An Elliptic-Parabolic Problem in Bingham Fluid Motion

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Sommario. - *Si studia un'equazione ellittico-parabolica che descrive la distribuzione degli sforzi all'interno di un viscosimetro rotante riempito con un fluido di Bingham. Si prova l'esistenza e l'unicità delle soluzioni e alcuni risultati sulla continuità della frontiera solido-liquido.*

Summary. - *In this paper, we consider an elliptic-parabolic equation ruling the stress distribution inside a rotating viscometer filled with a Bingham fluid. Existence and uniqueness of the weak solution and some results concerning the continuity of the boundary of the solid region are proved.*

1. Introduction

A Bingham fluid is a particular type of non-Newtonian fluid where the constitutive relation relating stress and strain is such that, roughly speaking, the continuum behaves like a rigid body for small stress and like a Newtonian viscous fluid for high stress. Moreover this transition between the two different behaviours is sharp, and idealized to happen suddenly when the modulus of the stress equals a

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fixed threshold value. A thoroughly account of the mathematics of Bingham fluids can be found in [5].

Equations of motion for a Bingham fluid are obtained from mass and momentum conservation plus a contitutive equation, in differential form, for the stress tensor. These give a system of partial differential equations. In some cases, assuming some symmetry of the solutions and that the motion is laminar, this system can be reduced to a scalar equation plus the constitutive equation.

Two interesting cases are the longitudinal motion of a Bingham fluid along a pipe or between two parallel slabs under the effect of a pressure gradient, and the tranversal motion between two rotating coaxial cylinders (viscometer) generated by the rotation of one of the two cylinders (typically the internal one, called "bob").

In the last two cases, the slab and the viscometer, the velocity depends only upon one spatial variable. For the slab we have

$$\rho v_t = \sigma_x + p \ , \tag{1}$$

where v is the longitudinal componenent of the velocity, σ is the shear stress and p the pressure gradient. The constitutive law for σ is

$$\sigma = \mu v_x + k\frac{v_x}{|v_x|} \text{ when } |v_x| > 0, \quad |\sigma| \leq k \text{ when } |v_x| = 0, \tag{2}$$

see [2, 11]. Combining (2) with (1), we see that the motion of a Bingham fluid can be either a rigid motion if $v_x = 0$, or the motion of a viscous fluid if $v_x \neq 0$. These two behaviors can coexist, as it can be easily seen computing the stationary solution of (1) between two slabs at rest, under a uniform pressure field: if the pressure gradient is not too strong, the fluid has a rigid core (moving with spatially constant speed) and two transition layers between the core and the slabs where the velocity has the classical parabolic profile of the viscous fluid motion. It is important to notice that v_x is continuous, and this is also the case for the solution of the evolution problem.

Equations (1) and (2) give a scalar parabolic equation of degenerate type, which can be treated either tranforming it into a variational inequality [5, 11], or rewriting the problem, in the region where $|v_x| > 0$ (the fluid zone), as a free boundary problem, [2], the free boundary being the boundary of the region where $|v_x| = 0$ (the solid zone). The free boundary conditions can be derived directly from the equation of motion of the solid zone. The resulting problem is Stefan-like and can be treated in the framework of the classical solutions of one-dimensional free boundary problems for parabolic equations as in [8], see [2].

For the viscometer, after substituting the constitutive law, we have

$$\rho\omega_t = \frac{1}{r^3}\left(\mu r^3\omega_r + kr^2\frac{\omega_r}{|\omega_r|}\right)_r, \tag{3}$$

where now ω is the angular velocity v/r, and r ranges between the radius of the internal cylinder and the radius of the external one. No pressure gradient is present in this case.

This problem can be transformed again into a free boundary problem for an uniformly parabolic equation, see [4]. The free boundary conditions are rather complex, and in general are of non local type. Moreover there exists a big qualitative difference between the behaviour of the solutions of (1) and those of (3). In fact the solutions of (1) have a rather simple structure, reducing in almost all cases to a problem with only one free boundary, where the parabolic equation is satisfied only on one side of the free boundary (like the one-phase Stefan problem). On the contrary, the solutions of (3) can exibit a very complex structure with many solid and fluid regions alternating inside the spatial domain, and this can occur under quite natural boundary conditions (it is enough to vary with time the bob speed in an appropriate way), see [3].

It is then useful to have a more flexible definition of solution in order to obtain global solutions indipendently of the *a priori* knowledge of the free boundaries.

We introduce here a transformation which will give an elliptic-parabolic equation for a new unknown closely related to the stress. This can be done for both (1) and (3).

Let consider equation (1). We introduce a new variable $w(x,t)$ defined by

$$w(x,t) = \begin{cases} \frac{\mu}{\rho}v_x - \frac{k}{\rho} & v_x < 0 \\ [-\frac{k}{\rho}, \frac{k}{\rho}] & v_x = 0 \\ \frac{\mu}{\rho}v_x + \frac{k}{\rho} & v_x > 0 \end{cases}. \tag{4}$$

We can now take the inverse of the monotone graph in (4) expressing

v_x as a function of w by

$$v_x = C(w) = \begin{cases} \dfrac{\rho}{\mu}\left(w + \dfrac{k}{\rho}\right) & w < -\frac{k}{\rho} \\ 0 & w \in \left[-\frac{k}{\rho}, \frac{k}{\rho}\right] \\ \dfrac{\rho}{\mu}\left(w - \dfrac{k}{\rho}\right) & w > \frac{k}{\rho} \end{cases} . \tag{5}$$

Finally, differentiating (1) with respect to x and substituting (5), we obtain an equation for w

$$\frac{\partial}{\partial t} C(w(x,t)) = w_{xx}(x,t), \tag{6}$$

which is an elliptic-parabolic equation, i.e. a partial differential equation which is of parabolic type when $w \notin \left[-\frac{k}{\rho}, \frac{k}{\rho}\right]$ and it is of elliptic type for values of w inside the interval. A quite similar mathematical problem arises from a model describing water filtration in saturated-insaturated soils, which was studied in [6]. The only difference is that now (5), (6) is a “two-phases” elliptic-parabolic problem, since the elliptic zone for the values of w is a bounded interval instead of an half line as in [6]. Another problem which shares some similarity with our problem is that studied in [10] in connection with the theory of superconductors, where the parabolic zone is the interior of an interval and the elliptic one the rest of the real line.

A similar transformation can be performed for the viscometer problem (3)

$$\omega_r = C(u,r) = \begin{cases} \dfrac{\rho}{\mu r^3}\left(u + \dfrac{k}{\rho} r^2\right) & u < -\frac{k}{\rho} r^2 \\ 0 & u \in \left[-\frac{k}{\rho} r^2, \frac{k}{\rho} r^2\right] \\ \dfrac{\rho}{\mu r^3}\left(u - \dfrac{k}{\rho} r^2\right) & u > \frac{k}{\rho} r^2 \end{cases} . \tag{7}$$

The resulting equation for u is now

$$\frac{\partial}{\partial t} C(u(r,t),r) = \left(\frac{1}{r^3} u(r,t)_r\right)_r , \tag{8}$$

which is again an elliptic-parabolic problem. From the mathematical point of view, the major novelty now is the dependence on the space variable of the relation expressing the derivative of the angular velocity ω_r in term of the "stress" u. This is the main reason for the more complex behavior of the solution and it also the source of mathematical difficulties.

Equation (8) has to be complemented with initial and boundary conditions. The initial condition is given for the spatial derivative of the angular velocity, $C(u(x,0)) = v_0(x)$. This is consistent with the assignment of the velocity field in the original problem. Both Dirichlet and Neumann conditions on the lateral boundaries make sense. Giving the value of u on a later boundary means that on this boundary the stress is controlled (this is the case for the interior cylinder in "stress controlled" experiments.) A Neumann condition comes out if the fluid velocity is assigned at the boundary, which is a natural condition, expecially at the exterior cylinder, which is generally kept still (a no-slip condition is assumed for the fluid motion). In fact, if $\omega(r_1,t) = \omega_1(t)$ is a given differentiable function, then the value of $u_r(r_1,t)$ is obtained from (3) differentiating $\omega_1(t)$. Notice that this takes automatically care of the sign of ω_r and works for $\omega_r = 0$ as well.

REMARK. Equation (8) and relation (7) give a simple way to catalogue stationary solutions of the viscometer problem. In fact from (8) it follows that all stationary solution have the form

$$W(x) = a_1 r^4 + a_2, \tag{9}$$

where a_1 and a_2 are arbitrary constant. Comparing (9) with (7) we obtain the "structure" of the stationary solution, i.e. we can determine where the Bingham fluid is actually in the rigid state ("solid") or in the fluid state ("liquid").

Accordingly we can have one of the following situation

1. the viscometer is filled with solid;
2. the viscometer is filled with liquid;
3. a unique solid zone exist near the internal radius, followed by a liquid zone;
4. a liquid zone exist near the internal radius, followed by a solid zone and then by a new liquid zone;

5. a liquid zone exist near the internal radius, followed by a solid zone extending up to the external radius.

However, only the first two cases and the last one are compatible with the choice of the constant $a_1 = 0$, i.e. with constant $W(x) = a_2$. The latter are the solutions which correspond to the condition $w_x = 0$ on both the lateral boundaries, i.e. those which correpond to constant angular velocity at the the bob $(r = r_1)$ and the cup $(r = r_2)$. These are then the "true" stationary solution of the viscometer problem. The velocity profile can be recover from integration of (7), imposing the boundary conditions on the velocity. For instance, imposing $\omega(r_2) = 0$, one obtain a relation between velocity of the bob $\omega(r_1)$ and the constant stress needed to substain this velocity gradient, see [13, 1].

In the two remaining cases, the costant stresses applied on the two lateral boundaries do not equilibrate and the solution exibit a constant incresing (or descresing) angular velocity.

2. Existence and Uniqueness of Weak Solutions

In this section we consider a generalization of the equations described in the introduction.

$$(C(u,x))_t = (a(x)u_x)_x, \quad \text{in } Q_T = [1,2] \times [0,T], \tag{1}$$

where

$$C(u,x) = \begin{cases} c_1(u,x) & u < \nu_1(x) \\ 0 & \nu_1(x) < u < \nu_2(x) \\ c_2(u,x) & \nu_2 < u \end{cases}, \tag{2}$$

c_1 and c_2 are two smooth functions, such that

$$0 < c_m \leq \frac{\partial c_i}{\partial u} \leq c_M, \ i = 1,2, \text{ for any } x\,, \tag{3}$$

$$c_i(\nu_i(x), x) = 0, \ i = 1,2, \tag{4}$$

and

$$\nu_1(x) < \nu_m < 0 < \nu_M < \nu_2(x)\,. \tag{5}$$

We consider the problem of finding a function $u(x,t)$ satisfying (1) (in a weak sense to be defined), the initial condition:

$$C(u(x,0)) = c_0(x), \tag{6}$$

and one of the following boundary conditions:

$$u(i,t) = f_i(t),\ \ i = 1,2\,, t \in [0,T]\,, \tag{7}$$

$$u_x(i,t) = g_i(t),\ \ i = 1,2\,, t \in [0,T]\,. \tag{8}$$

$$u(1,t) = f_1(t),\ \ u_x(2,t) = g_2(t),\ \ t \in [0,T]\,. \tag{9}$$

We assume that the data satisfy the following

$f_i, g_i \in W^1_\infty([0,T]),\ i = 1,2,\ \ c_0 \in W^1_\infty([1,2]),$ (10)
the zero order compatibility conditions (c.c.) (11)
are satisfied for b.c. (7)
the first order compatibility conditions (12)
are satisfied for b.c. (8)
the zero order c.c. at $x = 1$ and the first order c.c. (13)
at $x = 2$ are satisfied for b.c. (9)

In the following we prove that a unique "weak" solution exists for each of the three problems: (1), (6), (7) (which we refer as Dirichlet problem, (DP), in the following); (1), (6), (8) (Neumann problem (NP), in the following); and (1), (6), (9) (Dirichlet-Neumann problem, (DNP) in the following).

The definition of weak solution is quite standard: a function $u(x,t)$ defined a.e. in $\overline{Q_T}$ and such that $C(u(x,t),x) \in C(\overline{Q_T})$ is a weak solution of (DP) if $u - \overline{u} \in L^2(0,T;H^1_0(1,2))$ where $\overline{u}$ is a smooth function taking the values $f_i(t)$ on the boundaries, and u satisfies the identity:

$$\int\!\!\int_{Q_T} \{a(x)u_x\phi_x - C(u,x)\phi_t\}\,dx\,dt = \int_0^1 c_0(x)\phi(x,0)\,dx, \tag{14}$$

for all $\phi \in C^1(\overline{Q_T})$ which vanish on $x = 1, 2$ and on $t = T$. The function $u(x, t)$ is a solution of (NP) if $u \in L^2(0, T; H^1(1, 2))$, and u verifies:

$$\begin{aligned} \iint_{Q_T} \{a(x)u_x\phi_x - C(u, x)\phi_t\}\, dx\, dt \\ = \int_0^1 c_0(x)\phi(x, 0)\, dx + \int_0^T \{a(2)g_2(\tau)\phi(2, \tau) + \\ - a(1)g_1(\tau)\phi(1, \tau)\}\, d\tau \end{aligned} \tag{15}$$

for all $\phi \in C^1(\overline{Q_T})$ which vanish on $t = T$. Finally the function u is a solution of problem (NDP) if: $u \in L^2(0, T; H^1(1, 2))$,the trace of $u - f_1$ on $x = 1$ is zero and u satisfies the following relation:

$$\begin{aligned} \iint_{Q_T} \{a(x)u_x\phi_x - C(u, x)\phi_t\}\, dx\, dt \\ = \int_0^1 c_0(x)\phi(x, 0)\, dx + \int_0^T \{a(2)g_2(\tau)\phi(2, \tau)\}\, d\tau \end{aligned} \tag{16}$$

for all $\phi \in C^1(\overline{Q_T})$ which vanish on $x = 1$ and on $t = T$.

To prove the existence of a solution we follow the scheme settled in [6] for the filtration problem: we start with a standard parabolic regularization of the problem obtained substituting $C(u, x)$ by $C_n(u, x)$ such that: $C_n \in C^\infty(\mathbf{R}^2)$, $M > \partial C_n/\partial u > 1/n$, $|\partial C_n/\partial x| < M$ with M indip. of n, and the sequence $C_n(u, x)$ converges uniformly to $C(u, x)$.

Problems (DP), (NP) and (DNP), with C substituted by C_n, are easily solved using the classical nonlinear parabolic results of [12]. To each of the C_n then it corresponds a u_n which is a solution of the regularized versions of (1), which we denote by (1_n), with appopriate boundary conditions.

In order to pass to the limit, we need some a priori estimates, which we prove only for (DP) and (NP), the case (DNP) being treated similarly.

The first estimate concernes the uniform boundedness of u_n:

In the case of Dirichlet boundary conditions, a standard application of the maximum principle gives $|u_n| < \max_{\partial_p Q_T} |u_n|$. We will prove later the L^∞ estimate for the solutions of (NP).

The second estimate is the uniform boundedness of u_{nx}:

In the case of Neumann boundary conditions, it is sufficient to observe that $z = a(x)u_{nx}$ solves a parabolic equation to which the strong maximum principle applies, giving an uniform *a priori* estimate for $|u_{nx}|$.

In the case of Dirichlet boundary conditions we first need to construct barriers for u_{nx} on the lateral boundary. These can be choosen in the form

$$u_2^{\pm} = \pm A \int_x^2 \frac{\xi}{a(\xi)}\, d\xi + f_2(t). \tag{17}$$

(17) gives respectively a super-solution u_2^+ and a sub-solution u_2^-, for A large enough but independent of n, both equal to f_2 at $x = 2$. This gives a bound for $|u_{nx}(2,t)|$ independent of n.

Similarly, defining

$$u_1^{\pm} = \mp A \int_1^x \frac{\xi}{a(\xi)}\, d\xi \pm B \int_1^x \frac{1}{a(\xi)}\, d\xi + f_1(t) \tag{18}$$

we obtain a super-solution u_1^+ and a sub-solution u_1^-, for A large enough, and B even larger, but both independent of n. u_1^+ and u_1^- are both equal to f_1 at $x = 1$. This gives now a bound for $|u_{nx}(1,t)|$ independent of n. Finally we proceed as in the case of Neumann b.c. Using the technique of [6], (see proof of Lemma(4) there), we can prove that the boundedness of u_{nx} implies that

$$\left| \frac{C_n(u_n(x,t+h),x) - C_n(u_n(x,t),x)}{h^{1/2}} \right| \leq M \tag{19}$$

where M does not depend on n. Moreover M does not depend on the L^∞ norm of u_n. As a consequence we can deduce an a priori estimate for $|u_n|$ also for the Neumann case. Moreover, since C_n tends to infinity if u_n tends to infinity, $\{v_n = C(u_n,.)\}$ is a sequence of uniformly bounded, uniformly Hölder continuous functions.

The above estimates ensure the weak convergence (modulo subsequences) of the sequence $\{u_n - \overline{u}\}$ in the space $L^2(0,T;H_0^1(1,2))$ in the Dirichlet case and of $\{u_n\}$ in $L^2(0,T;H^1(1,2))$ in the Neumann case. Moreover $\{v_n\}$ converges in $C^0(\overline{Q}_T)$. Proceeding like in [6] we can prove that $v = \lim v_n = C(u,.)$ where $u = \lim u_n$ and that u is a solution of (14).

Some more regularity of the solution can be obtained multiplying (1_n) times u_{nt} and integrating by parts.

$$\int\int_{Q_T} \frac{\partial C_n(u_n,x)}{\partial u} u_{nt}^2 + 1/2 \sup_{t\in(0,T)} \int_{Q\times\{t\}} a(x) u_{nx}^2$$
$$\leq M + \left|\int_0^T a(1)u_{nx}(1,t)u_{nt}(1,t)\right| + \left|\int_0^T a(2)u_{nx}(2,t)u_{nt}(2,t)\right|$$

The left hand side of the above inequality is obviously bounded in the case of Dirichlet boundary conditions, while in the case of Neumann b.c. it suffices to integrate by part the integrals inside the absolute values, obtaining

$$\int\int_{Q_T} \frac{\partial C_n(u_n,x)}{\partial u} u_{nt}^2 + \sup_{t\in(0,T)} \int_{Q\times\{t\}} a(x) u_{nx}^2 \leq M. \qquad (20)$$

From (20) we get an uniform bound for the $L^2(Q_T)$-norm of u_{nxx}. Suppose now that u_1 and u_2 are two solutions of one of our problems. Then we can subtract the equations, in weak form, satisfied by u_1 and u_2 obtaining

$$\int\int_{Q_T} \{a(x)(u_{1x}-u_{2x})\phi_x - (C(u_1,x)-C(u_2,x))\phi_t\}\, dx\, dt = 0. \quad (21)$$

Uniqueness of the solution can be proved substituting a suitable test function,

$$\phi(x,t) = \begin{cases} \int_t^{t_1}(u_1(x,s)-u_2(x,s))ds, & 0\leq t\leq t_1 < T, \\ 0, & \text{otherwise}, \end{cases} \qquad (22)$$

where t_1 is a arbitrary time in $(0,T)$, see [6] for the details of the proof. Notice that the only relevant property required to prove uniqueness is the monotonicity of $C(u,x)$ with respect to u. Following the proof in [6] then we have $u_{1x} - u_{2x} = 0$ a.e., from which uniqueness follows for the Dirichlet problem. For the Neumann problem we come back to the equation and we get $(C(u_1,\cdot)-C(u_2,\cdot))_t = 0$, and then $C(u_1,\cdot)-C(u_2,\cdot) = 0$. This gives the uniqueness for the problem in the case of Neumann boundary condition. The stronger form of uniqueness, i.e. $u_1 = u_2$, can be obtained only if the values of the initial datum belong to the parabolic region for an interval, in which case we can reconstruct u starting from $C(u,\cdot)$.

Another consequence of the monotonicity of C is the (weak) comparison principle:

PROPOSITION 1. *If c_{0a}, f_{ia}, c_{0b}, f_{ib} are two sets of admissible initial and boundary data such that $c_{0a} \leq c_{0b}$, $f_{ia} \leq f_{ib}$ then the corresponding solutions u_a and u_b satisfy*

$$C(u_a(x,t),x) \leq C(u_b(x,t),x). \tag{23}$$

3. Regularity of the free boundary

In this section we investigate the qualitative properties of the "free boundary", i.e. the boundary of the set where $C(u(x,t),x) = 0$. Notice that this definition makes sense because of the continuity of $C(u(x,t),x)$. Here are some surprising differences with respect to the case of filtration equation in [7]. In the case of the filtration equation, some sort of strong maximum principle can be proved. This implies that no internal satured zone (which is the counterpart of our solid region) can exist, so that the saturated part is always connected to the lateral boundary. As a consequence, the most general situation at a given time t is the presence of a satured zone, followed by a unique unsatured zone, and finally a new satured one; each of the three can be missing. This reduces the analysis required for proving the regularity of the free boundary essentially to the case of a single free boundary, where the flux (u_x in our notation) is known as a function of the distance of the free boundary from the fixed lateral boundary. This makes relatively easy to find the barrier functions needed to prove the continuity of the free boundary, and the application of the inverse function theorem to prove its extra regularity, as a level set of the pressure function.

In our case none of the previous properties are true anymore. The "free boundary" is the boundary of a level set of $C(u,x)$ and, even assuming the continuity of u, we can have either $u = \nu_1(x)$ or $u = \nu_2(x)$ on the free boundary. Moreover, no strong maximum principle is available for C, and an internal zone with $C \equiv 0$ can exists even when C has the same sign on both sides of this zone.

This makes the task of proving regularity properties a rather difficult task.

Here we present some results about the regularity properties of the free boundary. We denote by $\mathcal{S}$ the region where $C(u(x,t),x) = 0$, which we will call the "solid" in the following. We will also call "fluid" the region $\mathcal{F}$ where $|C| > 0$. The free boundary is the boundary of the region $\mathcal{S}$, i.e. the transition points from solid to fluid.

Let $t > 0$ and denote by $\mathcal{F}(t)$ the set $\{x : |C(u(x,t),x)| > 0\}$. Since $C(u(x,t),x)$ is a continuous function, $\mathcal{F}(t)$ is open and therefore is the union of at most a countable number of intervals. Let $\mathcal{F}_i(t)$ denote one of this interval, and $\mathcal{F}_i$ the open connected component of $\mathcal{F}$ which contains $\mathcal{F}_i(t)$.

LEMMA 1. $\mathcal{F}_i \cap \partial_P Q_t \neq \emptyset$.

Proof. We can assume, without loss of generality, that $C > 0$ in $\mathcal{F}_i$. Then $v(x,t) = C(u(x,t),x)$, as a function of (x,t), satisfies the uniformly parabolic equation

$$\begin{aligned} v_t = {} & a_x(x)U_y(v(x,t),x)+ \\ & + a(x)U_{yy}(v(x,t),x)(a(x)U_c(v(x,t),x)v_x)_x+ \\ & + a(x)U_{cy}(v(x,t),x)v_x + a_x(x)U_y(v(x,t),x)+ \\ & + a(x)U_{yy}(v(x,t),x), \end{aligned} \tag{1}$$

in $\mathcal{F}_i$, where $U(c,y)$ denotes, for each y, the inverse function of $c_2(u,y)$ with respect to the variable u. In order to apply the maximum principle to (1), we assume that

$$\begin{gathered} a_x(x)U_y(v,x) + a(x)U_{yy}(v,x) = g(v,x) + f(x) \\ \text{with } g(0,x) \equiv 0 \text{ and } f(x) \leq 0, \forall x\,. \end{gathered} \tag{2}$$

Then if $\mathcal{F}_i \cap \partial_P Q_T = \emptyset$, $C = 0$ on $\partial_P F_i$ and consequently $C \leq 0$ in $\mathcal{F}_i$, which is a contradiction.

Notice that (2) is satisfied in the case of the viscometer equation (8) of Section 1.

REMARK. The previous lemma says that either a fluid zone exists at the initial time or it originates from the boundary of the domain. In other words, fluid zones cannot be generated inside a solid zone.

LEMMA 2. *Let $\overline{x}$ and $\delta > 0$ be such that $(\overline{x} - \delta, \overline{x} + \delta) \subset \mathcal{S}(t)$. Then there exists $\varepsilon > 0$ depending on δ, but not on t, such that $R_\delta = (\overline{x} - \delta/2, \overline{x} + \delta/2) \times (t, t+\varepsilon) \subset \mathcal{S}$.*

Proof. For the sake of simplicity we give the proof for $a(x) = x^{-3}$ and $\nu_i = (-1)^i x^2$, i.e. having the form of the corresponding data for the viscometer equation, see (7) and (8) of Sect. 1. The proof is easily extended to more general form of the coefficients, provided the condition $(-1)^i \nu_i(x) a(x)$ is a strictly decreasing function, holds. This excludes the case of constant coefficients. But in that case a solid interval can not exist in between two fluid regions having C of the same sign because of maximum principle (in this case u would be constant in the region $C = 0$ and this region would immediatly disappear.)
Define $v_+(x,t) = u(x,t) + x^2$ and $v_-(x,t) = u(x,t) - x^2$. Because of the assumptions on $C(u,.)$, a constant $A > 0$ can be found such that $v_+ \geq AC(u,.)$ when $u < 0$ and $v_- \leq AC(u,.)$ for $u > 0$. Moreover $v_+(x,t) \geq 0$ implies $C(u(x,t),x) \geq 0$ and $v_-(x,t) \leq 0$ implies $C(u(x,t),x) \leq 0$. Define also $z_+ = -\alpha x^4 + x^2 - \beta$ and $z_- = \alpha x^4 - x^2 + \beta = -z_+$. It can be easily checked that, for any α and β, z_+ and z_- are stationary solutions of the equations satisfied by v_+ and v_- respectively. We can choose $\alpha > 0$ and $\beta > 0$ in such a way that z_+ is positive in $(\overline{x} - \delta/2, \overline{x} + \delta/2)$ and negative outside and the converse is true for z_-. Because of the Hölder continuity of C we have that a $\varepsilon > 0$, depending only on δ, α, β and the Hölder constant of C, exists such that $AC(u(x \pm \delta/2, \tau), x \pm \delta/2)$ and, consequently, $v_+(\overline{x} \pm \delta/2, \tau)$ are greater than $z_+(\overline{x} \pm \delta/2)$ for any $\tau \in (t, t+\varepsilon)$. Then $v_+(y,\tau) > z_+(y,\tau) > 0$ in $(\overline{x} - \delta/2, \overline{x} + \delta/2) \times (t, t+\varepsilon)$, which implies that $C \geq 0$ in the same rectangle. Since $z_- = -z_+$, we can repeat the above argument to prove that $C \leq 0$ in the same domain. Consequently $R_\delta \subset \mathcal{S}$.

We now give a partial result concerning the continuity of the free boundaries, to this purpose we state the following extra-assumptions.

H_δ: *for any t, the distance between two points belonging to the fluid region which are separated by a solid region is always greater than δ, for a given positive δ.*

Note that, because of assumptions H_δ, the number of intervals which compose $\mathcal{S}(t)$ is a bounded function $n(t)$ with integer value. Then, at first, we can focus our attention on an interval (t_1, t_2) in which the function $n(t) = N$ is constant.

In (t_1, t_2) we can define the interfaces $\sigma_i(t)$ between solid and liquid counting the boundary points of $\mathcal{S}(t)$ starting from the left. Notice that the number of these interfaces is a constant, and then each σ_i is a function defined on the whole interval (t_1, t_2).

The first step is to prove that, for any $t \in (t_1, t_2)$, σ_i has limit at t both from below and from above. We assume, without loss of generality, that C vanishes on a right neighbourhood of σ_i. Suppose that there is no limit from below, i.e. $\liminf_{\tau\uparrow t} \sigma_i = x_1 < \limsup_{\tau\uparrow t} \sigma_i = x_2$. Let t_n be a sequence such that $t_n \to t$ and $\sigma_i(t_n) \to x_1$. Then from the assumption H_δ we have $C(u(x,t_n),t_n) = 0$ for $x \in (\sigma_i(t_n), \sigma_i(t_n) + \delta)$. Now lemma 2 ensures that C vanishes in $(\sigma_i(t_n) + \frac{\delta}{4}, \sigma_i(t_n) + 3\frac{\delta}{4})$ for a time interval $(t_n, t_n + \varepsilon)$ where $\varepsilon > 0$ depends only on δ. This contradicts the assumtion $\limsup_{\tau\uparrow t} \sigma_i > x_1$.

Similarly we can prove the existence of the limit from above.

Moreover the limit from above must be less or equal to the limit from below, otherwise we would have a contradiction with Lemma 2.

It remains to exclude the case in which $x_M = \lim_{\tau\uparrow t} \sigma_i(\tau) > \lim_{\tau\downarrow t} \sigma_i(\tau) = x_m$.

In this case we can find a $t' < t$ and x_1, $x_2 \in (x_m, x_M)$ such that $|C| > 0$ in $[x_1, x_2] \times [t', t)$, so that the equation satisfied by the function C is uniformly parabolic in this rectangle; it can be easily proved that the solution of this parabolic equation is analytic with respect to the space variable (at least in the physical case, more in general we have to assume that the functions $a(x)$, $C(u,\, x)$ are analytic) and hence $|C| \not\equiv 0$ on $(x_1, x_2) \times t$, which is a contraddiction.

This concludes the proof of the continuity of σ_i at t and hence, since t is arbitrary, in the whole interval (t_1, t_2).

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