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Regularity Considerations for Semilinear Parabolic Systems

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Dedicated to the Memory of Pierre Grisvard

0. Introduction, Notations

We consider semilinear parabolic systems

$$\partial_t u + A(t)u + M(t, x, u, Du, ..., D^m u) = 0 \tag{0.1}$$

over $[0, +\infty) \times \overline{\Omega} \subset \mathbb{R}^{n+1}$. $A(t)$ is an elliptic system of order $2m$ satisfying the Legendre-Hadamard condition, the nonlinear term M is subject to suitable growth conditions. Ω is a bounded domain of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ on which the vector u satisfies Dirichlet-0-conditions. Of course we prescribe the initial value

$$u(0, x) = \varphi(x),\ x \in \overline{\Omega}. \tag{0.2}$$

In the first part we work within the class of Hölder-continuous vectors, this is $C^{\alpha/2m,\alpha}([0, T] \times \overline{\Omega})$. For simplicity we assume that $M(t, ...)$ has the form $M(t, x, D^m u)$ and is quadratic in the m-th order derivatives $D^m u$. This is a direct approach to regularity and it yields the following result: If the maximal interval of existence $[0, T(\varphi))$ for (0.1,0.2) is finite then the oscillation

$$\sup_{|t-s|\leq\delta} \|u(t) - u(s)\|_{C^0(\overline{\Omega})} \tag{0.3}$$

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for every $\delta > 0$ exceeds a certain value $\varepsilon_0 > 0$ which can be determined a-priori. Thus we improve on the results in [W1] in several respects. An important rôle in our considerations is played by the interpolation inequality

$$\|u\|_{C^{\alpha/2m,0}([0,T]\times\overline{\Omega})} \leq c(T)\|\partial_t u\|^{(\alpha/2m)/(1+\alpha/2m)}_{C^{\alpha/2m,0}([0,T]\times\overline{\Omega})} \cdot \|u\|^{1-(\alpha/2m)/(1+\alpha/2m)}_{C^0([0,T]\times\overline{\Omega})} + \|u\|_{C^0([0,T]\times\overline{\Omega})}. \tag{0.4}$$

As it was brought to our attention by A. Lunardi (University of Parma) the constant $c(T)$ in (0.4) as $T \to 0$ blows up in a power-like way. We clarify its usage here in order to avoid non-controllable quantities. As an example we show that for a single second-order equation

$$\partial_t u - a_{ij}(t,x)\partial_{x_i}\partial_{x_j}u + M(t,x,\nabla u) = 0$$

with quadratic growth of M with respect to ∇u we have an a-priori bound on $\|u(t)\|_{C^0(\overline{\Omega})}$ and that this is sufficient to ensure global (in time) classical solvability.

In the results previously described we considered solutions in classes of Hölder continuous vectors; the critical quantity is the oscillation (0.3), the critical growth of M with respect to $D^m u$ is quadratic. This is different in the second part (Chapter 3) of the present paper. Here we switch over to weak solutions for which we have a reasonable notion of energy: $u \in L^\infty((0,T),L^2(\Omega)) \cap L^2((0,T),H^{m,2}(\Omega))$. In order to define weak solutions to systems like (0.1) different assumptions on the elliptic operator $A(t)$ are needed. Whereas in the first part it was sufficient to assume that the coefficient matrices in (0.1) are Hölder-continuous in (t,x), we now suppose that $A(t)$ in (0.1) has divergence-structure. The regularity of the coefficient-matrices is of such a type that $A(t)u$ can be written down pointwise if u permits it. For details we refer to [GW]. This assumption allows us to define the notion of a weak solution to (0.1) in the usual way and to ask for their regularity. The so called "controllable" growth conditions

$$|M(t,.,u,Du,...,D^m u)| \leq c\left(1+\sum_{\nu=0}^{m}|D^\nu u|^{\frac{n+4m}{n+2\nu}}\right) \tag{0.5}$$

are "critical" with respect to the "energy class" $u \in L^\infty\Big((0,T), L^2(\Omega)\Big) \cap L^2\Big((0,T), H^{m,2}(\Omega)\Big)$. It has been proved in [GW] that under this growth condition any weak solution is regular. A sign condition on M is not needed. Here we show, by means of a counterexample, that this result is optimal as it concerns the growth condition.

We introduce some notation. $C^{\frac{\alpha}{2m},\alpha}([T_1,T_2]\times\overline{\Omega})$ is the subspace of $C^0([T_1,T_2]\times\overline{\Omega})$ whose members u have finite semi-norm

$$[u]^{[T_1,T_2]\times\overline{\Omega}}_{\frac{\alpha}{2m},\alpha} = \sup_{\substack{(t,x)\neq(t',x')\\ (t,x),(t',x')\in[T_1,T_2]\times\overline{\Omega}}} \frac{|u(t',x')-u(t,x)|}{|t-t'|^{\alpha/2m}+|x-x'|^\alpha}$$

$(0\le\alpha<1)$. The norm of $C^{\frac{\alpha}{2m},\alpha}([T_1,T_2]\times\overline{\Omega})$ is then given by

$$\|u\|_{\frac{\alpha}{2m},\alpha} = \|u\|_{C^0([T_1,T_2]\times\overline{\Omega})} + [u]^{[T_1,T_2]\times\overline{\Omega}}_{\frac{\alpha}{2m},\alpha}.$$

All coefficient-matrices of $A(t)$ in (0.1) belong to this space. If we want to stress the underlying time-interval we also write $\|u\|^{[T_1,T_2]}_{\frac{\alpha}{2m},\alpha}$ for the norm of $C^{\frac{\alpha}{2m},\alpha}([T_1,T_2]\times\overline{\Omega})$. Instead of $\|.\|_{0,0}$ we use the symbol $\|.\|_0$. If no misunderstanding can arise $\|.\|_0$ is also employed for the norm of $C^0(\overline{\Omega})$. Analogously to $C^{\alpha/2m,\alpha}([T_1,T_2]\times\overline{\Omega})$ we define $C^{\gamma,k+\eta}([T_1,T_2]\times\overline{\Omega})$ for $0\le\gamma<1$, $k\in\mathbb{N}\cup\{0\}$, $0\le\eta<1$ and the norms $\|.\|_{\gamma,k+\eta}$, $\|.\|^{[T_1,T_2]}_{\gamma,k+\eta}$. Let

$$w\in C^1([T_1,T_2]\times\overline{\Omega}),$$

$$\partial_t w\in C^{\frac{\alpha}{2m},\alpha}([T_1,T_2]\times\overline{\Omega}),$$

$$w:[T_1,T_2]\to C^{2m+\alpha}(\overline{\Omega})$$

$$\text{with } \sup_{T_1\le t\le T_2}\|w(t)\|_{C^{2m+\alpha}(\overline{\Omega})}<+\infty.$$

Then we set

$$|||w|||_{[T_1,T_2]} = \|\partial_t w\|_{\frac{\alpha}{2m},\alpha} + \sup_{T_1\le t\le T_2}\|w(t)\|_{C^{2m+\alpha}(\overline{\Omega})}.$$

If $T_1 = 0$ we also write $|||w|||_{T_2}$ instead of $|||\,.\,|||_{[0,T_2]}$. Due to appropriate interpolation inequalities finiteness of $|||w|||_{[T_1,T_2]}$ implies finiteness of

$$\sum_{\substack{|\tilde\alpha|=j,\\ 1\le j\le 2m}} \|D^{|\tilde\alpha|}w\|^{[T_1,T_2]}_{(2m-j+\alpha)/2m,\alpha}$$

(cf. [W1]), together with the corresponding estimate.

1. General Theory for Semilinear Parabolic Systems in Hölder Spaces under Homogeneous Dirichlet-Conditions

We carry over the assumptions in [W1]: instead of equations $\partial_t u + A(t)u = f$, $u(0) = \varphi$, we can as well treat systems where the $a_{\tilde\alpha}(t,x)$ are $N \times N$-matrices. We then assume Legendre-Hadamard's condition to be fullfilled, this is (c_0 is some positive constant)

$$\mathcal{R}e(-1)^m \sum_{|\tilde\alpha|=2m} a_{\tilde\alpha}(t,x)\xi^{\tilde\alpha}\zeta\zeta^* \ge c_0|\xi|^{2m}|\zeta|^2,$$
$$\xi\in\mathbb{R}^n,\ \zeta\in\mathbb{C}^N,\ x\in\overline{\Omega},\ t\ge 0.$$

For simplicity we assume the ellipticity condition to be valid for all $t \ge 0$. The following quantities are assumed to be given:

$$m, n, N, c_0, \Omega, \|a_{\tilde\alpha}\|_{\frac{\alpha}{2m},\alpha}, \alpha.$$

Dependence of constants on these quantities is not explicitly mentioned. In contrast to that, dependence of the constants on the time interval $[0,T]$, the initial value φ and the right-hand side f is mentioned. We are going to consider semilinear problems

$$\partial_t u + A(t)u + M(t,.,D^m u) = 0,$$
$$u(0) = \varphi,\ (A(0)\varphi + M(0,.,D^m\varphi))|\partial\Omega = 0,$$
$$\frac{\partial^j}{\partial\nu^j}\varphi = 0,\ 0\le j\le m-1, \frac{\partial^j u}{\partial\nu^j} = 0 \text{ on } \partial\Omega,\ 0\le j\le m-1.$$

Therefore we fix our *assumptions on M*:
A1. Let

$$|M(t',x',p') - M(t,x,p)| \le c(T)\cdot|p'-p|\cdot(|p'|+|p|) +$$
$$+\, c(T)(1+|p'|^2+|p|^2)\cdot(|t'-t|^{\frac{\alpha}{2m}} + |x'-x|^\alpha),$$

$$0\le t',t\le T,\ x',x\in\overline{\Omega},\ p',p\in\mathbb{R}^{Ns_m},\ T\ge 0.$$

s_m is the number of multiindices $\tilde{\alpha}$ of $\mathbb{R}^n$ with $|\tilde{\alpha}| = m$. $c(.)$ depends monotonically non decreasing on $T \geq 0$. $\Big($If $w \in C^0([0,T], C^m(\overline{\Omega}))$ has the property $D^m w \in C^{\alpha/2m,\alpha}([0,T] \times \overline{\Omega})$ we arrive at

$$\|M(.,.,D^m w)\|^{[0,T]}_{\frac{\alpha}{2m},\alpha} \leq c(T)(\|D^m w\|^{[0,T]}_{\frac{\alpha}{2m},\alpha} \|D^m w\|^{[0,T]}_0 + 1).\Big)$$

A2. Let $w_i \in C^0([0,T], C^m(\overline{\Omega}))$, $D^m w_i \in C^{\frac{\alpha}{2m},\alpha}([0,T] \times \Omega)$, $i = 1,2$, $w_1(0) = w_2(0)$. Then we suppose that

$$\|M(.,.,D^m w_2) - M(.,.,D^m w_1)\|^{[0,T]}_{\frac{\alpha}{2m},\alpha} \leq \lambda(T,D) \cdot |||w_2 - w_1|||_T,$$

where $\lambda : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, $D \geq |||w_2|||_T + |||w_1|||_T$, $\lambda(T,D) \to 0$ as $T \to 0$ for every $D \geq 0$. (This requires a condition on $\partial M/\partial p$ analogous to the one for M in A1, but somewhat weaker.)

As a consequence of Assumption A2 we have

THEOREM 1.1. *Let $\varphi \in C^{2m+\alpha}(\overline{\Omega})$, let*

$$(A(0)\varphi + M(0,.,D^m\varphi))|\partial\Omega = 0, \ \frac{\partial^j \varphi}{\partial \nu^j} = 0 \text{ on } \partial\Omega, \ 0 \leq j \leq m-1.$$

Then there exists a $T(\varphi)$, $0 < T(\varphi) \leq +\infty$, such that there is a unique u with $|||u|||_T < +\infty$ for every $T < T(\varphi)$,

$$\partial_t u + A(t)u + M(t,.,D^m u) = 0, \ 0 \leq t \leq T < T(\varphi), \tag{1.1}$$

$$u(0) = \varphi, \tag{1.2}$$

$$\frac{\partial^j u}{\partial \nu^j}(t) = 0 \text{ on } \partial\Omega, \ 0 \leq t \leq T < T(\varphi). \tag{1.3}$$

If $T(\varphi) < +\infty$ then $|||u|||_T \to +\infty$ as $T \uparrow T(\varphi)$. $T(\varphi)$ is called the maximal interval of existence for the Problem (1.1,2,3). Let $F > 0$, let φ fulfil the previous assumptions. Let $F \geq \|\varphi\|_{2m+\alpha}$. Then there is a finite interval $[0, T_1(F)]$, $T_1(F) > 0$, such that Problem (1.1,2,3) has a unique solution u on $[0, T_1(F)]$ with $|||u|||_{T_1(F)} < +\infty$. $[0, T_1(F)]$ is called a first interval of existence.

Proof. In view of the linear estimates in [LSU, ch. VII], [W1, p. 437] being valid also for systems like ours the assertions of Theorem 1.1 can be easily shown to be true by making use of Banach's fixed point theorem. □

As for global existence we have

THEOREM 1.2. *Let $\varphi \in C^{2m+\alpha}(\overline{\Omega})$, $(A(0)\varphi + M(0,.,D^m\varphi))|\partial\Omega = 0$, $\partial^j\varphi|\partial\nu^j = 0$ on $\partial\Omega$, $0 \le j \le m-1$. Let $F \ge \|\varphi\|_{2m+\alpha}$. Let $T > T_1(F) > 0$. Then there exists a constant*

$$\varepsilon_0 = \varepsilon_0(T, T_1(F)) > 0$$

with the following property: Let u be a solution of Problem (1.1,2,3) on $[0,\tilde{T}]$ with $|||u|||_{\tilde{T}} < +\infty$, for every $\tilde{T}$, $0 < \tilde{T} < T$. If for some $\delta > 0$ we have

$$\|u(t+h) - u(t)\|_0 \le \varepsilon_0 \tag{1.4}$$

for all h, t, $0 \le h \le \delta$, $0 \le t \le t+h < T$, then u can be continued into T such that $|||u|||_T$ is finite and such that u solves Problem (1.1,2,3) on $[0,T]$. In particular we have $T(\varphi) > T$.

Proof. Set

$$v(t) = u(t) - u(t-\delta)$$

on $[\delta, \tilde{T}]$, $T-\delta \le \tilde{T} < T$. δ is positive, $< \frac{1}{4}T_1(F)$ and will be specified later on. We have

$$\begin{aligned}
\partial_t v + A(t)v &= -(A(t) - A(t-\delta))u(t-\delta) - \\
&\quad - (M(t,.,D^m u(t)) - M(t-\delta,.,D^m u(t-\delta))), \\
&= -(A(t) - A(t-\delta))u(t-\delta) - \\
&\quad - (M(t,.,D^m(u(t) - u(t-\delta)) + D^m u(t-\delta)) - \\
&\quad - M(t-\delta,.,D^m u(t-\delta))).
\end{aligned}$$

Then (observe that $\left|[\delta,\tilde{T}]\right| \ge T - 2\delta \ge T_1(F) - 2\delta > \frac{1}{2}T_1(F)$)

$$\begin{aligned}
|||v|||_{[\delta,\tilde{T}]} &\underset{\text{A1}}{\overset{\text{[W1, pp. 438, 439]}}{\le}} c(T)|||u|||_{T-\delta} + \\
&\quad + c(T)\Big(\|D^m(u(.) - u(.-\delta)) + D^m u(.-\delta)\|^{[\delta,\tilde{T}]}_{\frac{\alpha}{2m},\alpha} \cdot \\
&\qquad \cdot \|D^m(u(.) - u(.-\delta)) + D^m u(.-\delta)\|^{[\delta,\tilde{T}]}_0 + \\
&\quad + \|D^m u(.)\|^{[0,T-\delta]}_{\frac{\alpha}{2m},\alpha}\|D^m u(.)\|^{[0,T-\delta]}_0 + 2\Big) +
\end{aligned}$$

$$
\begin{aligned}
&+ c(T)(\|\varphi\|_{2m+\alpha} + \|u(\delta)\|_{2m+\alpha}),\\
\leq \quad & c(T)|||u|||_{T-\delta} +\\
&+ c(T)\|D^m(u(.) - u(. - \delta))\|^{[\delta,\tilde{T}]}_{\frac{\alpha}{2m},\alpha} \cdot\\
&\quad \cdot \|D^m(u(.) - u(. - \delta))\|^{[\delta,\tilde{T}]}_0 +\\
&+ c(T)\|D^m(u(.) - u(. - \delta))\|^{[\delta,\tilde{T}]}_{\frac{\alpha}{2m},\alpha} \cdot\\
&\quad \cdot \|D^m u(. - \delta))\|^{[\delta,\tilde{T}]}_0 +\\
&+ c(T)\|D^m u(. - \delta))\|^{[\delta,\tilde{T}]}_{\frac{\alpha}{2m},\alpha} \cdot\\
&\quad \cdot \|D^m(u(.) - u(. - \delta))\|^{[\delta,\tilde{T}]}_0 +\\
&+ c(T)\|D^m u(. - \delta))\|^{[\delta,\tilde{T}]}_{\frac{\alpha}{2m},\alpha}\|D^m u(. - \delta))\|^{[\delta,\tilde{T}]}_0 +\\
&+ c(T)\|D^m u(.)\|^{[0,T-\delta]}_{\frac{\alpha}{2m},\alpha}\|D^m u(.)\|^{[0,T-\delta]}_0 +\\
&+ 2c(T) + c(T)(\|\varphi\|_{2m+\alpha} + \|u(\delta)\|_{2m+\alpha}),
\end{aligned}
$$

[W1, pp. 438, 439]

$$
\begin{aligned}
\leq \quad & c(T)|||u|||_{T-\delta} +
\end{aligned}
$$

(0.4) on $[\delta,\tilde{T}]$

$$
\begin{aligned}
&+ c(T,T_1(F))|||v|||_{[\delta,\tilde{T}]}\, g(\|v(.)\|^{[\delta,\tilde{T}]}_0) +\\
&+ c(T,T_1(F))\left(|||v|||^{1-\gamma_1}_{[\delta,\tilde{T}]} h_1(\|v(.)\|^{[\delta,\tilde{T}]}_0) + 1\right) \cdot\\
&\quad \cdot \|D^m u(.)\|^{[0,T-\delta]}_0 +\\
&+ c(T,T_1(F))\left(|||v|||^{1-\gamma_2}_{[\delta,\tilde{T}]} h_2(\|v(.)\|^{[\delta,\tilde{T}]}_0) + 1\right) \cdot\\
&\quad \cdot \|D^m u(.)\|^{[0,T-\delta]}_{\frac{\alpha}{2m},\alpha} +\\
&+ c(T)\|D^m u(.)\|^{[0,T-\delta]}_{\frac{\alpha}{2m},\alpha}\|D^m u(.)\|^{[0,T-\delta]}_0 +\\
&+ 2c(T) + c(T)(\|\varphi\|_{2m+\alpha} + \|u(\delta)\|_{2m+\alpha}).
\end{aligned}
$$

Here γ_1, γ_2 denote fixed positive numbers with $\gamma_1, \gamma_2 \in (0,1)$. h_1, h_2 are some continuous functions from $\mathbb{R}^+$ into itself. g is a fixed continuous function from $\mathbb{R}^+$ into itself with $g(r) \to 0$ as $r \to 0$. Now we choose ε_0 in such a way that

$$c(T, T_1(F))g(\varepsilon_0) \leq \frac{1}{2}.$$

Since g is a function simply originating from the interpolation inequalities employed in [W1, p. 438] we have $\varepsilon_0 = \varepsilon_0(T, T_1(F))$. Assume now that (1.4) is valid. δ is taken from (1.4). Possibly we diminish it to satisfy $\delta < \frac{1}{4}T_1(F)$. Then $\|u(t)\|_0 \leq c(\delta; T)(1 + \|\varphi\|_0)$, $0 \leq t < T$. Employing the inequality $a^{1-\gamma}b \leq c(\gamma)(\rho a + (\frac{1}{\rho^{1-\gamma}}b)^{1/\gamma})$, $a, b \geq 0$, $\rho > 0$, $\gamma \in (0,1)$, we arrive at

$$\begin{aligned} |||v|||_{[\delta,\tilde{T}]} \leq\ & c(T)|||u|||_{T-\delta} + \\ & + c(T, T_1(F), \gamma_1, \gamma_2, h_1, h_2, \delta, \\ & \quad \|\varphi\|_{2m+\alpha}, \|u(\delta)\|_{2m+\alpha}, |||u|||_{T-\delta}). \end{aligned}$$

Letting $\tilde{T}$ tend to T we arrive at the assertion. □

2. An Application to Second Order Equations

We now consider the previous problem for $m = 1$ and for a single equation. Then we have

$$\begin{aligned} \partial_t u - a_{ij}(t,x)\partial^2_{x_i x_j} u + M(t, x, \nabla u) &= 0, \\ u(0) &= \varphi, \\ u(t) &= 0 \text{ on } \partial\Omega,\ t \geq 0. \end{aligned} \tag{2.1}$$

We omit the summation sign in the spatial elliptic part and set $A(t)u = -a_{ij}(t,x)\partial^2_{x_i x_j} u$, thereby assuming that $A(t)$ only contains second order derivatives. The first compatibility condition reads

$$A(0)\varphi + M(0, x, \nabla\varphi) = 0 \text{ on } \partial\Omega. \tag{2.2}$$

Instead of (2.1) we consider the problems

$$\begin{aligned} \partial_t u_\sigma - a_{ij}(t,x)\partial^2_{x_i x_j} u_\sigma + M(t, x, \nabla u_\sigma) - & \\ - M(0, x, \sigma\nabla\varphi) + \sigma M(0, x, \nabla\varphi) &= 0, \\ u_\sigma(0) &= \sigma\varphi, \\ u_\sigma(t) &= 0 \text{ on } \partial\Omega,\ t \geq 0, \end{aligned} \tag{2.3}$$

$0 \le \sigma \le 1$. Since

$$\begin{aligned}&A(0)\sigma\varphi + M(0,x,\nabla\sigma\varphi) - M(0,x,\sigma\nabla\varphi) + \sigma M(0,x,\nabla\varphi)\\&= \sigma(A(0)\varphi + M(0,x,\nabla\varphi)),\\&= 0 \text{ on } \partial\Omega,\ 0 \le \sigma \le 1,\end{aligned} \tag{2.4}$$

the first order compatibility condition is fullfilled for all problems (2.3), provided it is so for (2.1). For $\sigma = 1$ the unique solution of (2.3, $\sigma = 1$) is the function u under consideration, for $\sigma = 0$ the unique solution of (2.3, $\sigma = 0$) is $u_0 \equiv 0$. A minor generalisation of Theorem 1.1 shows that there is a joint first interval of existence $[0, T_1]$ for all problems (2.3), $0 \le \sigma \le 1$. The maximum-principle furnishes

$$\|u_{\sigma_2}(t) - u_{\sigma_1}(t)\|_0 \overset{[\mathrm{LSU, p.\ 13}]}{\le} |\sigma_2 - \sigma_1| \cdot c(\|\varphi\|_0, \|\nabla\varphi\|_0) \cdot e^t.$$

Let us set $\sigma_2 = \sigma_1 + \varepsilon$ for some $\varepsilon > 0$. Then

$$\begin{aligned}&\|u_{\sigma_2}(t+h) - u_{\sigma_2}(t)\|_0 \le\\&\le \|u_{\sigma_2}(t+h) - u_{\sigma_1}(t+h)\|_0 + \|u_{\sigma_1}(t+h) - u_{\sigma_1}(t)\|_0 +\\&\qquad + \|u_{\sigma_1}(t) - u_{\sigma_2}(t)\|_0\\&\le 2\varepsilon e^t c(\|\varphi\|_0, \|\nabla\varphi\|_0) + \|u_{\sigma_1}(t+h) - u_{\sigma_1}(t)\|_0.\end{aligned}$$

Let $T > 0$. Let $u_{\sigma_2}, u_{\sigma_1}$ solve (2.3, $\sigma = \sigma_2$), (2.3, $\sigma = \sigma_1$) resp. over any cylinder $[0, \tilde{T}] \times \overline{\Omega}$, $0 < \tilde{T} < T$. Let

$$2\varepsilon e^T c(\|\varphi\|_0, \|\nabla\varphi\|_0) \le \frac{1}{2}\varepsilon_0(T, T_1),$$

where $\varepsilon_0(T, T_1) > 0$ is the quantity constructed in Theorem 1.2. It can be chosen uniformly for $\sigma \in [0,1]$. If u_{σ_1} is uniformly continuous from $[0, T)$ into $C^0(\overline{\Omega})$ and thus, according to Theorem 1.2, exists on $[0, T] \times \overline{\Omega}$ by continuation as the unique solution of (2.3, $\sigma = \sigma_1$), Theorem 1.2 now shows: u_σ exists on $[0, T] \times \overline{\Omega}$ for all σ, $\sigma_1 \le \sigma \le \sigma_1 + \left(\frac{e^{-T}}{4}\varepsilon_0(T, T_1)/c(\|\varphi\|_0, \|\nabla\varphi\|_0)\right)$. Starting with $\sigma_1 = 0$ we exhaust $[0, 1]$ in finitely many steps. Since T can be chosen arbitrarily we end up with the global solution for (2.1).

3. On the Necessity of Controllable Growth Conditions in Regularity Theory

We consider the semilinear parabolic equation

$$u_t + (-\Delta)^m u = M(t, x, u) \text{ in } [0,1] \times \overline{B} \tag{3.1}$$

with smooth initial and boundary values. $B \subset \mathbb{R}^n$ denotes the (open) unit ball, M a Hölder continuous nonlinear function. In [GW] the sufficiency of controllable growth conditions

$$|M(t, x, u)| \le c(1 + |u|)^{1+\frac{4m}{n}} \tag{3.2}$$

for weak solutions $u \in L^\infty((0,1), L^2(B)) \cap L^2((0,1), H^{m,2}(B))$ of (3.1) to be smooth was shown. Here by means of a simple example we also demonstrate the necessity of (3.2). In [GW] for simplicity we considered homogeneous Dirichlet boundary data on $[0,1] \times \partial B$. But by simply subtracting the data, smoothly extended to $[0,1] \times \overline{B}$, it is sufficient to assume smooth initial and boundary data:

$$\begin{cases} \left(\frac{\partial}{\partial \nu}\right)^j u(t,.)|\partial B = \left(\frac{\partial}{\partial \nu}\right)^j \varphi(t,.)|\partial B \\ \qquad \text{for } j = 0, ..., m-1, \ t \in [0,1], \\ u(0,.) = \varphi(0,.), \end{cases} \tag{3.3}$$

with some $\varphi \in C^\infty([0,1] \times \overline{B})$.

For some $\gamma > 0$, to be specified below, we define on $[0,1] \times \overline{B}$:

$$u(t,x) = (1 - t + |x|^{2m})^{-\frac{\gamma}{2m}}.$$

Obviously u is arbitrarily smooth in $[0,1] \times \overline{B} \setminus \{(1,0)\}$ and develops a singularity in $(t,x) = (1,0)$. We want to show that for any $\delta > 0$ there is some $\gamma > 0$ such that $u \in L^\infty((0,1), L^2(B)) \cap L^2((0,1), H^{m,2}(B))$ weakly solves the equation (3.1) with an appropriate nonlinearity M, satisfying the growth condition

$$|M(t,x,u)| \le c(1 + |u|)^p$$

with "slightly supercritical" exponent: $1 + \frac{4m}{n} < p < 1 + \frac{4m}{n} + \delta$.

Let $\frac{\partial}{\partial r} = \sum_{i=1}^{n} \frac{x_i}{|x|} \frac{\partial}{\partial x_i}$ denote the radial derivative. By induction on j we find:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\gamma}{2m}(1-t+|x|^{2m})^{-\frac{\gamma}{2m}-1}, \\ \dfrac{\partial}{\partial r}\Delta^j u = \displaystyle\sum_{k=1}^{2j+1} c_{jk}|x|^{2mk-2j-1}(1-t+|x|^{2m})^{-\frac{\gamma}{2m}-k}, \\ \qquad\qquad m > j \geq 0, \\ \Delta^j u = \displaystyle\sum_{k=1}^{2j} d_{jk}|x|^{2mk-2j}(1-t+|x|^{2m})^{-\frac{\gamma}{2m}-k}, \\ \qquad\qquad m \geq j \geq 1, \end{cases} \tag{3.4}$$

$(t,x) \in [0,1] \times \overline{B} \setminus \{(1,0)\}$, $c_{jk}, d_{jk} \in \mathbb{R}$ are suitable numbers, depending on γ and m.

In particular, with suitable numbers $\tilde{d}_{mk} \in \mathbb{R}$, u is a classical solution on $[0,1) \times \overline{B}$ of the following equation:

$$\begin{aligned} u_t + (-\Delta)^m u &= \sum_{k=0}^{2m-1} \tilde{d}_{mk}|x|^{2mk}(1-t+|x|^{2m})^{-\frac{\gamma}{2m}-k-1} \\ &= \left(\sum_{k=0}^{2m-1} \tilde{d}_{mk}|x|^{2mk}(1-t+|x|^{2m})^{-k+\varepsilon\frac{\gamma}{2m}}\right) \\ &\quad \cdot(1-t+|x|^{2m})^{-\frac{\gamma}{2m}(1+\frac{2m}{\gamma}+\varepsilon)} \\ &=: g(t,x)u^p = g(t,x)|u|^{p-1}u, \end{aligned} \tag{3.5}$$

where the additional parameter $\varepsilon > 0$ will also be specified below and $p = p(\gamma, \varepsilon) = 1 + \frac{2m}{\gamma} + \varepsilon$. The function

$$g(t,x) := \sum_{k=0}^{2m-1} \tilde{d}_{mk}|x|^{2mk}(1-t+|x|^{2m})^{-k+\varepsilon\frac{\gamma}{2m}}$$

is Hölder continuous on $[0,1] \times \overline{B}$. We set

$$M(t,x,u) = g(t,x)|u|^{p-1}u. \tag{3.6}$$

Now we want to investigate the integrability properties of the solution u. We additionally assume

$$\gamma < \frac{n}{2}. \tag{3.7}$$

For $t \in [0,1]$ we find

$$\|u(t)\|^2_{L^2(B)} = \int_B (1-t+|x|^{2m})^{-\frac{\gamma}{m}}\, dx \leq \int_B |x|^{-2\gamma} < \infty$$

uniformly on $[0,1]$. Moreover by Lebesgue's theorem we see that

$$u \in C^0([0,1], L^2(B)). \tag{3.8}$$

Observing the radial symmetry of u and the estimates (**??**) we calculate by means of Fubini-Tonelli:

$$\begin{aligned}\int_0^1 \|u(t)\|^2_{H^{m,2}}\, dt &\leq c\int_0^1\int_B (1-t+|x|^{2m})^{-\frac{\gamma}{m}-1}\, dx\, dt\\ &\leq c\int_B |x|^{-2\gamma}\, dx < \infty,\end{aligned}$$

$$u \in L^2((0,1), H^{m,2}(B)). \tag{3.9}$$

Due to the properties (3.8) and (3.9) of u and

$$\begin{aligned}\int_0^1\int_B |M(t,x,u)|\, dx\, dt &\leq c\int_0^1\int_B (1-t+|x|^{2m})^{-\frac{\gamma}{2m}-1}\, dx\, dt\\ &\leq \int_B |x|^{-\gamma}\, dx < \infty,\end{aligned}$$

we conclude that u is a singular weak solution to (3.1) on $[0,1] \times \overline{B}$. Admissible testing functions are e.g. differentiable once with respect to t and $2m$-times with respect to x.

To conclude we let $\gamma \nearrow \frac{n}{2}$ and $\varepsilon \searrow 0$ and find that

$$p = p(\gamma,\varepsilon) = 1 + \frac{2m}{\gamma} + \varepsilon \searrow 1 + \frac{4m}{n}$$

approaches the "critical exponent" in our regularity result [GW].

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