

A Model of Frost Heave with Sharp Interface between the Unfrozen and the Frozen Soils

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SOMMARIO. - *Quando un suolo umido è sottoposto ad un processo di congelamento, si osserva in generale un'espansione di volume. L'aumento di volume è dovuto principalmente ad una migrazione di acqua dalla base del suolo verso il fronte di congelamento, che separa la zona non congelata, in basso, da quella, superiore, completamente congelata. Il processo accoppiato di trasferimento di massa e calore si accompagna, in determinate condizioni, alla formazione di strati segregati di ghiaccio puro ("lenti di ghiaccio"). In tal caso, il fronte di congelamento rimane fermo. Se il processo di congelamento è troppo rapido o il peso sovrastante che agisce sul campione di suolo è rimarchevole, la crescita della lente di ghiaccio non avviene e il fronte di congelamento si sposta in basso verso la base del suolo ("penetrazione del ghiaccio"). In questo lavoro si discute un modello che ammette una netta interfaccia fra le regioni del suolo.*

SUMMARY. - *When a moist soil freezes, a volume expansion can be generally observed. The increase of volume is mainly due to a water migration from the base of the soil up to the freezing front, which separates the lower unfrozen part of the soil from the upper frozen one. The coupled heat-mass transfer process is accompanied, under certain conditions, to the formation of pure ice segregated layers (ice lenses). In this, case, the freezing front keeps at rest. If the freezing process is too fast or the overburden pressure acting on the column of soil is relevant, the ice lens growth does not occur and the freezing front moves towards the base of the soil (frost penetration). In this paper a model admitting a sharp interface between the two regions is discussed.*

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1. Introduction

In a porous medium saturated by water and cooled on the top by a freezing temperature, two regions will be observed: a lower unfrozen part (soil grains and water) and the upper frozen soil (soil grains and ice). The size of both regions changes with respect to time. In the unfrozen soil a water migration coming from the permeable base of the soil occurs. Sometime, the growth of a pure ice layer (ice lens) can be observed in proximity of the area where the change of phase takes place. The increase of the total volume of the medium, due to the incoming water which goes up and freezes, is known as frost heave. In this paper, we will assume that the unfrozen and frozen soils are separated by a sharp interface, corresponding to the freezing front. Referring to the classification introduced in [5], [4], we will deal with a primary frost heave model. The main assumptions of the one-dimensional model we are going to study from the mathematical point of view are the following (for more details, see [6], sect. 4.1, [3]):

- (H_1) The porous matrix of the unfrozen soil is non-deformable.
- (H_2) The volumetric flux of water in the unfrozen part is governed by the Darcy's law.
- (H_3) The change of temperature with respect to time is very slow and the process can be approximated as a sequence of quasi-steady states.
- (H_4) The freezing temperature is related to the water pressure at the interface by means of the Clausius-Clapeyron equation.
- (H_5) The interfacial effects between water and ice at the interface are described by the Kelvin-Laplace equation.
- (H_6) The mechanism by which a lens starts to form is based on the evaluation of the water pressure (hence of the freezing temperature, by (H_4)) at the freezing front and of the front speed. More precisely, when the freezing temperature attains a given value depending on the soil, the freezing front penetrates the soil and no ice lens will form (frost penetration).
- (H_7) In case of frost penetration the speed of the ice particles and of the soil grains immediately over the freezing front is the same.

In [2] and [1] the evolutive case of the model (i.e. quitting (H_3)) is detected, respectively in the case of lens formation and frost penetration; the existence of the solution is proved. In [3] the model $(H_1) - (H_6)$ is studied, adding the assumption (which replaces (H_7)) that the soil grains are at rest even in the frozen soil, whenever frost penetration occurs.

2. Statement of the problem

Let us choose an upward increasing coordinate z such that $z = 0$ coincides with the fixed base of the soil, $z_S(t)$ is the freezing front and $z_T(t)$ is the top of the soil. Call $\Omega_u = \{0 \leq z < z_S(t)\}$ the unfrozen soil and $\Omega_f = \{z_S(t) < z \leq z_T(t)\}$ the frozen soil. In [6], sect. 4.1 we showed that the equations corresponding to the model $(H_1) - (H_7)$ are the following:

$$z_S(t)q_w(t) = -K_0p_w(z_S(t), t) \quad (2.1)$$

$$T(z_S(t), t) = \frac{T_0 p_w(z_S(t), t)}{L \rho_w} \quad (2.2)$$

$$\rho_i \dot{z}_T(t) - \rho_w q_w(t) = (\rho_i - \rho_w)\varepsilon_0 \dot{z}_S(t) \quad (2.3)$$

$$\begin{aligned} (q_w(t) - \varepsilon_0 \dot{z}_S(t))\rho_w L = & -k_f \frac{g(t) - T(z_S(t), t)}{z_T(t) - z_S(t)} + \\ & + k_u \frac{T(z_S(t), t) - h(t)}{z_S(t)} \end{aligned} \quad (2.4)$$

$$(p_w(z_S(t), t) - p_c)\dot{z}_S(t) = 0 \quad (2.5)$$

$$\dot{z}_S(t) \leq 0, \quad p_w(z_S(t), t) - p_c \geq 0 \quad (2.6)$$

$$q_w(t) \geq 0 \quad (2.7)$$

$$z_S(t) \geq 0 \quad (2.8)$$

$$z_S(0) = b > 0, \quad z_T(0) = c > b \quad (2.9)$$

$$\begin{cases} T(0, t) = h(t) \\ T(z_T(t), t) = g(t) \end{cases} \quad (2.10)$$

The unknown quantities of the problem are the free boundaries $z_S(t)$ and $z_T(t)$, the freezing temperature $T(z_S(t), t)$, the water pressure at the interface $p_w(z_S(t), t)$, the volumetric water discharge $q_w(t)$. The positive constant K_0 (hydraulic conductivity), L (latent heat of water

per unit mass), $T_0 (= 273, 15)$, ρ_w, ρ_i (water and ice densities), k_u, k_f (thermal conductivities of the unfrozen and frozen soil), ε_0 (porosity of the unfrozen soil) are known. The critical value p_c , which we assume to be negative and constant, is related to the characteristics of the soil. Equation (2.1) comes from (H_1) and (H_2) , while (2.2) corresponds to (H_4) . Equations (2.3) and (2.4) are the mass and the heat balances at the freezing front z_S . Equation (2.5) together with the constraints (2.6) are the switching conditions for the process of lens formation and frost penetration: indeed, whenever the front z_S is at rest, an ice lens is forming at the height $z = z_S$; if $\dot{z}_S < 0$, a frost penetration process is occurring: in this case, the water pressure at the freezing front and the freezing temperature $T(z_S(t), t)$ are known, owing to (2.5) and (2.2). Conditions (2.7) and (2.8) are introduced in order to get solutions which are consistent from the physical point of view. Finally, (2.9) and (2.10) settle the initial position of the freezing front, the initial height of the soil and the boundary temperatures, at the base and on the top of the soil.

REMARK 2.1. Once system (2.1)-(2.10) has been solved, the temperature in each point of the soil and the water pressure in the unfrozen soil can be evaluated by means of the formulas (see [6], eq. (4.16)):

$$T(z, t) = \begin{cases} T(z_S(t), t) \frac{z}{z_S(t)} + T(0, t) \frac{z_S(t) - z}{z_S(t)}, & z \in \Omega_u \\ T(z_S(t), t) \frac{z - z_T(t)}{z_S(t) - z_T(t)} + T(z_T(t), t) \frac{z_S(t) - z}{z_S(t) - z_T(t)}, & z \in \Omega_f \end{cases}$$

$$p_w(z, t) = \frac{-q_w(t)z}{K_0}, \quad z \in \Omega_u$$

We find it convenient to write the system (2.1)-(2.10) in terms of $z_S(t)$, $z_T(t)$ and $T_S(t) = T(z_S(t), t)$ only:

$$\rho_i \dot{z}_T(t) + K_0 \rho_w^2 L \frac{T_S(t)}{T_0 z_S(t)} = (\rho_i - \rho_w) \varepsilon_0 \dot{z}_S(t) \quad (2.11)$$

$$\left(-K_0 \rho_w L \frac{T_S(t)}{T_0 z_S(t)} - \varepsilon_0 \dot{z}_S(t) \right) \rho_w L = -k_f \frac{g(t) - T_S(t)}{z_T(t) - z_S(t)} + k_u \frac{T_S(t) - h(t)}{z_S(t)} \quad (2.12)$$

$$(T_S(t) - T_f) \dot{z}_S(t) = 0 \quad (2.13)$$

$$T_S(t) - T_f \geq 0 \quad (2.14)$$

$$\dot{z}_S(t) \leq 0, \quad z_S(t) \geq 0 \quad (2.15)$$

$$T_S(t) \leq 0 \quad (2.16)$$

$$z_S(0) = b > 0, \quad z_T(0) = c > b, \quad (2.17)$$

$$T(0, t) = h(t), \quad T(z_T(t), t) = g(t)$$

In (2.14) we defined

$$T_f = \frac{T_0 p_c}{L \rho_w} < 0 \quad (2.18)$$

As for the boundary temperatures h and g , we make the following assumptions (*HG*):

$$i) \quad h(t), g(t) \in C^1[0, \infty);$$

$$ii) \quad h(t) \geq 0, g(t) < T_f;$$

A solution of the above written problem in the interval $[0, \tau]$ will be a triplet $(z_S(t), z_T(t), T_S(t))$ with $z_S, z_T \in C^1[0, \tau]$, $T_S \in C^0[0, \tau]$ and such that eqq. (2.11)-(2.17) are satisfied. If a solution of (2.11)-(2.17) is such that $\dot{z}_S = 0$ (respect. $\dot{z}_S < 0$) in some interval, we will call it a lens formation-type solution, or *LF* (respect. frost penetration-type solution, or *FP*).

3. Constant boundary temperatures

In order to discuss the solvability of system (2.1)-(2.10), we start by stating the following result, dealing with the case when the boundary temperatures (2.10) are constant.

PROPOSITION 3.1. *Given the initial conditions (2.9) and the temperatures at the extremities of the soil $T(0, t) = h_0 \geq 0$ and $T(z_T(t), t) = g_0 < T_f$, where h_0 and g_0 are constant, we have:*

i) if

$$k_u h_0 (c - b) + k_f b g_0 > 0, \quad (3.1)$$

then system (2.11)-(2.13) has no solution consistent with (2.14)-(2.17);

ii) if

$$k_u h_0(c - b) \leq -k_f b g_0 \leq k_u h_0(c - b) + \eta \quad (3.2)$$

where

$$\eta(b, c) = -\frac{K_0 p_c}{b} ((c - b) \times \\ \times (L \rho_w b + C_0(b) k_u) + k_f C_0(b) b) > 0, \quad (3.3)$$

$$C_0(b) = \frac{T_0 b}{\rho_w L K_0}. \quad (3.4)$$

then the unique solution of (2.1)-(2.10) describes a LF process for any $t \geq 0$ and the height of the soil to the value z_T^∞ , where

$$\begin{cases} z_T^\infty = \left(1 - \frac{k_f g_0}{k_u h_0}\right) b > c & \text{if } h_0 > 0; \\ z_T^\infty = +\infty & \text{if } h_0 = 0; \end{cases} \quad (3.5)$$

iii) if

$$-k_f b g_0 > k_u h_0(c - b) + \eta, \quad (3.6)$$

then the unique solution of (2.1)-(2.10) is of FP-type up to a finite time \bar{t} , when $\dot{z}_S(\bar{t}) = 0$. For $t = \bar{t}$ there is a switch to a LF process and the lens formation process proceeds as in point ii).

Proof. We start by remarking that \dot{z}_S is a linear function of T_S , as we deduce from (2.12):

$$\dot{z}_S = \alpha T_S + \beta \quad (3.7)$$

where

$$\begin{cases} \alpha(z_S, z_T) = -\frac{1}{\varepsilon_0 \rho_w L} \left(\frac{K_0 \rho_w L}{T_0} \frac{1}{z_S} + \frac{k_f}{z_T - z_S} + \frac{k_u}{z_S} \right) \\ \beta(z_S, z_T, h_0, g_0) = \frac{1}{\varepsilon_0 \rho_w L} \left(\frac{k_f g_0}{z_T - z_S} + \frac{k_u h_0}{z_S} \right) \end{cases} \quad (3.8)$$

Hence, for any given z_S, z_T and any pair of data h_0, g_0 the equation $\dot{z}_S = 0$ admits the unique solution

$$\bar{T} = -\beta/\alpha. \quad (3.9)$$

The physical problem makes sense as long as $z_T > z_S$ (actually, we will show that this is a property of the solution of (2.11)-(2.17)); therefore:

$$\dot{z}_S < 0 \text{ (} > 0 \text{) if } T_S > \bar{T} \text{ (} < \bar{T} \text{)}. \quad (3.10)$$

Let us consider now the problem (2.11)-(2.13). Formally, we have the two solutions of such system, corresponding to $T_S(t) = T_f$ and $T_S(t) = \bar{T}$. Indeed, if $T_S = T_f$ eqq. (2.11)-(2.12) give a pair of O.D.E.s for z_S, z_T , which determine univoquely the two boundaries. On the other hand, if $T_S = \bar{T}$, we can express by means of (2.11) T_S as a function of \dot{z}_T and, by substitution, we see that eq. (2.12) reduces to an O.D.E. for z_T .

Our next aim is to show that the assumptions (2.14)-(2.17) make us exclude one of the two formal solutions of (2.11)-(2.13), so that the uniqueness of the whole problem will be achieved.

Putting together (3.10) and (2.14)-(2.16), we get:

$$\begin{cases} \dot{z}_S(t) = 0 & \text{if } T_f \leq \bar{T} = T_S \leq 0 \\ \dot{z}_S(t) < 0 & \text{if } T_S = T_f > \bar{T} \end{cases} \quad (3.11)$$

Evaluating \bar{T} for $t = 0$, we have:

$$\bar{T}|_{t=0} = \bar{T}(b, c, h_0, g_0) = \frac{k_f g_0 b + k_u h_0 (c - b)}{k(c - b) + k_f b} \quad (3.12)$$

where

$$k = \frac{K_0 \rho_w L}{T_0} + k_u > 0$$

We have the following possibilities (let us call $\bar{T}(b, c, h_0, g_0) = \bar{T}(0)$):

- a) $\bar{T}(0) > 0$;
- b) $T_f < \bar{T}(0) \leq 0$;
- c) $\bar{T}(0) < T_f < 0$;
- d) $T_f = \bar{T}(0)$.

In the first case, none of the formal solutions of (2.11)-(2.13) can be accepted, owing to (3.11). In case b) the solution of (2.11)-(2.13) corresponding to $T_S = T_f$ can not be accepted, once more by virtue of (3.11) (we recall that T_S is required to be continuous up to $t = 0$).

Similarly, in the case *c*) the solution with $T_S = \bar{T}$ must be excluded. In order to discuss the occurrence *d*), we write eqq. (2.11)-(2.12) in the following form:

$$\begin{cases} \dot{x}(t) = \frac{A}{x(t)} + \frac{B}{y(t)} \\ \dot{y}(t) = \frac{C}{x(t)} + \frac{D}{y(t)} \\ x(0) = b, \quad y(0) = c - b \end{cases} \quad (3.13)$$

where $x(t) = z_S(t)$, $y(t) = z_T(t) - z_S(t)$ and

$$\begin{cases} A = A(T_S, h_0) = \frac{L\rho_w\gamma - k_u(T_S(t) - h(t))}{\varepsilon_0\rho_w L} > 0 \\ B = B(T_S, g_0) = \frac{k_f}{\varepsilon_0\rho_w L}(g_0 - T_S) < 0 \\ C = C(T_S, h_0) = \frac{k_u(T_S - h_0)(\rho_i(1 - \varepsilon_0) + \rho_w\varepsilon_0)}{\varepsilon_0\rho_w\rho_i L} + \\ \quad - \frac{(1 - \varepsilon_0)\gamma L}{\varepsilon_0 L} < 0 \\ D = D(T_S, g_0) = -\frac{k_f(g_0 - T_S)(\rho_i(1 - \varepsilon_0) + \rho_w\varepsilon_0)}{\varepsilon_0\rho_w\rho_i L} > 0. \end{cases} \quad (3.14)$$

where we set

$$\gamma = -K_0 p_c = -K_0 \rho_w \frac{LT_f}{T_0} > 0. \quad (3.15)$$

After easy computations, we can see that

$$AD - BC = -\frac{B\rho_i\gamma}{\rho_w} > 0, \quad (3.16)$$

This means that as long as $\dot{x} = \dot{z}_S < 0$, it is $\dot{y} > 0$. Hence, since h and g are constant, we have from (3.9), (3.8):

$$\frac{d\bar{T}}{dt} = \frac{k_f(\dot{x}y - x\dot{y})(kg_0 - k_u h_0)}{(ky + k_f x)^2} > 0 \text{ if } \dot{x} \leq 0. \quad (3.17)$$

Thus, the solution of (2.11)-(2.14) with $T_S(t) = T_f$ can not be accepted; actually, for $t > 0$ we would have $\dot{z}_S(t) > 0$. We conclude that when $\bar{T}(0) = T_f$ the solution of (2.11)-(2.14) is of *LF*-type (i.e. $T_S(t) = \bar{T}$, $\dot{z}_S(t) = 0$).

Recalling (3.12), (2.18) it easily seen that the conditions $\bar{T}(0) > 0, T_f \leq \bar{T}(0) \leq 0, \bar{T}(0) < T_f < 0$ correspond respectively to conditions (3.1), (3.2) and (3.6).

We are going now to integrate the system (2.11)-(2.17) in order to conclude the proof of proposition 3.1.

Assume that for $t = 0$ (3.2) holds. The solution of (2.11)-(2.17) is, at least in some right neighborhood of $t = 0$, of LF -type; as we mentioned above, the mathematical problem is reduced to solving the following O.D.E. for the top of the soil z_T :

$$\begin{cases} \dot{z}_T(t) = \frac{\rho_w}{\rho_i} \frac{-k_f b g_0 - k_u h_0 (z_T(t) - b)}{(z_T(t) - b)(L\rho_w b + C_0 k_u) + k_f C_0 b} \\ z_T(0) = c \end{cases} \quad (3.18)$$

where $C_0(b)$ is defined by (3.4). Once (3.18) has been integrated, the temperature $T_S(t) = \bar{T}$ is evaluated by means of (2.11).

Expressing the conditions (2.14) and (2.16) in terms of z_T , one gets:

$$0 \leq \dot{z}_T(t) \leq -\frac{\rho_w K_0 p_c}{\rho_i b}. \quad (3.19)$$

Obviously, the previous formula computed for $t = 0$ gives again (3.2). If it is $\dot{z}_T(0) = 0$ (that is $\bar{T}(0) = 0$), the unique solution of (3.18) is $z_T(t) \equiv c$; this entails that the water pressure in Ω_u and the freezing temperature $T(b, t)$ are identically zero.

If $\dot{z}_T(0) > 0$ (that is $\bar{T}(0) < 0$), the solution of (3.18) is the following, when $h_0 > 0$:

$$\begin{aligned} & -\frac{L\rho_w b + C_0 k_u}{k_u h_0} (z_T(t) - c) + \frac{k_f b \rho_i}{\rho_w} \times \\ & \times \frac{C_0 k_u (g_0 - h_0) + L\rho_w b g_0}{(k_u h_0)^2} \ln \left| \frac{k_u h_0 (z_T(t) - b) + k_f b g_0}{k_u h_0 (c - b) + k_f b g_0} \right| = t. \end{aligned} \quad (3.20)$$

Examining the implicit equation (3.20), we easily see that the solution $z_T(t)$ of (3.18) is strictly increasing and tends to the asymptotic value defined in (3.5). The thickness of the lens tends to $z_T^\infty - c$. The initial value of the hydraulic flux is

$$q_w(0) = \frac{-k_f b g_0 - k_u h_0 (c - b)}{(c - b)(L\rho_w b + C_0 k_u) + k_f C_0 b}$$

and is such that $\lim_{t \rightarrow +\infty} q_w(t) = 0^+$; therefore, if (2.7) (or, equivalently, (2.16)) holds for $t = 0$, it holds for any $t > 0$. Correspondingly, the water pressure tends to the stationary profile $p_w^\infty(z) \equiv 0$, while $T_S(t) = T(b, t)$ tends to zero by increasing and starting from the value $T(b, 0) = -T_0 b q_w(0) / L \rho_w K_0$. The asymptotic profile of the temperature T is

$$T(z, t) = \begin{cases} \frac{h_0}{b-z} & z \in \Omega_u, \\ g_0 \frac{b-z}{b-z_T^\infty} & z \in \Omega_f. \end{cases} \quad (3.21)$$

Consider now the condition, equivalent to the second one in (2.6), $T_f \leq \bar{T}$. If it holds for $t = 0$, (that is if the second inequality of (3.2) holds), then it will be fulfilled for any $t > 0$: indeed, $T_f \leq \bar{T}$ is equivalent to the second inequality in (3.19) and $\dot{z}_T(t)$ achieves its positive maximum for $t = 0$.

If we take $h_0 \equiv 0$, the solution of (3.18) is

$$z_T(t) = b + \frac{-\phi_2 + \sqrt{\phi_2^2 - \phi_1 \left(\frac{\rho_w k_f b g_0}{\rho_i} 2t - (\phi_1(c-b) + 2k_f C_0 b)(c-b) \right)}}{\phi_1}$$

where

$$\phi_1 = L \rho_w b + C_0 k_u > 0, \quad \phi_2 = k_f C_0 b > 0.$$

The upper surface z_T does not tend to a finite value but increases as $t^{1/2}$. Nevertheless, the water flux, the freezing temperature and the water pressure have the same properties as in the case $h_0 > 0$. In particular, from (3.21) we deduce that the temperature tends to vanish everywhere in Ω_u ; the same property holds for each point fixed in Ω_f .

We conclude that the solution of (2.11)-(2.17), whenever (3.2) holds, is globally a *LF*-type solution, without switches to *FP* solutions.

Let us pass now to examine the solutions of (2.11)-(2.17) of *FP*-type. Assume that (3.6) holds. Taking $T_S(t) = T_f$ and writing (2.11)-(2.12) as in (3.13), we see that the coefficients defined in (3.14) are

constants. By virtue of (3.10), condition $T_f > \bar{T}(0)$ is equivalent to

$$\frac{y(t)}{x(t)} < -\frac{B(T_f, g_0)}{A(T_f, h_0)}. \tag{3.22}$$

Condition (3.6) is nothing but (3.22) evaluated for $t = 0$.

When h and g are constant, system (3.13) is autonomous and the orbits are given by:

$$\begin{aligned} & -\frac{1}{2} \ln \left| \frac{-A_f u^2 + (C_f - B_f)u + D_f}{-A_f \left(\frac{c-b}{b}\right)^2 + (C_f - B_f)\frac{c-b}{b} + D_f} \right| - \frac{B_f + C_f}{4A_f \omega} \times \\ & \times \ln \left| \frac{u + \frac{B_f - C_f}{2A_f} - \omega \frac{c-b}{b} + \frac{B_f - C_f}{2A_f} + \omega}{u + \frac{B_f - C_f}{2A_f} + \omega \frac{c-b}{b} + \frac{B_f - C_f}{2A_f} - \omega} \right| = \ln \left| \frac{x}{b} \right| \end{aligned}$$

where $u(t) = \frac{y(t)}{x(t)}$ and

$$A_f = A(T_f, h_0), B_f = B(T_f, g_0), C_f = C(T_f, h_0), D_f = D(T_f, g_0),$$

$$\omega = \sqrt{\frac{4A_f D_f + (B_f - C_f)^2}{4A_f^2}}$$

Recalling (3.16), we see that

$$-A_f u^2 + (C_f - B_f)u + D_f = -(A_f u + B_f)u + (C_f u + D_f) > 0, \tag{3.23}$$

if $u < -\frac{B_f}{A_f}$. Noticing that

$$\frac{|B_f - C_f|}{2A_f} < \omega, u + \frac{B_f - C_f}{2A_f} - \omega < 0, u < -\frac{B_f}{A_f}, \tag{3.24}$$

it is easily seen that the orbits starting from one point $(x(0), y(0))$ verifying (3.6) can be written in the following form:

$$x = \sqrt{\frac{-A_f(c-b)^2 + (C_f - B_f)(c-b)b + D_f b^2}{-A_f u^2 + (C_f - B_f)u + D_f}} \times \tag{3.25}$$

$$\begin{aligned}
& \times \left(\frac{\left(\frac{c-b}{b} + \frac{B_f - C_f}{2A_f} + \omega \right) \left(u + \frac{B_f - C_f}{2A_f} - \omega \right)}{\left(\frac{c-b}{b} + \frac{B_f - C_f}{2A_f} - \omega \right) \left(u + \frac{B_f - C_f}{2A_f} + \omega \right)} \right)^{-\frac{B_f + C_f}{4A_f \omega}} \\
& = F(u)
\end{aligned}$$

The orbit (3.25) satisfies (3.22) as far as $z_S = x$ is greater than \bar{x} , say for $t = \bar{t} < +\infty$, with $\bar{x} = F(-B_f/A_f)$. By virtue of (3.23) and (3.24), we have $x(t) > 0$ for $0 \leq t \leq \bar{t}$. We have $\dot{x}(\bar{t}) = \dot{z}_S(\bar{t}) = 0$ and $\dot{x}(t) > 0$ for $t > \bar{t}$: thus, the solution can be accepted only as long as $t < \bar{t}$, since (2.5) is violated for $t \geq \bar{t}$. Finally, we remark that the thickness of the frozen part $y(t) = z_T(t) - z_S(t)$, $0 \leq t < \bar{t}$ is increasing, by virtue of (3.16). The height of the soil for $t = \bar{t}$ is

$$z_T(\bar{t}) = y(\bar{t}) + x(\bar{t}) = (1 - B_f/A_f)\bar{x}.$$

It can be checked that for $t \geq \bar{t}$ a process of lens formation occurs. Actually, consider the system (3.18), updating the data at the time $t = \bar{t}$, that is replacing b with \bar{x} and c with $z_T(\bar{t})$. For $t = \bar{t}$ we have

$$-k_f g_0 \bar{x} = k_u h_0 (z_T(\bar{t}) - \bar{x}) + \eta(\bar{x}, z_T(\bar{t})); \quad (3.26)$$

in other words (3.2) holds. The equality in (3.26) means that for $t = \bar{t}$ the water pressure at the front z_S is the critical one p_c (or, equivalently, the freezing temperature is T_f). The system describing the process of lens formation which sets up at $t = \bar{t}$ is obviously:

$$\begin{cases} \dot{z}_T(t) = \frac{\rho_w}{\rho_i} \frac{-k_f \bar{x} g_0 - k_u h_0 (z_T(t) - \bar{x})}{(z_T(t) - \bar{x})(L\rho_w \bar{x} + \bar{C}_0 k_u) + k_f \bar{C}_0 \bar{x}} \\ z_T(\bar{t}) = \left(1 - \frac{B_f}{A_f}\right) \bar{x} \end{cases}$$

where $\bar{C}_0 = \frac{T_0 \bar{x}}{\rho_w L K_0}$. According to (3.26), we have $\dot{z}_T(\bar{t}) > 0$. The process of lens formation develops as we described above. The asymptotic height of the soil is

$$z_T^\infty = \left(1 - \frac{k_f g_0}{k_u h_0}\right) \bar{x}$$

Notice that $T_S(t), \dot{z}_S(t), \dot{z}_T(t)$ are continuous at the switch time \bar{t} : namely, the triplet (T_S, z_S, z_T) obtained by attaching the *FP*-type solution with the *LF*-type one at $t = \bar{t}$ is solution of (2.11)-(2.17), as the requirements specified in section 2 are satisfied. \diamond

4. Boundary temperatures depending on time

Assume now that the boundary temperatures depend on time. We have:

PROPOSITION 4.1. *Let h and g satisfy the properties (HG) listed in section 2. Then:*

i) if

$$k_u h(0)(c - b) + k_f b g(0) > 0, \quad (4.1)$$

then system (2.11)-(2.13) has no solution consistent with (2.14)-(2.17);

ii) if

$$k_u h(0)(c - b) + k_f b g(0) \leq 0, \quad (4.2)$$

then (2.11)-(2.17) has, at least locally, a unique solution for $t \geq 0$.

*In particular, defining η and $C_0(b)$ as in (3.3) and (3.4), we have that the process starts with a *LF*-type solution in the following cases:*

$$k_u h(0)(c - b) \leq -k_f b g(0) < k_u h(0)(c - b) + \eta, \text{ or} \quad (4.3)$$

$$-k_f b g(0) = k_u h(0)(c - b) + \eta \text{ and } G(0) > 0 \quad (4.4)$$

where

$$\begin{aligned} G(t) = & (k_f \dot{g}(t)x(t) + k_u \dot{h}(t)y(t))(ky(t) + k_f x(t)) + \\ & + k_f \left(\left(\frac{A(T_f, h(t))}{b} + \frac{B(T_f, g(t))}{c - b} \right) y(t) + \right. \\ & \left. - \left(\frac{C(T_f, h(t))}{b} + \frac{D(T_f, g(t))}{c - b} \right) x(t) \right) (kg(t) - k_u h(t)) \end{aligned} \quad (4.5)$$

and A, B, C, D are defined as in (3.14); the process starts with a *FP*-type solution if:

$$-k_f b g(0) > k_u h(0)(c - b) + \eta \text{ or} \quad (4.6)$$

$$-k_f b g(0) = k_u h(0)(c - b) + \eta \text{ and } G(0) < 0. \quad (4.7)$$

Proof. As in the previous case, $\dot{z}_S(t)$ is a linear function of T_S , but the function β , defined in (3.8), depends on t through $h(t)$ and $g(t)$. Thus, we have $\bar{T} = \bar{T}(z_S, z_T, h(t), g(t))$, where \bar{T} has the same meaning as in (3.9). Moreover, (3.10) still holds. Each of the two solutions of (2.13) $T_S = T_f$ and $\dot{z}_S = 0$ (that is $T_S = \bar{T}$) determines univoquely the solution of (2.11)-(2.12), by virtue of the assumption $HG, i)$ stated in section 1.

Consider now $\bar{T}(0)$, that is defined as in (3.12), where, instead of h_0 and g_0 , $h(0)$ and $g(0)$ must be replaced. Cases (4.1), (4.3), (4.6) correspond respectively to $\bar{T}(0) > 0$, $T_f < \bar{T}(0) \leq 0$, $\bar{T}(0) < T_f < 0$ and they can be proved exactly as we did for cases $a)$, $b)$, $c)$ of proposition 3.1, by excluding one of the formal solution of (2.11)-(2.13).

Let us pass to examine the case $T_f = \bar{T}(0)$. Eqq. (2.11)-(2.13) can be still written in the form (3.13), with the only difference that the coefficients A, B, C, D depend on time through $h(t)$ and $g(t)$ (i.e. $A = A(T_S, h(t))$, etc.). Making the derivative of \bar{T} w. r. t. time, one gets, instead of (3.17):

$$\frac{d\bar{T}}{dt} = \frac{k_f(\dot{x}y - x\dot{y})(kg - k_u h) + (k_f \dot{g}x + k_u \dot{h}y)(ky + k_f x)}{(ky + k_f x)^2}. \quad (4.8)$$

Inequality (3.16) (and the consequence written just below that formula) is still valid.

Now, it is sufficient to remark that if

$$\frac{d\bar{T}}{dt}(0) < 0 \text{ (respect. } > 0), \quad (4.9)$$

then the solution of (2.11)-(2.13) of LF -type (respect. of FP - type) must be excluded, since T_S would be smaller than T_f (respect. \dot{z}_S would be positive).

By virtue of (3.14), it easily seen that inequalities (4.9) are equivalent respectively to conditions (4.4) and (4.7) (notice that $G(0)$ can be computed once the data (2.17) are known).

Let us examine now the case $T_f = \bar{T}(0), G(0) = 0$. We have that system (2.11),(2.12),(2.17) has at most one solution which is physically consistent, i.e. which satisfies (2.13)-(2.16). Indeed, assume, contrary to our claim, that two distinct triplets $(z_S^{(F)}, z_T^{(F)}, T_S^{(F)} = T_f)$ (FP -type solution) and $(z_S^{(L)} = b, z_T^{(L)}, T_S^{(L)} = \bar{T}_L)$ (LF -type solution)

can be found such that eqq. (2.11)-(2.17) are fulfilled. In particular, it would be (cfr. (3.11),(2.14)):

$$\bar{T}_F \leq T_f \leq \bar{T}_L, \quad \bar{T}_F \neq T_f, \quad \bar{T}_L \neq T_f \quad (4.10)$$

where (cfr. (3.9))

$$\begin{cases} \bar{T}_F = - \frac{\beta(z_S^{(F)}, z_T^{(F)}, h(t), g(t))}{\alpha(z_S^{(F)}, z_T^{(F)})} = - \frac{\beta_F}{\alpha_F} \\ \bar{T}_L = - \frac{\beta(b, z_T^{(L)}, h(t), g(t))}{\alpha(b, z_T^{(L)})} = - \frac{\beta_L}{\alpha_L}. \end{cases} \quad (4.11)$$

Since $\dot{z}_S^{(F)} \leq 0$ (cfr. (3.11)), it is

$$z_S^{(L)} = b \geq z_S^{(F)}. \quad (4.12)$$

The comparison theorem applied to equation (2.11) yields, recalling (4.10) and that $\rho_i < \rho_w$, the inequality $z_T^{(L)} \leq z_T^{(F)}$, hence

$$y^{(L)} = z_T^{(L)} - b \leq y^{(F)} = z_T^{(F)} - z_S^{(F)} \quad (4.13)$$

and therefore (see (3.8))

$$\beta_L \leq \beta_F. \quad (4.14)$$

We calculate now the derivative w. r. t. time of α :

$$\frac{d\alpha}{dt} = \frac{k}{\varepsilon_0 \rho_w L} \frac{\dot{x}}{x^2} + \frac{k_f}{\varepsilon_0 \rho_w L} \frac{1}{y^2} \left(\frac{C}{x} + \frac{D}{y} \right). \quad (4.15)$$

From (3.14) and (4.10), we see that the following inequalities hold:

$$C(T_f, h) \leq C(\bar{T}_L, h) < 0, \quad D(\bar{T}_L, g) \geq D(T_f, g) > 0. \quad (4.16)$$

From (4.12), (4.13), (4.15) and (4.16) and taking into account that $\alpha_L(b, c) = \alpha_F(b, c)$ we deduce:

$$\alpha_L \geq \alpha_F. \quad (4.17)$$

From (4.14) and (4.17) it would follow $\bar{T}_F \geq \bar{T}_L$, which is not consistent with (4.10).

On the other hand, it can be proved that it is not possible that any of the two formal solutions of (2.11)-(2.13), (2.17) is not consistent with the constraints (2.14)-(2.16).

Actually, let us assume, by contrary, that in some interval $(0, \tau]$ it is

$$-\frac{\beta_F}{\alpha_F} = \bar{T}_F > T_f, \quad -\frac{\beta_L}{\alpha_L} = \bar{T}_L < T_f. \quad (4.18)$$

Owing to (3.11), it would be

$$z_S^{(L)} = b < z_S^{(F)} = x^{(F)}, \quad \dot{x}^{(F)} > 0, \quad t \in (0, \tau]. \quad (4.19)$$

From (4.18) and (2.11) we deduce, arguing as in the previous case, $\dot{y}^{(L)} > \dot{y}^{(F)}$, hence

$$y^{(L)} > y^{(F)}, \quad \beta_L > \beta_F, \quad t \in (0, \tau]. \quad (4.20)$$

Equation (4.18) yields $C(\bar{T}_L, h) < C(T_f, h) < 0$, $0 < D(\bar{T}_L, g) < D(T_f, g)$ and, together to (4.19), (4.20):

$$\frac{C(\bar{T}_L, h)}{b} + \frac{D(\bar{T}_L, g)}{y^{(L)}} \leq \frac{C(T_f, h)}{x^{(F)}} + \frac{D(T_f, g)}{y^{(F)}}.$$

Therefore:

$$\alpha_L < \alpha_F, \quad (4.21)$$

but (4.20), (4.21) are not consistent with (4.18). \diamond

REMARK 4.1. The last part of the proof of proposition 4.1 provides existence and uniqueness of the solution of (2.11)-(2.17) even in the case $\bar{T}(0) = T_f$, $G(0) = 0$. Nevertheless, it can not be predicted, at least maintaining the present assumptions, which type of solution will take place (LF or FP), or if it will oscillates between the two formal solutions of (2.11)-(2.13). Obviously, additional conditions involving higher derivatives of the data (which have to be required more regular) can be given in order to predict if $\bar{T}(t)$ will be greater or smaller than T_f in a right neighbourhood of $t = 0$.

The rest of this section is devoted to the qualitative analysis of the evolution in time of the solution introduced in proposition 4.1. In particular, we are interested in showing that, contrary to the case h ,

g constant, subsequent switches from LF -type solutions to FP -type solutions and *vice versa* are possible.

Let us assume, as an example, that condition (4.6) holds. Owing to proposition 4.1, we are induced to solve (3.13) by imposing $T_S(t) = T_f$ in (3.14) and looking for the boundaries $x(t), y(t)$.

Secondly, we pass to solve the following system with respect to t :

$$\begin{cases} \frac{y(t)}{x(t)} = -\frac{B(T_f, g(t))}{A(T_f, h(t))} \\ z_S(t) \geq 0 \end{cases} \quad (4.22)$$

It is worth to notice that it is not possible for $x(t)$ to reach the base of the soil $x = 0$ in a finite time (even if h vanishes). Actually, as long as $\dot{x} < 0$, condition (3.22) holds (with $h(t), g(t)$ instead of h_0, g_0); thus, the temperature $g(t)$ should go to $-\infty$ in a finite time, contrary to assumption HG .

Therefore, if (4.22) has not any solution, we have that

$$\lim_{t \rightarrow +\infty} z_S(t) = \ell, \quad 0 \leq \ell < b. \quad (4.23)$$

From (2.11) we see that this case is consistent only with z_T unbounded.

In order to construct an example for (4.23), we could consider a function $g(t)$ with the property, for each time t :

$$\dot{g}(t) < \min \{F_1(t), F_2(t)\}$$

where

$$\begin{aligned} F_1(t) &= \frac{B^2 \varepsilon_0 \rho_w L}{(c-b)k_f A} \left(\frac{A}{b} + \frac{B}{c-b} \right) \\ F_2(t) &= -\frac{A \varepsilon_0 \rho_w L}{bk_f} \left(\frac{C}{b} + \frac{D}{c-b} \right) \end{aligned}$$

where A, B, C, D are computed in $T_f, h(t)$ and $g(t)$. Moreover, we take h constant and $g(0)$ so that (4.6) is fulfilled. After some computations, one can see that $\dot{x} < 0$ for all $t \geq 0$ and that ℓ defined in (4.23) is zero.

Assume now that (4.22) has solutions and call t_1 the smallest time such that (4.22) is verified. For $t = t_1$ (3.22) (with $h(t_1)$ and

$g(t_1)$ instead of h_0, g_0) is violated, since $\dot{x}(t_1) = 0$. If $\dot{x}(t)$ for $t > t_1$ is again negative, then the process of frost penetration goes on (the temperature \bar{T} is equal to T_f only for $t = t_1$, then it becomes again lower than T_f).

If, on the contrary,

$$\dot{x}(t) > 0 \text{ for } t > t_1, \quad (4.24)$$

(that is $\bar{T} > T_f$ for $t > t_1$), then the solution of (3.13) we computed is no longer acceptable.

We are induced to examine the possibility that for $t = t_1$ a lens starts to form. Call $b_l = x(t_1)$, $c_l = y(t_1) + b_l$. Since $\bar{T} > T_f$ for $t > t_1$ (cfr. (4.24) and (3.10)), we are in the case (4.4) of proposition 4.1, obviously taking $t = t_1$ as the initial time. The solution of (2.11)-(2.17) is found by solving the O.D.E. (3.18) in the unknown $z_T(t) = y(t) + b_l$, with the following modifications: $h(t), g(t)$ instead of h_0, g_0 , $b_l, C_0(b_l)$ (see (3.4)) instead of b, C_0 .

Notice that $T_S(t)$, \dot{z}_S and \dot{z}_T are continuous at $t = t_1$, so that the *FP*-type solution starting at $t = 0$ together to the *LF*-type solution starting at $t = t_1$ give the solution of (2.11)- (2.17) satisfying the requirements of section 2.

The *LF*-type solution of (3.18) (modified) develops according to one of the following possibilities:

- i) $0 < \dot{z}_T(t) < -\frac{\rho_w K_0 p_c}{\rho_i b_l}, t > t_1$;
- ii) $\dot{z}_T(t^*) = 0$ for a certain $t^* > t_1$ and $0 < \dot{z}_T(t) < -\frac{\rho_w K_0 p_c}{\rho_i b_l}, t_1 < t < t^*$;
- iii) $\dot{z}_T(t_2) = -\frac{\rho_w K_0 p_c}{\rho_i b_l}$ for a certain $t_2 > t_1$ and $0 < \dot{z}_T(t) < -\frac{\rho_w K_0 p_c}{\rho_i b_l}, t_1 < t < t_2$.

The first case is equivalent to (cfr. (3.19)) $T_f < \bar{T} < 0$ for each time $t > t_1$; the process of lens formation goes on for each time $t > t_1$.

In the second case, $\bar{T}(b_l, z(t^*), h(t^*), g(t^*)) = 0$. If the front speed $\dot{z}_T(t)$ is again positive for $t > t^*$, then the lens formation process goes on; if, on the contrary, $\dot{z}_T(t) < 0$ when $t > t^*$, the solution is

no longer acceptable. Obviously, no *FP*-type solutions can exist for $t \geq t^*$, since $T_S(t)$ is required to be continue.

Finally, in the third case the water pressure reaches the critical value p_c for $t = t_2$, that is $\bar{T}(b_l, z(t_2), h(t_2), g(t_2)) = T_f$.

If $p_w(b_l, t_2)$ is simply a minimum point, the pressure $p_w(b_l, t)$ will verify (2.6) also for $t > t_2$ (i.e. $\bar{T} < T_f$ for $t > t_2$). If, on the contrary, we have

$$\dot{z}_T(t) > -\frac{\rho_w K_0 p_c}{\rho_i b_l}, \quad t > t_2, \tag{4.25}$$

then the constrain $p_w(b_l, t) \geq p_c$ is no longer satisfied.

In that case, one examines the possibility that a new frost penetration process will occur for $t > t_2$.

Actually, we are in the case (4.7) of proposition 4.1, setting the initial time at $t = t_2$.

Hence, for $t = t_2$ a second process of frost penetration starts. The development of such a process is exactly as we already described for the first one. It is evident that choosing opportunely the boundary temperatures h and g we can simulate a process where the formation of pure ice layers alternates with the penetration of the freezing front towards the base of the soil. The solution of (2.11)-(2.17) is constructed by attaching the *LF* solutions and the *FP* solutions, since the regularity required is preserved.

The frost heave process stops when one of the following circumstances occurs:

i) the boundary $z_S(t)$ tends asymptotically to the base of the soil $z_S = 0$, as it happens in the example following (4.23);

ii) there is a time

$$\tau_2 > 0 \text{ such that } T_S(\tau_2) = \bar{T}(z_S(\tau_2), z_T(\tau_2), h(\tau_2), g(\tau_2)) > T_f$$

and $z_T(\tau_2)$ has a maximum (in that case the last process is lens formation).

The graph in fig. 4.1 shows the simulation of a transition process where two lenses form, alternating with the penetration of the freezing front. The system (2.11)-(2.13) has been solved by using *Mathematica* software and imposing the following boundary temperatures:

$$\begin{cases} h(t) = e^{-t} + 1, & t \geq 0 \\ g(t) = -5e^{\frac{1}{4}t^4} - \frac{8}{3}t^3 + \frac{19}{2}t^2 - 12t, & t \geq 0 \end{cases} \tag{4.26}$$

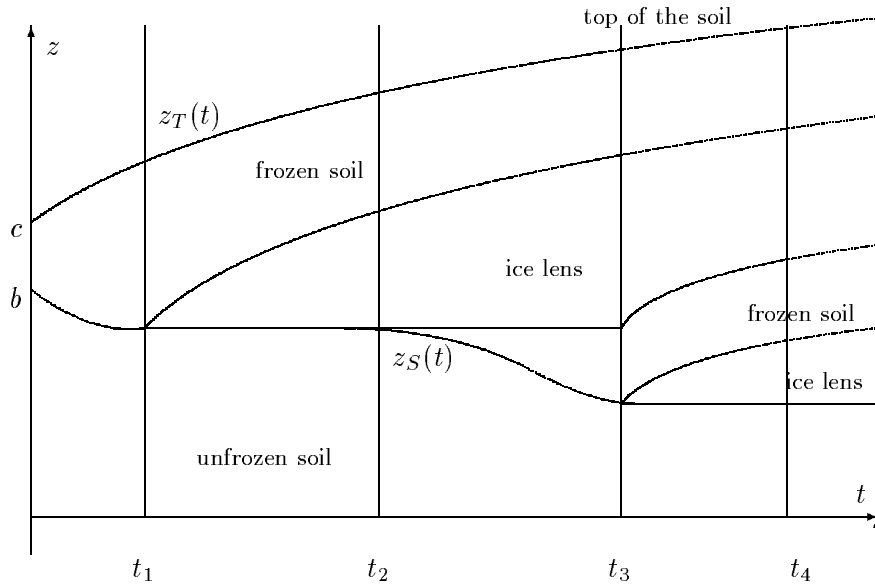


Figure 4.1: a transition process obtained by a numerical simulation. The lower moving boundary is the freezing front z_S , the upper one is the height of the soil z_T . The intermediate parts are formed by pure ice (lenses). In the interval of time $[0, t_1]$ a first process of frost penetration takes place; for $t = t_1$ the growth of the first lens starts. When $t = t_2$ the front penetrates again through the soil, until $t = t_3$, when it is $\dot{z}_S(t_3) = 0$ and the formation of the second lens starts. For $t > t_4$ the last process of frost penetration occurs. The diagram has been plotted by solving numerically the systems (2.11)-(2.17) and (4.22), once the boundary temperatures have been assigned as in (4.26).

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