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The Semigroup Approach to Non-Linear Age-Structured Equations

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1. Introduction

This paper is concerned with some non-linear evolution problems motivated by age-structured and size-structured population dynamics. Namely we will consider the problem:

$$\begin{cases} \frac{\partial}{\partial t} l(a,t) + \frac{\partial}{\partial a}\varphi(l(a,t)) + \mu_0(a) l(a,t) + D(l(\cdot,t)) = 0 \\ \varphi(l(0,t)) = E(l(\cdot,t)) \\ l(a,0) = l_0(a) \end{cases} \tag{1.1}$$

where $t \geq 0$, $a \in [0, a_\dagger]$ and $D(\cdot)$, $E(\cdot)$ are operators in $L^1(0, a_\dagger)$. Here the main feature is the presence of the non-linear function φ and of the non-local boundary condition at $a = 0$.

Moreover we will consider:

$$\begin{cases} \frac{\partial}{\partial t} l(a,t) + \frac{\partial}{\partial a} l(a,t) + D(l(\cdot,t)) + B l(a,t) \ni 0 \\ l(0,t) = E(l(\cdot,t)) \\ l(a,0) = l_0(a) \end{cases} \tag{1.2}$$

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where $l(a,t)$ belongs to a banach space Y and the attention is focused on the presence of the non-linear multivalued operator $B : Y \to 2^Y$.

Though these two problems are motivated in the context of population dynamics, we are here concerned with the mathematical treatment by the method of non-linear semigroup theory, more than with modeling aspects. However problem (1.1) may occur when dealing with size structured growth while problem (1.2) is related to non-linear diffusion of a structured population. References to these problems can be found in [5], [6], [7]; our methods are rather related to those in [7], within the context of the theory of nonlinear semigroups (see [1]–[4]). Possible examples of (1.1) and (1.2) are the following

$$\begin{cases} \dfrac{\partial}{\partial t} l(a,t) + \dfrac{\partial}{\partial a}\varphi(l(a,t)) + \mu_0(a) l(a,t) + \mu(a,P(t)) l(a,t)) = 0 \\[2mm] l(0,t) = \displaystyle\int_0^{a_\dagger} \beta(a,P(t)) l(a,t)\, da \\[2mm] l(a,0) = l_0(a) \end{cases} \tag{1.3}$$

where $P(t) = \displaystyle\int_0^{a_\dagger} l(a,t)\,da$,

$$\begin{cases} \dfrac{\partial}{\partial t} l(a,t,x) + \dfrac{\partial}{\partial a} l(a,t,x) \\ \qquad\qquad + \mu(a,x,P(t,x)) l(a,t,x)) - \Delta\varphi(l(a,t,x)) = 0 \\[2mm] l(0,t,x) = \displaystyle\int_0^{a_\dagger} \beta(a,x,P(t,x)) l(a,t,x)\, da \\[2mm] l(a,0,x) = l_0(a,x) \end{cases} \tag{1.4}$$

where $P(t,x) = \displaystyle\int_0^{a_\dagger} l(a,t,x)\,da$, $x \in \Omega \subset \mathbf{R}^n$ and some significant conditions on the boundary $\partial\Omega$ must be added.

2. Existence for problem (1.1)

Here we shall study (1.1) under the following assumptions:

$$\begin{cases} \varphi : \mathbf{R} \to \mathbf{R} \quad \textit{is continuous, increasing and such that} \\ \varphi(0) = 0, \quad \lim_{|x| \to +\infty} |\varphi(x)| = +\infty \end{cases} \tag{2.1}$$

$$\mu_0(\cdot) \in L^1_{loc}(0, a_\dagger) \quad \textit{is non-negative and} \quad \int_0^{a_\dagger} \mu_0(a)da = +\infty \tag{2.2}$$

$$\begin{cases} D : L^1(0, a_\dagger) \to L^1(0, a_\dagger) \quad \textit{is locally Lipschitz continuous,} \\ D(0) = 0, \quad \textit{and the following inequality is satisfied} \\ \int_0^{a_\dagger} D(l)(a) \ \operatorname{sign} l(a)da \geq -\delta |l|_{L^1(0,a_\dagger)} \quad \forall l \in L^1(0, a_\dagger) \end{cases} \tag{2.3}$$

$$\begin{cases} E : L^1(0, a_\dagger) \to \mathbf{R} \quad \textit{is locally Lipschitz continuous, and} \\ |E(l)| \leq C|l|_{L^1(0,a_\dagger)} \quad \forall l \in L^1(0, a_\dagger) \end{cases} \tag{2.4}$$

Actually, instead of (2.3) and (2.4), we will first consider the more restrictive assumptions:

$$D : L^1(0, a_\dagger) \to L^1(0, a_\dagger) \quad \textit{is Lipschitz continuous and} \quad D(0) = 0 \tag{2.5}$$

$$E : L^1(0, a_\dagger) \to \mathbf{R} \quad \textit{is Lipschitz continuous and} \quad E(0) = 0 \tag{2.6}$$

We first write (1.1) as an abstract Cauchy problem in the Banach space $X = L^1(0, a_\dagger)$:

$$\begin{cases} \frac{d}{dt} l(t) + Al(t) + Dl(t) = 0 \\ l(0) = l_0 \end{cases} \tag{2.7}$$

where $A : D(A) \subset X \to X$ is defined as follows:

$$\begin{cases} D(A) = \left\{ f \in X \mid \frac{d}{da}\varphi(f(\cdot)) + \mu_0(\cdot) f(\cdot) \in X \; ; \; \varphi(f(0)) = E(f) \right\} \\ Af = \frac{d}{da}\varphi(f(a)) + \mu_0(a) f(a) \end{cases} \tag{2.8}$$

We have:

PROPOSITION 2.1. *Let (2.1), (2.2), (2.5), (2.6) be satisfied. Then the operator $A : D(A) \subset X \to X$ is ω-m-accretive for some $\omega > 0$.*

Here we recall that the operator A is said to be ω-m-accretive if

$$\text{Range } (\lambda I + A) = X \quad \textit{for some} \quad \lambda > \omega \tag{2.9}$$

$$|\lambda(x_1 - x_2) + Ax_1 - Ax_2| \geq (\lambda - \omega)|x_1 - x_2| \quad \forall \lambda > \omega \tag{2.10}$$

Proof of Proposition 2.1. Let $\lambda > 0$, $g \in X$ and consider the equation

$$\lambda u + Au = g \tag{2.11}$$

i.e. the problem:

$$\begin{cases} \lambda u(a) + \frac{d}{da}\varphi(u(a)) + \mu_0(a) u(a) = g(a) \quad ; \quad a \in [0, a_+) \\ \varphi(u)(0) = E(u) \end{cases} \tag{2.12}$$

To solve this we first consider the auxiliary problem:

$$\begin{cases} \lambda v(a) + \dfrac{d}{da}\varphi(v(a)) + \mu_0(a)v(a) = g(a) \quad ; \quad a \in [0, a_+) \\ v(0) = \psi(x) \end{cases} \tag{2.13}$$

where $x \in \mathbf{R}$ and $\psi = \varphi^{-1}$ (defined on $\mathbf{R}$). Setting $w(a) = \varphi(v(a))$ we transform (2.13) into

$$\begin{cases} \dfrac{d}{da}w(a) = -(\lambda + \mu_0(a))\psi(w(a)) + g(a) \\ w(0) = x \end{cases} \tag{2.14}$$

Since ψ is an increasing function on $\mathbf{R}$, problem (2.14) has a unique maximal solution w which is continuous on $[0, a_\dagger)$ and such that

$$|w(a)| \leq |x| + |g|_{L^1(0,a_+)} \quad , \quad \frac{dw}{da} \in L^1_{loc}(0, a_\dagger) \tag{2.15}$$

Thus if we set $v(a) = \psi(w(a))$, $a \in [0, a_\dagger)$, we get a solution to (2.13), on the interval $[0, a_\dagger)$, such that $v \in D(A)$.

Now we recall that if f and g are absolutely continuous in $[\alpha, \beta]$ we have:

$$\int_\alpha^\beta [f'(a) - g'(a)]\,\mathrm{sign}[\psi(f(a)) - \psi(g(a))]da \geq -|f(\alpha) - g(\alpha)| \tag{2.16}$$

so that, from (2.14) we get

$$|v - \bar{v}|_{L^1(0,a_\dagger)} \leq \frac{1}{\lambda}|x - \bar{x}| \tag{2.17}$$

where v and $\bar{v}$ are the solutions to (2.13) with respective initial data x and $\bar{x}$.

If we now define the mapping $\Gamma : (-\infty, +\infty) \to L^1(0, a_\dagger)$ by setting:

$$(\Gamma x)(a) = v(a) \tag{2.18}$$

where $v(a)$ is the solution to (2.13) with datum x, our final goal of solving problem (2.12) is equivalent to solving the following equation in $\mathbf{R}$:

$$x = E(\Gamma x) \tag{2.19}$$

Since, by (2.17)

$$|E(\Gamma x) - E(\Gamma \bar{x})| \leq \frac{|E|_{Lip}}{\lambda}|x - \bar{x}| \tag{2.20}$$

equation (2.19) has a unique solution $x^* \in \mathbf{R}$ so that $\Gamma x^* \in D(A)$ is the unique solution to (2.11).

Thus equation (2.11) has a unique solution $u \in D(A)$ for any $g \in X$; if moreover we consider $g, \bar{g} \in X$ and call $u, \bar{u}$ the respective solutions of (2.11), using again (2.16) we get

$$|u - \bar{u}|_{L^1(0,a_\dagger)} \leq \frac{1}{\lambda - |E|_{Lip}}|g - \bar{g}| \tag{2.21}$$

thus proving that the operator A is ω-m-accretive with $\omega = |E|_{Lip}$.
□

We note that, defining

$$X_+ = \{f \in L^1(0, a_\dagger) \mid \quad f(a) \geq 0 \quad a.e. \ in \quad (0, a_\dagger)\} \tag{2.22}$$

and considering the assumption

$$E(X_+) \subset [0, +\infty) \tag{2.23}$$

from the proof of the previous proposition we can derive the following additional result

COROLLARY 2.1. *Let* (2.23) *be satisfied, then, under the assumptions of proposition 2.1, if* $g \in X_+$ *we have* $(\lambda + A)^{-1}g \in X_+$.

□

The proof is immediate by noticing that in (2.14), if $x \geq 0$ and g is non-negative a.e. in $[0, a_\dagger)$, then the solution w is also non-negative.

Our next result concerns the density of the domain $D(A)$:

PROPOSITION 2.2. *Let (2.1), (2.2), (2.5), (2.6) be satisfied. Then* $\overline{D(A)} = X$.

Proof. Let $f \in L^1(0, a_\dagger)$ and consider the sequence $f_n \in C^\infty([0, a_\dagger))$ such that $f_n \to f$ in X. Let $\rho \in C^\infty(\mathbf{R}^+)$ be such that $\rho(r) = 0$ for $0 \leq r \leq \frac{1}{2}$, and $\rho(r) = 1$ for $r \geq 1$. Then take R such that $R > \left(1 + |f_n|_{L^1(0,a_\dagger)}\right), \forall n$, and consider the mapping $T : B(0, R) \to B(0, R)$ defined as

$$(Tg)(a) = \rho(na) f_n(a) + (1 - \rho(na))\psi(E(g)) \qquad (2.24)$$

where $n > \frac{1}{R}\psi(|E(0)| + |E|_{Lip}R)$ and $B(0, R)$ is the ball in X, with center 0 and radius R.

For any fixed n, T is a compact mapping so that it has a fixed point $g_n \in B(0, R)$, i.e. we have a sequence $\{g_n\} \subset B(0, R)$ such that

$$g_n(a) = \rho(na) f_n(a) + (1 - \rho(na))\psi(E(g_n)) \qquad (2.25)$$

It is easy to see that $g_n \in D(A)$ and that

$$|f_n - g_n|_{L^1(0,a_\dagger)} \leq \int_0^{\frac{1}{n}} |f_n(a)| da + \frac{1}{n}\psi(|E(0)| + |E|_{Lip}R) \qquad (2.26)$$

so that

$$\lim_{n \to +\infty} g_n = \lim_{n \to +\infty} f_n = f$$

and the thesis is proved. □

The previous results show that the operator $A : D(A) \subset X \to X$ is the generator of a nonlinear quasi-contractive semigroup on the space X. This semigroup is generated in the sense of Crandall and Ligget [4] and, since the mapping $D(\cdot)$ is Lipschitz continuous, the Cauchy problem (2.7) has a solution in the following generalized sense

DEFINITION 2.1. *$l(t)$ is a generalized solution to* (2.7), *if there exist $l_\epsilon(t)$ (with $\epsilon > 0$) such that*

i) $l(t) = \lim_{\epsilon \to 0} l_\epsilon(t)$ *in* $C([0,T],X)$ *for any* $T > 0$;

ii) $\forall \epsilon > 0,$ $l_\epsilon(t)$ *is the solution of the difference equation*

$$\begin{cases} \frac{1}{\epsilon}[l_\epsilon(t) - l_\epsilon(t-\epsilon)] + Al_\epsilon(t) + Dl_\epsilon(t) = 0 \quad \forall t \geq \epsilon \\ l_\epsilon(t) = l_0 \quad \forall t < \epsilon \end{cases} \tag{2.27}$$

We also see that, by Corollary 2.1, with the further assumption:

$$](\lambda + D)^{-1}[(X_+) \subset X_+ \tag{2.28}$$

the solution of (2.7) belongs to X_+ if $l_0 \in X_+$.

We summarize the results in the following

PROPOSITION 2.3. *Under the assumptions* (2.1), (2.2), (2.5), (2.6), *for each $l_0 \in X$ problem* (2.7) *has a generalized solution $l(t)$ in the sense of Definition 2.1. Moreover, if $l_0 \in X_+$ and* (2.23), (2.28) *hold, then $l(t) \in X_+$ for all $t \geq 0$.*

□

We now consider the general case in which the mappings D and E satisfy the assumptions (2.3) and (2.4).

For each $N > 0$ we define the Lipschitz continuous operators $D_N : X \to X$ and $E_N : X \to \mathbf{R}$

$$D_N(f) = \begin{cases} D(f) & \text{if} \quad |f|_X \leq N \\ D\left(\frac{N}{|f|_X} f\right) & \text{if} \quad |f|_X > N \end{cases} \tag{2.29}$$

$$E_N(f) = \begin{cases} E(f) & \text{if} \quad |f|_X \leq N \\ E\left(\frac{N}{|f|_X} f\right) & \text{if} \quad |f|_X > N \end{cases} \tag{2.30}$$

and finally we denote by $A_N : D(A_N) \subset X \to X$ the corresponding operator (2.8).

Now, by Theorem 2.3 the problem

$$\begin{cases} \dfrac{d}{dt}l(t) + A_N l(t) + D_N l(t) = 0 \\ \\ l(0) = l_0 \end{cases} \tag{2.31}$$

has a unique generalized solution $l_N(t) \in C(\mathbf{R}^+; X)$. In other words

$$l_N(t) = \lim_{\epsilon \to 0^+} l_\epsilon^N(t), \quad \mathit{in} \quad C([0,T]; X), \quad \mathit{for\ any} \quad T > 0$$

where $l_\epsilon^N(t)$ satisfies

$$\begin{cases} \dfrac{1}{\epsilon}[l_\epsilon^N(t) - l_\epsilon^N(t-\epsilon)] + A_N l_\epsilon^N(t) + D_N l_\epsilon^N(t) = 0 \quad \forall t \geq \epsilon \\ \\ l_\epsilon^N(t) = l_0 \quad \forall t < \epsilon \end{cases} \tag{2.32}$$

By (2.3) and (2.4), (2.32) yields

$$\begin{aligned} \frac{1}{\epsilon}|l_\epsilon^N(t)|_X &\leq \frac{1}{\epsilon}|l_\epsilon^N(t-\epsilon)|_X + |E(l_\epsilon^N(t))|_X + \delta|l_\epsilon^N(t)|_X \\ &\leq \frac{1}{\epsilon}|l_\epsilon^N(t-\epsilon)|_X + (C+\delta)|l_\epsilon^N(t)|_X \end{aligned} \tag{2.33}$$

so that we have

$$|l_\epsilon^N(t)|_X \leq \frac{1}{1-(\delta+C)\epsilon}|l_\epsilon^N(t-\epsilon)|_X \qquad \forall \epsilon > 0 \tag{2.34}$$

and consequently

$$|l_\epsilon^N(t)|_X \leq \epsilon^{(C+\delta)t}|l_0(t)|_X \qquad \forall t \geq 0 \tag{2.35}$$

The previous estimate allows to solve our main problem; in fact we fix any l_0 and $T > 0$, then we choose N sufficiently large so that

$$\epsilon^{(C+\delta)T}|l_0(t)|_X \leq N \tag{2.36}$$

consequently $l(t) = l_N(t)$ is a generalized solution of the original problem.

We summarize these last results in the following

THEOREM 2.1. *Under the assumptions* (2.1)–(2.4), *for each* $l_0 \in X$ *problem* (2.7) *has a generalized solution in the sense of Definition 2.1. Moreover, if* $l_0 \in X_+$ *and* (2.23), (2.28) *hold, then* $l(t) \in X_+$ *for all* $t \geq 0$.

□

3. Existence for problem (1.2)

As mentioned earlier we now consider a real Banach space Y with the norm denoted $|\cdot|_Y$. We shall denote by X the space $L^1(0, a_\dagger; Y)$ endowed with the usual norm $|\cdot|_X$. We shall study problem (1.2) under the following assumptions on operators B, D and E.

$$B : Y \to 2^Y \quad \textit{is m-accretive}, \quad \overline{D(B)} = Y, \quad 0 \in B0 \tag{3.1}$$

$$\begin{cases} D : X \to X \quad \textit{is locally Lipschitz}, \quad D(0) = 0 \quad \textit{and} \\ (\lambda - C)|u|_X \leq |\lambda u - D(u)|_X \quad \forall u \in Y,\ \lambda > 0 \end{cases} \tag{3.2}$$

$$\begin{cases} E : X \to Y, \quad \textit{is locally Lipschitz}, \quad E(0) = 0 \quad \textit{and} \\ |E(f)|_Y \leq \alpha |f|_X \quad \forall f \in X \end{cases} \tag{3.3}$$

Proceeding as in the previous case we shall study first problem (1.2) under the hypothesis

$$D \in Lip(X, X) \quad , \quad D(0) = 0 \tag{3.4}$$

$$E \in Lip(X, Y) \quad , \quad E(0) = 0 \tag{3.5}$$

In order to formulate our problem in an appropriate abstract form we have to consider the following initial value problem in Y

$$\begin{cases} \dfrac{d}{da}l(a) + Bl(a) \ni g(a), \quad a \in [0, a_\dagger) \\ \\ l(0) = l_0 \end{cases} \tag{3.6}$$

where $l_0 \in Y$ and $g(\cdot) \in L^1((0, a_\dagger); Y)$. We recall the following result (see [2])

THEOREM 3.1. *Let $B : D(B) \subset Y \to Y$ satisfy (3.1). Then, for any $g \in X$, problem (3.6) has a unique mild (integral) solution $l(\cdot)$ in the sense that it is continuous, $l(0) = l_0$ and it satisfies:*

$$|l(a) - u|_Y \leq |l(s) - u|_Y + \int_s^a [g(\sigma) - v, l(\sigma) - u]_* d\sigma \tag{3.7}$$

for all $u \in D(B)$, $v \in Bu$, $0 < s \leq a < +\infty$, where

$$[x, y]_* = \lim_{\lambda \to 0^+} \frac{1}{\lambda} \left(|x + \lambda y|_Y - |x|_Y \right)$$

□

Now we define the operator $A : D(A) \subset X \to X$:

$$\begin{cases} D(A) = \Big\{ f \in C([0, a_\dagger]; Y), \\ \qquad\qquad f \quad \text{satisfies (3.7) for some} \ \ g \in X, f(0) = E(f) \Big\} \\ \\ Af = g \end{cases} \tag{3.8}$$

so that problem (1.2) has the following abstract formulation

$$\begin{cases} \dfrac{d}{dt}l(t) + Al(t) + Dl(t) \ni 0 \\ \\ l(0) = l_0 \end{cases} \tag{3.9}$$

Then we have the following

PROPOSITION 3.1. *Let (3.1), (3.4), (3.5) be satisfied, then the operator $A : D(A) \subset X \to X$ is ω-m-accretive with $\omega = |E|_{Lip}$.*

Proof. Let $\lambda > 0$ and $f \in X$. Since the operator

$$(\lambda I + B) : D(B) \subset Y \to Y$$

is m-accretive, the problem

$$\begin{cases} \lambda l(a) + \dfrac{d}{da} l(a) + Bl(a) \ni f(a), \quad a \in [0, a_\dagger) \\ \\ l(0) = l_0 \end{cases} \tag{3.10}$$

has a unique mild solution for any datum l_0. Moreover, if l and $\bar{l}$ are two solutions with respective data $(l_0,\, f)$ and $(\bar{l}_0,\, \bar{f})$, we also have

$$|l - \bar{l}|_X \leq \frac{1}{\lambda}\left(|l_0 - \bar{l}_0|_Y + |f - \bar{f}|_X\right) \tag{3.11}$$

Thus if we define the mapping $\Gamma : Y \to X$, by setting $(\Gamma x)(a) = l(a)$ where $l(a)$ is the solution to (3.10), with datum x, we have that, for $\lambda > |E|_{Lip}$, the equation

$$x = E(\Gamma x) \tag{3.12}$$

has a unique solution x^*, such that Γx^* is the (unique) mild solution to the problem (3.10) satisfying the condition $l(0) = E(l(\cdot))$.
Thus, since Γx^* satisfies (3.7) with $g = f - \lambda \Gamma x^*$, it belongs to $D(A)$ so that (2.9) is satisfied. Moreover (2.10) follows from (3.11) and the thesis is proved. □

From this point on we can proceed like in the study of problem (1.1) in the previous section, both concerning the special case of the assumptions (3.4), (3.5) and the general case of (3.2), (3.3). Moreover defining the set

$$\mathcal{K} = \{f \in X; f(a) \in K \quad a.\ e. \quad a \in [0, a_\dagger)\}$$

where K is a closed convex cone of Y, we can consider the following additional assumptions

$$(\lambda I + B)^{-1} K \subset K \qquad \forall \lambda > 0 \tag{3.13}$$

$$](\lambda + D)^{-1}[(\mathcal{K}) \subset \mathcal{K} \quad , \quad E(\mathcal{K}) \subset K \tag{3.14}$$

and have solutions in $\mathcal{K}$. The final results are summarized in the following

THEOREM 3.2. *Under the assumptions* (3.1), (3.2), (3.3), *for each* $l_0 \in X$ *problem* (3.9) *has a generalized solution in the sense of Definition 2.1. Moreover, if* $l_0 \in \mathcal{K}$ *and* (3.13), (3.14) *hold, then* $l(t) \in \mathcal{K}$ *for all* $t \geq 0$.

□

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