Almost Periodic Problems and a Property by Eric Séré

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Sommario. - *Si dimostra l'esistenza di infinite soluzioni per un problema di Neumann omogeneo quando il termine non lineare é almost periodico. Il risultato estende quanto é noto nel caso in cui la nonlinearità é periodica e la molteplicità si ottiene banalmente tramite traslazione. L'argomento é variazionale e si basa su una proprietà introdotta da E. Séré.*

Summary. - *We prove the existence of infinitely many solutions for a homogeneous Neumann problem where the nonlinear term is an almost periodic function. This result is an extension of the case where the nonlinearity is periodic and multiple solutions are trivially given by translations. The argument is variational and is based on a property developed by E. Séré.*

0. Introduction and statement of the main result

In this paper we describe a method which can be used to prove existence of multiple solutions to variational differential equations where the nonlinear terms are almost periodic functions. The method is quite simple and applies to cases which generalize problems with periodic nonlinearities.

Definition 0.1. Let be $g : \mathbf{R} \to \mathbf{R}$ a continuous function. Given $\varepsilon > 0$, a number $T \in \mathbf{R}$ is called an ε–period of g if

$$\sup_{t\in\mathbf{R}} |g(t+T) - g(t)| \ \leq \ \varepsilon$$

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A function g is called almost periodic if, for every $\varepsilon > 0$, there exists $\lambda_\varepsilon > 0$ such that each interval of length λ_ε contains at least one ε–period of g.

All periodic functions are also almost periodic (one can even allow $\varepsilon = 0$ in definition 0.1). The simplest almost periodic, but not periodic function is thus the sum of two periodic functions with periods T and S and $\frac{S}{T} \notin \mathbf{Q}$. For more interesting examples of almost periodic functions, see e.g. [**Be**].

To illustrate how our method works, we look for solutions of the homogeneous Neumann problem

$$(P) \qquad \begin{cases} -\Delta u(x) \;=\; g(u(x)) + h(x) & x \in \Omega \\ \frac{\partial u}{\partial n}(x) \;=\; 0 & x \in \partial\Omega \end{cases}$$

where Ω is a smooth open and bounded subset of $\mathbf{R}^N$, $N \geq 1$, and $\frac{\partial u}{\partial n}$ is the normal derivative of u with respect to the boundary of Ω; $g : \mathbf{R} \to \mathbf{R}$ and $h : \Omega \to \mathbf{R}$ are smooth functions (here we are not interested in regularity questions). Let $G(x) = \int_0^x g(t)\, dt$ be a primitive of g. Our main result is the following

THEOREM 0.1. *Assume that*

- *(H1)* $\quad$ *g and G are almost periodic functions,*
- *(H2)* $\quad$ $\int_\Omega h(x)\, dx = 0.$

Then problem (P) admits infinitely many solutions.

Some comments are in order about the conclusions of Theorem 0.1. This result is nontrivial only when the function g is almost periodic but not periodic. Indeed, in the periodic case, whenever one has a solution u, one can construct infinitely many others simply by considering $u_k = u + kT$, for all $k \in \mathbf{Z}$, where T is the period of g. These solutions cannot be considered geometrically distinct since the all differ from each other by multiples of T. When g is almost periodic but not periodic this is no longer true, for tranlations of a solution do not solve the equation.

To prove theorem 0.1 we use variational arguments, that is, we think of (weak) solutions to Problem (P) as critical points of the smooth functional $f : H^1(\Omega) \to \mathbf{R}$ defined by

$$f(u) \;=\; \frac{1}{2}\int_\Omega |\nabla u(x)|^2 dx - \int_\Omega G(u(x))\, dx - \int_\Omega h(x)u(x)\, dx$$

Let us sketch a possible argument for the proof of Theorem 0.2.

Note that, due to (H2)

$$f(u) \;\geq\; C_1\int_\Omega |\nabla u(x)|^2 dx - C_2 \qquad \forall u \in H^1(\Omega)$$

for some suitable constants positive constants C_1, C_2.

Since the functional f is bounded from below in $H^1(\Omega)$, we consider a minizing sequence u_n:

$$f(u_n) \to c := \inf_{H^1(\Omega)} f$$

Assume that u_n is bounded in $H^1(\Omega)$. Then we can extract a subsequence u_{n_k} such that:

$$u_{n_k} \rightharpoonup u \qquad \text{in} \quad H^1(\Omega)$$

Due to the compactness of the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$, the functional f is weakly lower semicontinuous, and u is shown to be an absolute minimum point, so that at least one critical point is found. If u is not a strict minimum point we have infinitely many solutions. Otherwise there is an open bounded set B containing u and such that $f(u) < \inf_{\partial B} f$; then we translate B by a suitable ε-period of G so that the strict inequality is preserved, and we minimize f on this translated set. The same argument yields a new (local) minimizer. Iterating this procedure the theorem follows.

From the above argument we see that the proof of the theorem is trivial if minimizing sequences for f are bounded. However simple examples (see Remark 1.2 below) show that this need not be the case, so that the procedure has to be modified in order to face a possible lack of compactness. This is accomplished by means of some ideas introduced in [**CZES**], [**STT**].

More precisely, as one immediately sees, if u_n is a minimizing sequence for f, then ∇u_n is bounded in L^2. Therefore the only loss of compactness of the minimizing sequences is due to the possible unboundedness of the mean values $\overline{u}_n$. This fact is not a problem in the case of periodic nonlinearities, since one can normalize u_n by requiring that the mean values $\overline{u}_n$ lie in the interval $[0, T]$; this normalization yields a new minimizing sequence which is bounded.

If G is *not* periodic this is no longer true, unless the sequence of mean values of u_n is close to a sequence of ε_n–periods of G, with $\varepsilon_n \to 0$. The main part of the paper is thus devoted to show that it is not restrictive to assume that this property is satisfied, at the price of losing some information on the levels.

The main result leading to the proof of Theorem 0.2 is the following.

THEOREM 0.2. *Let f be as above, and let $c = \inf_{H^1(\Omega)} f$. Then for every $\varepsilon > 0$, f has a critical value $c_\varepsilon \in [c, c+\varepsilon)$.*

In order to prove this result, we need to make slight changes in the previous arguments.

Since possibly $c_\varepsilon \neq c$, weak lower semicontinuity of the functional f cannot be used to prove that c_ε is a critical value. Nevertheless, once more due to the compactness of the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$, not only f is weakly lower semicontinuous, but it satisfies the PS condition on bounded subsets of $H^1(\Omega)$ (see the proof of Lemma 2.3). Since we will be able to contruct a bounded PS sequence at the level c_ε (see Theorem 1.1 and the proof of Theorem 0.3), at least one critical point is found. Finally, to obtain the multiplicity result we can argue as we did above.

In Section 1 we prove an abstract result related to Theorem 0.3. Section 2 is devoted to the proofs of Theorem 0.3 and of the main result Theorem 0.2, while in Section 3 we add some remarks and comments.

NOTATIONS. $H := H^1(\Omega)$ denotes the Sobolev space of L^2 real valued function on Ω whose distributional derivative is (represented by) an L^2 function. This is an Hilbert space which we endow with the norm $||u||^2 = \int_\Omega |\nabla u(x)|^2 dx + |\overline{u}|^2$, where $\overline{u} = |\Omega|^{-1} \int_\Omega u(x) dx$ denotes the mean value of $u \in H$ ($|\Omega|$ is the finite Lebesgue measure of Ω).

By "$\cdot$" we will denote both the scalar product in $\mathbf{R}^N$ and in H. The context will always rule out possible ambiguities.

1. An abstract result

In this section we restate a result essentially due to E. Séré (see [**CZES**], [**STT**]) in the form which will apply to Problem (P). The proof of Theorem 1.1 below is given only for the convenience of the reader, since nearly all the arguments used can be found in the works [**CZES**], [**STT**].

In this section, $(H, ||\cdot||)$ will denote a real Hilbert space, and $f : H \to \mathbf{R}$ will be a smooth functional.

THEOREM 1.1. *Assume that $c := \inf f > -\infty$. Then for every $\varepsilon > 0$ there exists a sequence u_n in H and a real number $c_\varepsilon \in [c, c+\varepsilon)$ such that*

$$f(u_n) \to c_\varepsilon$$
$$\nabla f(u_n) \to 0$$
$$||u_n - u_{n-1}|| \to 0$$

Proof. Let $\eta(u,t)$ be the global flow on H induced by the field $-\nabla f/(1+||\nabla f||)$, that is, the unique global solution of the Cauchy problem

$$\begin{cases} \frac{\partial\eta}{\partial t}(u,t) = \frac{-\nabla f(\eta(u,t))}{1+||\nabla f(\eta(u,t))||} \\ \\ \eta(u,0) = u \end{cases}$$

Given $\varepsilon > 0$, choose $u \in H$ such that $f(u) < c+\varepsilon$, and define $\varphi(t) = f(\eta(u,t))$. By definition φ is a non increasing, smooth function which is bounded from below by the value c. Therefore it has a finite limit when $t \to +\infty$: call c_ε this limit. Choose now a sequence of real numbers s_n such that

$$s_n \to +\infty \qquad\qquad |s_n - s_{n-1}| \to 0.$$

Applying Ekeland's variational principle to φ yields a sequence t_n such that

$$|t_n - s_n| \to 0 \qquad \varphi(t_n) \to c_\varepsilon \qquad \varphi'(t_n) \to 0$$

Now set $u_n = \eta(u,t_n)$. Since $\eta(u,\cdot)$ is a Lipschitz map with Lipschitz constant 1, we have

$$||u_n - u_{n-1}|| = ||\eta(u,t_n) - \eta(u,t_{n-1})|| \leq |t_n - t_{n-1}| \to 0$$

Furthermore

$$o(1) = \varphi'(t_n) = -\frac{||\nabla f(u_n)||^2}{1+||\nabla f(u_n)||}$$

that is $||\nabla f(u_n)|| \to 0$, and the proof is complete. ◇

With the terminology of [**CZES**], we can say that for all $\varepsilon > 0$ there exists a $\overline{PS}$ sequence for f at some level between c and $c+\varepsilon$. It is also quickly seen that such an estimate on the level cannot be improved, in the sense that there need not exist any $\overline{PS}$ sequences at level c, as the following remark shows.

REMARK 1.1. Let $H = \mathbf{R}$ and $f(u) = \cos(u)+\cos(\pi u)$; then $\inf_{\mathbf{R}} f = -2$, and it is not attained. Let u_n be a minimizing sequence for f, so that $f(u_n) \to -2$; (u_n) is unbounded, since f is a continuous map strictly greater than -2. Clearly $\cos(u_n) \to -1$ and $\cos(\pi u_n) \to -1$, which implies $\mathrm{dist}(u_n, (2\mathbf{Z}+1)\pi) \to 0$ and $\mathrm{dist}(u_n, 2\mathbf{Z}+1) \to 0$. It follows that $|u_n - u_{n-1}| \to \infty$ along a subsequence, that is, u_n cannot be a $\overline{PS}$ sequence.

2. Proofs of Theorem 0.2 and Theorem 0.3

We begin by stating two simple properties concerning the function G and the behavior of the functional f when it is evaluated at translations of a given function.

LEMMA 2.1. *If (H1) holds, there exist real numbers $T_k \to +\infty$ such that*

$$\sup_{t\in\mathbf{R}} |G(t+T_k) - G(t)| + \sup_{t\in\mathbf{R}} |g(t+T_k) - g(t)| \ \to \ 0 \qquad \text{as } k \to \infty.$$

This means that there exist sequences consisting of ε_k–periods ($\varepsilon_k \to 0$) *both* for G and for its derivative. For the proof, see [**Be**].

The next lemma is a direct consequence of almost periodicity. Since its proof is trivial, we omit it.

LEMMA 2.2. *Let T be an ε–period of G and g. Then for every $u \in H$ the following inequalities hold*

$$|f(u+T) - f(u)| \leq \varepsilon|\Omega|$$
$$||\nabla f(u+T) - \nabla f(u)|| \leq \varepsilon|\Omega|^{1/2}$$

Finally, we recall a property which enables one to find local minima for a functional like f.

LEMMA 2.3. *Let B be an open, bounded subset of H such that*

$$\inf_{\overline{B}} f < \inf_{\partial B} f$$

Then f has at least one local minimum in B.

Proof. Note first that the Palais–Smale condition holds for f on bounded subsets of H. Indeed if u_n is a bounded Palais–Smale sequence for f, then it contains some subsequence, still denoted u_n, such that $u_n \rightharpoonup u$ weakly in H, $u_n \to u$ strongly in $L^2(\Omega)$, and $u_n \to u$ almost everywhere. Then, since g is bounded we can pass to the limit in $\nabla f(u_n)$, to obtain that $\nabla f(u) = 0$. At this point we have

$$\begin{aligned}\int_\Omega |\nabla u_n - \nabla u|^2 dx &= \int_\Omega |\nabla u_n - \nabla u|^2 dx - \int_\Omega (g(u_n) - g(u))dx + o(1)\\ &= (\nabla f(u_n) - \nabla f(u)) \cdot (u_n - u) + o(1)\\ &= o(1)\end{aligned}$$

that is, $u_n \to u$ strongly in H.

To complete the proof, pick a Palais–Smale sequence u_n in B such that $f(u_n) \to \inf_B f$: since u_n is bounded, and the PS condition holds on bounded subsets of H we can assume (passing if necessary to a subsequence) that $u_n \to u \in \overline{B}$ strongly in H. By continuity $f(u) = \inf_B f < \inf_{\partial B} f$, which also shows that $u \in B$. ♢

We are now ready to prove the theorems stated in the introduction. We start with Theorem 0.3.

Proof of Theorem 0.3 . The application of Theorem 1.1 yields a $\overline{PS}$ sequence u_n for f at a level $c_\varepsilon \in [c, c+\varepsilon)$, namely a sequence such that

$$f(u_n) \to c_\varepsilon$$
$$\nabla f(u_n) \to 0$$
$$||u_n - u_{n-1}|| \to 0.$$

Moreover ∇u_n is bounded in $L^2(\Omega)$, independently of n. If $\overline{u}_n$ is bounded as well, then u_n contains a subsequence, still denoted u_n, such that $u_n \to u$ strongly in H, since PS holds on bounded subsets of H. By continuity, $\nabla f(u) = 0$ and $f(u) = c_\varepsilon$. Note that up to this point we have not used the fact that $||u_n - u_{n-1}|| \to 0$.
It remains to discuss the case of a $\overline{PS}$ sequence with unbounded mean values. Assume then that u_n is a $\overline{PS}$ sequence and $\overline{u}_n$ is unbounded. Choose a sequence T_k of $\frac{1}{k}$–periods for G and g (this is possible by Lemma 2.1); since $||u_n - u_{n-1}|| \to 0$ implies $|\overline{u}_n - \overline{u}_{n-1}| \to 0$, it is quickly seen that there exists a subsequence u_{n_k} of u_n such that $\text{dist}(\overline{u}_{n_k}, \{T_k, -T_k\}) \to 0$. Passing to another subsequence if necessary, we can assume for definiteness that $|u_{n_k} - T_k| \to 0$ as $k \to \infty$. If now we define $v_k = u_{n_k} - T_k$, Lemma 2.2 proves that

$$\begin{aligned} f(v_k) &= f(u_{n_k}) + o(1) \to c_\varepsilon \\ \nabla f(v_k) &= \nabla f(u_{n_k}) + o(1) \to 0 \qquad \text{as} \quad k \to \infty \end{aligned}$$

Now, however, $\overline{v}_k$ is bounded (actually, $\overline{v}_k \to 0$); therefore v_k is precompact since the PS condition holds on bounded subsets of H and we find a critical point at a level c_ε. ♢

We are now in a position to prove also Theorem 0.2.

Proof of Theorem 0.2. Let $c = \inf_H f$. If the value c is not attained, then the conclusion follows trivially from Theorem 0.3, for in this case we find infinitely many critical levels above c.
Suppose now that there exists $u^* \in H$ such that $f(u^*) = c$. In this case we can also assume that u^* is a strict global minimum point for f, because otherwise the conclusion holds by construction. With the

aid of Lemma 2.3 we now produce infinitely many new local minima for f. First of all, we can find $\delta^* > 0$ such that $0 < \delta < \delta^*$ implies

$$\inf_{||u-u^*||=\delta} f(u) > f(u^*)$$

If not, for every $k > 0$ we can find $0 < \delta_k \leq \frac{1}{k}$ such that

$$\inf_{||u-u^*||=\delta_k} f(u) = c$$

Applying Ekeland's variational principle yields a sequence u_n^k such that, as $n \to +\infty$

$$||u_n^k - u^*|| \to \delta_k, \qquad f(u_n^k) \to c, \qquad \nabla f(u_n^k) \to 0$$

Since PS condition holds on bounded subsets of H, by passing to a subsequence we can assume $u_n^k \to u^k$ in H as $n \to +\infty$. Clearly $||u^k - u^*|| = \delta_k$ and $f(u^k) = c$. Since $\delta_k \to 0$ as $k \to \infty$, this contradicts the assumption that u^* is a strict global minimum point.

Take now a sequence T_n of $\frac{1}{n}$–periods of G and g, ($T_n \to \infty$) and define $\delta_k = \frac{\delta^*}{k}$. Applying Lemma 2.2 yields a subsequence T_{n_k} for which the same strict inequalities hold around $u^* + T_{n_k}$, namely

$$\inf_{||u-(u^*+T_{n_k})||=\delta_k} f(u) > f(u^* + T_{n_k}).$$

The application of Lemma 2.3 in the sets $B_k = \{u \ / \ ||u - (u^* + T_{n_k})|| \leq \delta_k\}$ yields a sequence u_k of local minima of f such that $||u_k - (u^* + T_{n_k})|| \leq \delta_k \to 0$. To conclude the proof, we have to show that the sequence u_k contains infinitely many different points of H. This follows from $||u_k|| = ||u^* + T_{n_k}|| + o(1) \to +\infty$, for $T_{n_k} \to +\infty$ by construction. $\diamondsuit$

3. Further comments

As we already said in the introduction, nearly the same arguments can be used to deal with different problems.

For instance, the method used to solve Problem (P) applies without changes to the search of periodic solutions for a single second order equation with periodic forcing term. More precisely we can state:

THEOREM 3.1. *Let $g, h : \mathbf{R} \to \mathbf{R}$ be continuous. Assume*

- *i) g and G (a primitive of g) are almost periodic;*
- *ii) h is T–periodic and $\int_0^T h(t)dt = 0$.*

Then the problem

$$\begin{cases} -\ddot{u}(t) = g(u(t)) + h(t) \\ u(t+T) = u(t) \qquad \forall t \end{cases}$$

admits infinitely many solutions.

The proof of this result is exactly like that of Theorem 0.2, once one chooses $H = H^1_{T-per}(\mathbf{R};\mathbf{R})$, the Sobolev space of L^2 real valued T–periodic functions whose derivative is (represented by) an L^2 function, and

$$f(u) = \frac{1}{2}\int_0^T |\dot{u}(t)|^2 dt - \int_0^T G(u(t))dt - \int_0^T h(t)u(t)dt$$

Note also that this type of problem is a generalization of the forced simple pendulum, and that no assumption on the norm of h or the period are made.

Another problem which we can face using $\overline{PS}$ sequences, instead of just the PS ones, is looking for non trivial solutions to

$$\begin{cases} -\ddot{u}(t) + u(t) = \alpha(t)\nabla G(u(t)) \\ \lim\limits_{t\to\pm\infty} u(t) = \lim\limits_{t\to\pm\infty} \dot{u}(t) = 0 \end{cases}$$

where $\alpha : \mathbf{R} \to \mathbf{R}$ is a smooth, almost periodic and positive function which is bounded away from zero, and $G : \mathbf{R}^N \to \mathbf{R}$ is a smooth, globally super–quadratic function (see [**STT**]). In this case $H = H^1(\mathbf{R};\mathbf{R}^N)$ and the functional

$$f(u) = \frac{1}{2}\int_{\mathbf{R}} |\dot{u}(t)|^2 dt + \frac{1}{2}\int_{\mathbf{R}} |u(t)|^2 dt - \int_{\mathbf{R}} \alpha(t)G(u(t))dt$$

has a mountain–pass level at a positive level c. Due to the super–quadraticity of G, each PS sequence u_n at the level c is bounded in $H^1(\mathbf{R};\mathbf{R}^N)$. Hence, it is not restrictive to assume that $u_n \rightharpoonup u$ in $H^1(\mathbf{R};\mathbf{R}^N)$. It is easy to show that such a u is a critical point of f, but possibly is the trivial one ($u \equiv 0$).

Now, due to $c > 0$ the PS sequence u_n cannot vanish on $\mathbf{R}$ (see [**STT**], [**CZR**]) and $u \equiv 0$ means that u_n vanishes on compact subsets of $\mathbf{R}$. Roughly speaking, this happens if the "mass" of u_n concentrates at infinity when $n \to +\infty$.

To find a non trivial critical point also in the case $u_n \rightharpoonup 0$, we can try to follow the "mass" of u_n by looking for a sequence of real numbers τ_n such that

$$u_n(\cdot + \tau_n) \text{ is a } PS \text{ sequence}$$

$$|u_n(\tau_n)| \geq \delta > 0$$

Indeed, if we can produce such a sequence, let be $v_n(t) := u_n(t + \tau_n)$. Of course $||v_n|| = ||u_n||$ and, due to the boundeness of u_n, it is not restrictive to assume that $v_n \rightharpoonup v$ in $H^1(\mathbf{R}; \mathbf{R}^N)$. Once more v is a critical point of f (as a consequence of the first one of the above requirements), but now $v \neq 0$ (as a consequence of the second one).

When α is a periodic function, such τ_n can be easily found in the form $\tau_n = k_n T$, where k_n is an integer number and T is the period of α (see [**CZR**]).

In [**STT**] the authors prove that the sequence τ_n can be found also in the case α is an almost periodic functions, as a consequence of the fact that one can work with a $\overline{PS}$ sequence u_n (at a level which is as close to c as we like), instead of just a PS one (at the level c). First of all, a considerable amount of work in [**STT**] is devoted to prove that a choice of τ_n which fulfils the second one of the above requirements can be done in some uniform way, in the sense that

$$||u_n - u_{n-1}|| \to 0 \qquad \text{implies} \qquad |\tau_n - \tau_{n-1}| \to 0$$

Of course τ_n is unbounded (otherwise $u_n \not\rightharpoonup 0$), and if we choose a sequence T_k of ε_k–periods of α, with $\varepsilon_k \to 0$ and $T_k \to +\infty$, there exists a subsequence τ_{n_k} of τ_n such that:

$$\operatorname{dist}(\tau_{n_k}, \{T_k \ , \ -T_k\}) \ \to \ 0$$

i.e. τ_{n_k} is a sequence of ε'_k–periods of α, with $\varepsilon'_k \to 0$. Such a subsequence τ_{n_k} satisfies the first one of the above requirements. For what we said before, at least one non trivial critical point of f is found.

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