2. Main results and examples

Let $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ be an open set, $p_0 \in [1, \infty)$ and let T be a C_0-semigroup on $L^{p_0}(\Omega)$ with generator A. We always identify $L^{p_0}(\Omega)$ with a subspace of $L^{p_0}(\mathbb{R}^n)$ by extending functions by zero. Given $m \in (1, \infty)$ we define a constant $c_{mn} > 0$ by

$$\frac{1}{c_{mn}} \int_{\mathbb{R}^n} \exp\left(\frac{-|x|^{m/(m-1)}}{4}\right) dx = 1.$$

Moreover, define the family $(G_{p_0}(t))_{t \geq 0}$ of operators on $L^{p_0}(\mathbb{R}^n)$ by $G_{p_0}(t)f := k_t * f$, where

$$k_t(x) := \frac{1}{c_{mn}} \frac{1}{t^{m/n}} \exp\left(\frac{-|x|^{m/(m-1)}}{4t^{1/(m-1)}}\right) \qquad (t > 0,\ x \in \mathbb{R}^n).$$

Since $k_t \in L^1(\mathbb{R}^n)$ for all $t > 0$ it follows from Young's inequality that $\|G_{p_0}(t)f\|_{p_0} \leq \|k_t\|_1 \|f\|_{p_0}$. We say that the semigroup T satisfies an *upper Gaussian estimate of order* m if there exist constants $a \geq 0$, $M > 0$, $b > 0$ such that

$$|T(t)f| \leq Me^{at}G_{p_0}(bt)|f| \qquad (t \geq 0)$$

for all $f \in L^{P_0}(\Omega)$.

Furthermore, we assume that E and F are Banach spaces and that there exists a topological vector space G such that $E \hookrightarrow G$ and $F \hookrightarrow G$. Then two operators $S_E \in \mathcal{L}(E)$ and $S_F \in \mathcal{L}(F)$ are called *consistent* if $S_E x = S_F x$ for all $x \in E \cap F$. We call two semigroups T_E and T_F on E and F consistent if $T_E(t)$ and $T_F(t)$ are consistent for all $t > 0$. Assuming that T is a C_0-semigroup on $L^{p_0}(\Omega)$ which satisfies an upper Gaussian estimate of order m it is not difficult to verify that there exist consistent semigroups T_p on $L^p(\Omega)$, $(1 \leq p < \infty)$, such that $T = T_{p_0}$ and

$$|T_p(t)f| \leq Me^{at}G_p(bt)|f| \qquad (f \in L^p(\Omega),\ t \geq 0), \tag{2.1}$$

(see [Hi2, Lemma 3.1]). For $\theta \in [0, \pi)$ put

$$\begin{aligned} S_\theta &:= \{z \in \mathbb{C} \setminus \{0\}; |\arg z| \leq \theta\} \cup \{0\}, \\ S_\theta^0 &:= \{z \in \mathbb{C} \setminus \{0\}; |\arg z|, \theta\}. \end{aligned}$$

Moreover, we call an operator $S \in \mathcal{L}(L^p(\Omega), L^q(\Omega))$, $(1 \leq p, q \leq \infty)$, an *integral operator*, if there exists a measurable function $K : \Omega \times \Omega \to \mathbb{C}$ such that for all $f \in L^p(\Omega)$, $K(x, \cdot)f(\cdot) \in L^1(\Omega)$ x-a.e. and

$$(Sf)(x) = \int_\Omega K(x,y)f(y)dy \qquad x\text{-a.e.}$$

In that case S is represented by the kernel and we write $S \sim K$. If in addition $|K|$ defines also an integral operator in $\mathcal{L}(L^p(\Omega), L^q(\Omega))$, then S is called a *regular integral operator*. It follows by standard arguments that $T_p(t)$ is an integral operator, say $T_p(t) \sim K(t, \cdot, \cdot)$. We denote by A_p the generator of T_p. Considering $e^{-\alpha t}T(t)$ instead of $T(t)$, we may always assume that (2.1) is satisfied with $a = 0$.

Suppose now that T is a bounded analytic C_0-semigroup on $L^p(\Omega)$ of angle φ satisfying a Gaussian estimate of order m with $a = 0$. Let $l \in \mathbb{N}$, $\theta \in [0, \varphi + \pi/2)$ and $\lambda \in S_\theta^0$. Then by [Hi2, Thm. 2.2], $(\lambda - A_p)^{-}l$ is a regular integral operator with kernel

$K_R^l(\lambda,\cdot,\cdot)$. Moreover, if $l > \frac{n}{m}$, then there exist constant $M, c > 0$ such that

$$|K_R^l(\lambda,x,y)| \leq Me^{-c|\lambda|^{\frac{1}{m}}|x-y|}|\lambda|^{\frac{n}{m}-l} \tag{2.2}$$

for all $x, y \in \Omega$, all $\lambda \in S_\theta^0$.

Finally, let A be a closed, densely defined operator in a Banach space X and let $\omega \in [0,\pi)$. Denote by $\sigma(A)$ and $\rho(A)$ the spectrum and resolvent set of A, respectively. The operator A is called *of type* ω if $\sigma(A) \subset S_\omega$ and for $\theta \in (\omega,\pi)$ there exists a constant M such that

$$\|(\lambda I - A)^{-1}\| \leq \frac{M}{|\lambda|}, \qquad (\lambda \in \mathbb{C} \setminus S_\theta).$$

Assume that $0 \in \rho(A)$. If A is of type ω and $\alpha > 0$, then A^α is defined by $A^\alpha := (A^{-\alpha})^{-1}$, where $A^{-\alpha}$ is given by

$$A^{-\alpha} = \frac{1}{2\pi i}\int_\Gamma \lambda^{-\alpha}(\lambda - A)^{-1} d\lambda$$

and Γ is a suitable path of integration. We note that A^α is a closed operator with dense domain.

For the time being, assume that A is the generator of a bounded holomorphic C_0-semigroup on X of angle φ. Let $\alpha \geq 1$ and $\varphi > \frac{\pi}{2}(1-\frac{1}{\alpha})$. We show in Proposition 3.1 that $-e^{-i\pi\alpha}A^\alpha$ generates an holomorphic semigroup S on X of angle θ, where $\theta < \frac{\pi}{2} - (\frac{\pi}{2} - \varphi)\alpha$. Our first result deals with upper Gaussian estimates for the semigroup S generated by $-e^{-i\pi\alpha}A^\alpha$.

THEOREM 2.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set, $m \in (1,\infty)$, $p_0, \alpha \in [1,\infty)$ and $\varphi > \frac{\pi}{2}(1-\frac{1}{\alpha})$. Let T be a bounded holomorphic C_0-semigroup on $L^{p_0}(\Omega)$ of angle φ with generator A and let S be the semigroup on $L^{p_0}(\Omega)$ generated by $-e^{-i\pi\alpha}A^\alpha$. If T satisfies an upper Gaussian estimate of order m with $a = 0$, then S satisfies an upper Gaussian estimate of order $m\alpha$.*

Gaussian estimates are closely related to the problem of p-independence of $\sigma(A_p)$, the spectrum of A_p. We first give a result dealing with generators A_p of consistent semigroups defined on $L^p(\Omega)$, where Ω is a bounded open subset of $\mathbb{R}^n$.

PROPOSITION 2.2. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set and let $m \in (1,\infty)$. Assume that A generates a C_0-semigroup on $L^{p_0}(\Omega)$ which satisfies an upper Gaussian estimate of order m. Then $\sigma(A_p)$ is independent of $p \in [1,\infty)$.*

For unbounded sets Ω, the situation is more complicated. Denote by $\rho_\infty(A_p)$ the connected component of the resolvent set of A_p which contains a right halfplane. Modifying the arguments given by Arendt [Ar, Thm. 4.2] we obtain the following result.

THEOREM 2.3. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $m \in (1,\infty)$. Assume that A generates a C_0-semigroup on $L^{p_0}(\Omega)$ which satisfies an upper Gaussian estimate of order m. Then $\rho_\infty(A_p)$ is independent of $p \in [1,\infty)$.*

COROLLARY 2.4. *Assume that A_2 is self-adjoint and that T_2 admits an upper Gaussian estimate of order $m > 1$. Then $\sigma(A_p)$ is independent of $p \in [1,\infty)$.*

COROLLARY 2.5. *Assume that T_2 is a bounded analytic C_0-semigroup on $L^2(\Omega)$ which satisfies an upper Gaussian estimate of order $m > 1$ with $a = 0$. Let $\alpha \geq 1$.*
a) Then $\rho_\infty(A_p^\alpha)$ is independent of $p \in [1,\infty)$.
b) If A_2 is self-adjoint, then $\rho_\infty(A_p^\alpha)$ is independent of $p \in [1,\infty)$.

In the following we give two types of examples to which our theorems apply.

EXAMPLES 2.6.

A) Operators associated to elliptic boundary value problems on $L^p(\Omega)$, $1 \leq p < \infty$, where Ω is bounded

Let Ω be a bounded domain in $\mathbb{R}^N$ such that $\partial\Omega \in C^{2+\rho}$ for some $\rho \in (0,1)$. Consider a differential operator A of the form

$$A(x,\partial) := -\sum_{1\leq i,j\leq N} a_{ij}(x)\partial_i\partial_j + \sum_{1\leq i\leq N} a_i(x)\partial_i + a_0(x)$$

where a_{ij}, a_i, $a_0 \in BUC^\rho(\Omega)$ and

$$\sum_{1\le i,j\le N} a_{ij}(x)\xi_i\xi_j \ge c|\xi|^2$$

for all $x \in \mathbb{R}^N$, $\xi = (\xi_1,\dots,\xi_N) \in \mathbb{R}^N$ and some constant $c > 0$. Let $B(x,\partial) := b(x)\cdot\nabla + b_0(x)$ be the boundary operators such that $b = (b_1,\dots,b_n)$, $b_i, b_0 \in C^\rho(\Omega)$ and $b(x)\cdot\nu(x) \ge c_0 > 0$, where $\nu(x)$ is the unit outward normal vector to $\partial\Omega$ at the point $x \in \partial\Omega$. Given $p \in (1,\infty)$, the operator

$$D(\mathcal{A}_p) := \{u \in W_p^2(\Omega); Bu = 0\} \qquad \mathcal{A}_p u := Au$$

is called the L^p-realization of the boundary value problem (A,B). Set

$$\varphi_A := \max_{x\in\overline{\Omega},\,\xi\in S^{N-1}} \arctan \frac{|\,\Im m\, a_\pi(x,\xi)|}{\Re e\, a_\pi(x,\xi)},$$

where a_π denotes the symbol of the principal part of A. Let $\varphi \in (\varphi_A, \pi/2)$. Then $-\mathcal{A}_p$ generates an analytic semigroup T_p on $L^p(\Omega)$, $1 < p < \infty$ of angle $\pi/2 - \varphi$ (cf. [A-D-N] or [Am]). Furthermore, it is shown in [Iv] and [So] that the semigroup T_p generated by $-\mathcal{A}_p$ satisfies an upper Gaussian estimate of order 2. Let T_1 be the consistent semigroup on $L^1(\Omega)$ and denote by A_1 its generator. Then it follows from Theorem 2.2 that $\sigma(A_p)$ is independent of $p \in [1,\infty)$.

B) Elliptic operators on $L^p(\mathbb{R}^n)$ with Hölder continuous coefficients

Let $A = \sum_{|\alpha\le m|} a_\alpha(x)D^\alpha$, $\rho \in (0,1)$, $a_\alpha \in BUC^\rho(\mathbb{R}^n,\mathbb{C})$ for $|\alpha| = m$ and $a_\alpha \in L^\infty(\mathbb{R}^n,\mathbb{C})$ for $|\alpha| \le m$. Suppose that there exists a constant $\delta > 0$ such that

$$\sup_{|\xi|=1} \Re e \sum_{|\alpha|=m} a_\alpha(x)(i\xi)^\alpha < -\delta \qquad \text{for all} \quad x \in \mathbb{R}^n.$$

Given $p \in (1,\infty)$, we define the L^p-realization $\mathcal{A}_p$ of A by

$$\begin{aligned} D(\mathcal{A}_p) &:= W_p^m(\mathbb{R}^n) \\ \mathcal{A}_p &:= Af \qquad \text{for all} \quad f \in D(\mathcal{A}_p)\,. \end{aligned} \tag{2.3}$$

Then it is well-known that $\mathcal{A}_p$ generates an analytic C_0-semigroup T_p on $L^p(\mathbb{R}^n)$ $(1 < p < \infty)$ of some angle $\varphi \in (0, \pi/2]$ (cf. [Am]). Furthermore, it was shown by Friedman [Fr, Thm. 9.4.2] that T_p satisfies an upper Gaussian estimate of order m. Denote by T_1 the consistent semigroup on $L^1(\mathbb{R}^n)$ and by A_1 its generator. Theorem 2.3 implies now implies now that $\rho_\infty(A_p)$ is independent of $p \in [1, \infty)$.

3. Proofs

We start this section with an auxiliary result. Here and in the following we use the convention that M denotes a positive constant whose value may vary from line to line.

PROPOSITION 3.1. *Let $\alpha \geq 1$. Let A be the generator of a bounded analytic C_0-semigroup on a Banach space X of angle φ. If $\varphi > \frac{\pi}{2}(1 - \frac{1}{\alpha})$, then $-e^{-i\pi\alpha}A^\alpha$ generates an holomorphic C_0-semigroup on X of angle θ, where $\theta < \frac{\pi}{2} - (\frac{\pi}{2} - \varphi)\alpha$.*

Proof. Let $\theta \in (0, \frac{\pi}{2} - (\frac{\pi}{2} - \varphi)\alpha)$ and choose $z \in S_\theta^0$. We define

$$S(z) := \frac{1}{2\pi i} \int_\Gamma e^{e^{-i\pi(\alpha-1)}\lambda^\alpha z} (\lambda - A)^{-1} d\lambda\,,$$

where $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$; $\Gamma_{1,3} = \{re^{\pm i\beta}; r \geq 1\}$, $\Gamma_2 = \{e^{i\theta}; |\theta| \leq \beta\}$ and β is chosen such that

$$\frac{\pi}{2} + \pi(\alpha - 1) + \theta < \alpha\beta < \pi\,.$$

Then

$$\begin{aligned} \left\| \frac{1}{2\pi i} \int_{\Gamma_1} e^{e^{-i\pi(\alpha-1)}\lambda^\alpha z} (\lambda - A)^{-1} d\lambda \right\| &\leq \\ \leq M \int_1^\infty e^{r^\alpha \Re e(z e^{i(\alpha\beta - \pi(\alpha-1))})} \frac{1}{r} dr &\leq M \end{aligned}$$

for some constant $M > 0$ and all $z \in S_\theta^0$. In the same way one shows that the terms corresponding to Γ_2 and Γ_3 define bounded operators on X. The proof of the fact that $S(z)_{z \in S_\theta^0}$ is strongly continuous on X is straightforward and therefore omitted.

Next let $\lambda > 1$. Then

$$\begin{aligned}\int_0^\infty e^{-\lambda t} S(t)dt &= \frac{1}{2\pi i}\int_0^\infty e^{-\lambda t}\int_\Gamma e^{e^{-i\pi(\alpha-1)}\mu^\alpha t}(\mu - A)^{-1} d\mu\, dt \\ &= \frac{1}{2\pi i}\int_\Gamma\int_0^\infty e^{t(-\lambda+e^{-i\pi(\alpha-1)}\mu^\alpha)}dt(\mu - A)^{-1}d\mu \\ &= \frac{1}{2\pi i}\int_\Gamma(\lambda - e^{-i\pi(\alpha-1)}\mu^\alpha)^{-1}(\mu - A)^{-1}d\mu \\ &= (\lambda - e^{-i\pi(\alpha-1)}A^\alpha)^{-1}.\end{aligned}$$

Hence $(S(t))_{t\geq 0}$ is a C_0-semigroup on X with generator $-e^{-i\pi\alpha}A^\alpha$. Since $(S(t))_{t\geq 0}$ admits a bounded analytic extension to the sector S_θ^0 which is strongly continuous, it follows that S is a holomorphic semigroup on X of angle θ, where $\theta < \frac{\pi}{2} - (\frac{\pi}{2} - \varphi)\alpha$. □

In the following proposition we collect some well known facts on integral operators which will be used later on (see [Sch, Ch. IV] and [A-B] for proofs and references). For $1 \leq p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, we put

$$L^\infty[L^{p'}] := \Big\{K : \Omega \times \Omega \to \mathbb{C} \ \text{ measurable };\ \operatorname*{ess\,sup}_{y\in\Omega}\Big(\int_\Omega |K(x,y)|^{p'}dx\Big)^{\frac{1}{p'}} < \infty\Big\}.$$

Proposition 3.2. *a) Let $1 \leq p, q \leq \infty$ and let $S \in \mathcal{L}(L^p, L^q)$ be an integral operator represented by K. Let $S_0 \in \mathcal{L}(L^p, L^q)$ be such that*

$$|S_0 f| \leq S|f| \qquad (f \in L^p(\Omega)).$$

Then S_0 is a regular integral operator and $|K_0(x,y)| \leq K(x,y)$ x-a.e., where $K_0 \sim S_0$.

b) Let $1 \leq p < \infty$ and consider the mapping

$$(S_K f)(x) = \int_\Omega K(x,y)f(y)dy \qquad (f \in L^p(\Omega)).$$

Then the mapping $K \mapsto S_K$ establishes an isometric isomorphism of $L^\infty(\Omega \times \Omega)$ onto $\mathcal{L}(L^1(\Omega), L^\infty(\Omega))$, and of $L^\infty[L^{p'}]$ onto $\mathcal{L}(L^p(\Omega), L^\infty(\Omega))$ for $1 < p < \infty$.

LEMMA 3.3. *There exists a constant $M > 0$ such that for all $s \geq 1$ we have*

$$\int_0^{2\pi} e^{s\cos\theta} d\theta \leq M \frac{e^s}{s^{1/2}} .$$

Proof. Observe that

$$\int_0^{2\pi} e^{s\cos\theta} d\theta = \int_0^{2\pi} \sum_{n=0}^{\infty} \frac{s^n}{n!} (\cos\theta)^n d\theta =$$
$$= \sum_{n=0}^{\infty} \frac{s^n}{n!} \int_0^{2\pi} (\cos\theta)^n d\theta = \sum_{n=0}^{\infty} \frac{s^2 n}{2^{2n}(n!)^2} 2\pi .$$

In order to prove the claim we verify that

$$\frac{s^{(2n+1)/2}}{2^{2n}(n!)^2} \leq \frac{s^{2n}}{(2n)!} + \frac{s^{2n+1}}{(2n+1)!}$$

for all $n \in \mathbb{N} \cup \{0\}$ and all $s \geq 1$. □

Proof of Theorem 2.1 Fix $l \in \mathbb{N}$ such that $l > n/m + 1$. Let $\theta \in [0, \varphi + \pi/2)$. For $\lambda \in S_\theta^0$ and $t > 0$ define

$$F_{l,t,\alpha}(\lambda) := \int_0^\lambda \frac{(\lambda - s)^{l-2}}{(l-2)!} e^{e^{-i\pi(\alpha-1)} s^\alpha t} ds .$$

The theorem of the residues implies that

$$S(t) = \frac{(l-1)!}{2\pi i} \int_\Gamma F_{l,t,\alpha}(\lambda)(\lambda - A)^{-1} d\lambda ,$$

where $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$; $\Gamma_{1,3} = \{re^{\pm i\beta}; r \geq 1\}$, $\Gamma_2 = \{Re^{i\theta}; |\theta| \leq \beta\}$ for suitable $R > 0$ and $\beta \in (\frac{1}{\alpha}(\frac{\pi}{2} + \pi(\alpha - 1)), \pi)$. By (2.2), $(\lambda - A)^{-l}$ is a regular integral operator whose kernel $K_R^l(\lambda, \cdot, \cdot)$ satisfies

$$|K_R^l(\lambda, x, y)| \leq M e^{-c|\lambda|^{\frac{1}{m}}|x-y|} |\lambda|^{\frac{n}{m} - l} \qquad (x, y \in \Omega, \lambda \in S_\theta^0)$$

for suitable $M, c > 0$. For $x, y \in \Omega$ and $t > 0$ we set

$$|K_S(t, x, y)| := \frac{(l-1)!}{2\pi i} \int_\Gamma F_{l,t,\alpha}(\lambda)(K_R(\lambda, x, y))^l d\lambda .$$

In the sequel we show that there exist constants $M, \gamma > 0$ such that

$$|K_S(t,x,y)| \le \frac{M}{t^{N/m\alpha}} \exp\left(-\frac{(\gamma|x-y|)^{\frac{m\alpha}{m\alpha-1}}}{t^{\frac{1}{m\alpha-1}}} \right) \tag{3.1}$$

for $x, y \in \Omega$ and $t > 0$. It then follows from Proposition 3.2a and Fubini's theorem that $S(t)$ is a regular integral operator whose kernel satisfies (3.1).

In order to prove assertion (3.1) we consider first the term corresponding to Γ_1 and Γ_3, respectively. Then we have

$$\begin{aligned} |K_S(t,x,y)| &\le M \int_R^\infty \Big| \int_0^{re^{i\beta}} (re^{i\beta}-s)^{l-2} e^{e^{-i\pi(\alpha-1)} s^\alpha t} ds \Big| \\ &\qquad r^{n/m-l} e^{-cr^{1/m}|x-y|} dr \\ &\le M e^{-cR^{1/m}|x-y|} \int_R^\infty r^{l-1} e^{r^\alpha t \cos(\alpha\beta - \pi(\alpha-1))} r^{n/m-l} dr \\ &\le M \frac{e^{-cR^{1/m}|x-y|}}{t^{n/m\alpha}} \int_{R^\alpha t}^\infty e^{u\cos(\alpha\beta-\pi(\alpha-1))} u^{\frac{n}{m\alpha}-1} du\,. \end{aligned}$$

Inspired by an argument due to Auscher [Au] we choose

$$R^\alpha := \max\left\{ \left(\frac{c|x-y|}{2t} \right)^{\frac{m\alpha}{m\alpha-1}}, \frac{1}{t} \right\}. \tag{3.2}$$

For the case $R^\alpha = 1/t$ we have

$$\int_{R^\alpha t}^\infty e^{u\cos(\alpha\beta-\pi(\alpha-1))} u^{\frac{n}{m\alpha}-1} du \le M\,.$$

Hence

$$\begin{aligned} |K_S(t,x,y)| &\le \frac{M}{t^{\frac{n}{m\alpha}}} \exp\left(-\frac{c|x-y|}{t^{1/m\alpha}} \cdot \frac{t^{\frac{1}{m\alpha(m\alpha-1)}}}{t^{\frac{1}{m\alpha(m\alpha-1)}}} \right) \\ &\le \frac{M}{t^{\frac{n}{m\alpha}}} \exp\left(-\frac{c|x-y|}{t^{\frac{1}{m\alpha-1}}} \cdot \frac{(c|x-y|)^{\frac{1}{m\alpha-1}}}{2^{\frac{1}{m\alpha-1}}} \right) \\ &\le \frac{M}{t^{\frac{n}{m\alpha}}} \exp\left(-\frac{(c|x-y|)^{\frac{m\alpha}{m\alpha-1}}}{(2t)^{\frac{1}{m\alpha-1}}} \right) \end{aligned}$$

for all $x, y \in \Omega$ and $t > 0$. For the case $R^\alpha = \left(\frac{c|x-y|}{2t}\right)^{\frac{m\alpha}{m\alpha-1}}$ we have

$$\begin{aligned}
|K_S(t,x,y)| &\leq \frac{M}{t^{\frac{n}{m\alpha}}} \exp\left(-R^{1/m} c|x-y|\right) \\
&\leq \frac{M}{t^{\frac{n}{m\alpha}}} \exp\left(-c|x-y| \left(\frac{c|x-y|}{2t}\right)^{\frac{1}{m\alpha-1}}\right) \\
&\leq \frac{M}{t^{\frac{n}{m\alpha}}} \exp\left(-\frac{(c|x-y|)^{\frac{m\alpha}{m\alpha-1}}}{(2t)^{\frac{1}{m\alpha-1}}}\right).
\end{aligned}$$

In the next step we consider the term corresponding to Γ_2. Then

$$\begin{aligned}
&|K_S(t,x,y)| \leq \\
&\leq M \int_{-\beta}^{\beta} \left|(Re^{i\theta} - s)^{l-2} e^{e^{-i\pi(\alpha-1)} s^\alpha t}\right| \\
&\qquad\qquad R^{n/m-l} e^{-cR^{1/m}|x-y|} R\, ds\, d\theta \\
&\leq M \int_{-\beta}^{\beta} R^{l-1} e^{R^\alpha t \cos(\alpha\theta - \pi(\alpha-1))} R^{n/m-l+1} e^{-cR^{1/m}|x-y|} d\theta \\
&\leq M \frac{e^{-cR^{1/m}|x-y|}}{t^{\frac{n}{m\alpha}}} \int_{-\beta}^{\beta} e^{R^\alpha t \cos(\alpha\theta - \pi(\alpha-1))} (R^\alpha t)^{\frac{n}{m\alpha}} d\theta\,.
\end{aligned}$$

If $R^\alpha t \geq 1$, we obtain by Lemma 3.2

$$\begin{aligned}
|K_S(t,x,y)| &\leq M \frac{e^{-cR^{1/m}|x-y|}}{t^{\frac{n}{m\alpha}}} \frac{e^{R^\alpha t} (R^\alpha t)^{\frac{n}{m\alpha}}}{(R^\alpha t)^{1/2}} \\
&\leq M \frac{e^{-cR^{1/m}|x-y|}}{t^{\frac{n}{m\alpha}}} e^{(1+\varepsilon) R^\alpha t}
\end{aligned}$$

for some $\varepsilon \in (0, 1/2)$. Choosing now R^α as in (3.2) we verify that for the case $R^\alpha = 1/t$ we have

$$\begin{aligned}
|K_S(t,x,y)| &\leq M \frac{e^{-cR^{1/m}|x-y|}}{t^{n/m\alpha}} \\
&= \frac{M}{t^{n/m\alpha}} \exp\left(-\frac{c|x-y|}{t^{1/m\alpha}} \cdot \frac{t^{\frac{1}{m\alpha(m\alpha-1)}}}{t^{\frac{1}{m\alpha(m\alpha-1)}}}\right) \\
&\leq \frac{M}{t^{n/m\alpha}} \exp\left(-\frac{(c|x-y|)^{\frac{m\alpha}{m\alpha-1}}}{(2t)^{\frac{1}{m\alpha-1}}}\right)
\end{aligned}$$

Finally consider the case where $R^\alpha = \left(\frac{c|x-y|}{2t}\right)^{\frac{m\alpha}{m\alpha-1}}$. Then

$$\begin{aligned}
|K_S(t,x,y)| &\leq \\
&\leq Mt^{n/m\alpha}\exp\left(-cR^{\frac{1}{m}}|x-y|\right)\exp\left((1+\varepsilon)R^\alpha t\right) \\
&\leq Mt^{n/m\alpha}\exp\left(-c|x-y|\left(\frac{c|x-y|}{2t}\right)^{\frac{1}{m\alpha-1}}\right)\exp\left((1+\varepsilon)R^\alpha t\right) \\
&\leq Mt^{n/m\alpha}\exp\left(-\frac{(c|x-y|)^{\frac{m\alpha}{m\alpha-1}}}{(2t)^{\frac{1}{m\alpha-1}}}+\frac{(1+\varepsilon)(c|x-y|)^{\frac{m\alpha}{m\alpha-1}}}{2^{\frac{m\alpha}{m\alpha-1}}t^{\frac{1}{m\alpha-1}}}\right) \\
&\leq Mt^{n/m\alpha}\exp\left(\frac{(c|x-y|)^{\frac{m\alpha}{m\alpha-1}}}{t^{\frac{1}{m\alpha-1}}}[-\frac{1}{2^{\frac{1}{m\alpha-1}}}+\frac{(1+\varepsilon)}{2^{\frac{m\alpha}{m\alpha-1}}}]\right) \\
&\leq Mt^{n/m\alpha}\exp\left(\frac{\gamma(c|x-y|)^{\frac{m\alpha}{m\alpha-1}}}{t^{\frac{1}{m\alpha-1}}}\right),
\end{aligned}$$

where $\gamma = -\frac{1}{2^{\frac{1}{m\alpha-1}}}+\frac{(1+\varepsilon)}{2^{\frac{m\alpha}{m\alpha-1}}} < 0.$ □

Proof of Proposition 2.2. Let $1 \leq p,q < \infty$ and $\mu \in \rho(A_p)$. We claim that $\mu \in \rho(A_q)$. By [Ar, Prop. 2.3] it suffices to show that $\|R(\mu,A_p)\|_{\mathcal{L}(L^q)} < \infty$. Since

$$R(\mu,A_p) = \int_0^1 e^{\mu t}T_p(t)dt + e^{\mu}T_p(1)R(\mu,A_p)$$

we only have to prove that

$$\|T_p(1)R(\mu,A_p)\|_{\mathcal{L}(L^q)} < \infty\,.$$

To this end note that

$$T_p(1)R(\mu,A_p) = T_p(1/2)R(\mu,A_p)T_p(1/2)\,.$$

It then follows from Young's inequality that $\|T_p(1/2)\|_{\mathcal{L}(L^1,L^p)} < \infty$ and Proposition 3.2b implies that $\|T_p(1/2)\|_{\mathcal{L}(L^p,L^\infty)} < \infty$. Hence

$$\|T_p(1)R(\mu,A_p)\|_{\mathcal{L}(L^1,L^\infty)} < \infty\,.$$

It follows from Proposition 3.2b that $T_p(1)R(\mu,A_p)$ may be represented as an integral operator with bounded kernel. Since Ω is bounded, the proof is complete. □

Proof of Theorem 3.3. The proof of Theorem 3.3 parallels the one given by Arendt [Ar, Theorem 4.2] for the case $m = 2$. The only fact which needs comment is that $\tilde{T}_{\varepsilon,p}(t)$ given by

$$\tilde{T}_{\varepsilon,p}(t) = U_{\varepsilon,p}^{-1} T_p(t) U_{\varepsilon,p}$$

is bounded for the L^p norm.

Here we use the following notation. Let $\varepsilon \in \mathbb{R}^n$, $x \in \mathbb{R}^n$ and set $\varepsilon x = \sum_{j=1}^n \varepsilon_j x_j$. Define $L^p_\varepsilon := L^p_\varepsilon(\Omega)$ by

$$\begin{aligned} L^p_\varepsilon(\Omega) &:= L^p(\Omega, e^{-p\varepsilon x} dx) \\ &= \left\{ f : \Omega \to \mathbb{C};\ \int_\Omega |f(x)|^p e^{-p\varepsilon x} dx < \infty \right\}. \end{aligned}$$

Then $(U_{\varepsilon,p} f)(x) = e^{-\varepsilon x} f(x)$ defines an isometric isomorphism of L^p_ε onto L^p and $\tilde{T}_{\varepsilon,p}$ defines a C_0-semigroup on L^p_ε. It follows that $\tilde{T}_{\varepsilon,p}$ is an integral operator whose kernel $K_\varepsilon(t,\cdot,\cdot)$ is given by

$$K_\varepsilon(t,x,y) = e^{\varepsilon(x-y)} K(t,x,y)\,.$$

Let $\tilde{S}_{\varepsilon,p}(t) := U_{\varepsilon,p}^{-1} T_p(t) U_{\varepsilon,p}$. Then

$$\begin{aligned} |(\tilde{S}_{\varepsilon,p}(t) f)(x)| &\le \int_{\mathbb{R}^n} e^{\varepsilon(x-y)} |K(t,x,y)|\,|f(y)|\,dy \\ &\le \frac{M}{t^{n/m}} \int_{\mathbb{R}^n} e^{\varepsilon(x-y)} \exp\left(-\frac{c|x-y|^{\frac{m}{m-1}}}{t^{\frac{1}{m-1}}} \right) |f(y)|\,dy\,. \end{aligned}$$

Observe that there exists $w_1 \ge 0$ such that

$$\exp\left(-\frac{c}{2} \left(\frac{|x|^m}{t} \right)^{\frac{1}{m-1}} \right) \exp(|\varepsilon||x|) \le \exp(w_1 |\varepsilon|^m t)$$

for all $x \in \Omega$. Hence it follows from Young's inequality that

$$\begin{aligned} \|\tilde{S}_{\varepsilon,p}(t) f\|_{L^p} &\le e^{w_1|\varepsilon|^m t} \int_{\mathbb{R}^n} \frac{e^{|\varepsilon||x|}}{t^{n/m}} e^{-\frac{c}{2}\left(\frac{|x|^m}{t}\right)^{\frac{1}{m-1}}} dx\, \|f\|_{L^p} \\ &\le M e^{w_1|\varepsilon|^m t} \|f\|_{L^p}\,. \end{aligned}$$

□

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