An Introduction to the Microlocal Analysis of Hypercomplex Functions

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SOMMARIO. - *Questo lavoro fornisce una panoramica sullo stato dell'arte nel settore dell'analisi microlocale di funzioni ipercomplesse e sul tipo di problemi studiati in questo contesto. L'introduzione è essenzialmente autonoma.*

SUMMARY. - *We provide a survey on the state of the art in the area of microlocal analysis of hypercomplex functions and on the kind of problems which appears in this context. We intend to propose an essentially self-contained introduction.*

1. Introduction

A key result in the classical theory of analytical functionals is the duality theorem (algebraic) which associates to every analytical functional (with compact real carrier K) a function, the Fantappié indicatrix, holomorphic in $C \setminus K$, [Fa].

Köethe, in [Ko] has subsequently extended this same result to the realm of topological vector spaces. Later it has been observed (essentially by Martineau) that this duality theorem actually establishes an isomorphism between the space B_K of the real hyperfunctions which have support in the compact set K and the space $(A(K))'$ (of analytic functionals carried by K).

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P. Levy, in his classic [L], and independently Fichera in [Fi], proposed the construction of an analogous theorem for the case in which the holomorphic functions are substituted by regular functions of a quaternionic variable.

Gentili, Struppa and the author in the recent [FGS], have shown how this is possible and have constructed a quaternionic analogue of the sheaf of hyperfunctions; this sheaf is used, specifically, to demonstrate a Köethe duality theorem for the quaternionic case. The same type of approach used in [FGS] has been employed by the author in a recent work [F], in which monogenic functions with values in a Clifford algebra from the point of view of sheaf of theory have been considered. Specifically the author constructs a sheaf of boundary values of monogenic functions.

In this work we have gathered many basic results in this field, so that it can be considered as a first step for the systematic study of Clifford hyperfunctions.

The interest in such an approach is also justified by several elementary problems in the theory of ordinary differential operators which act upon such hyperfunctions. These problems, specifically, lead us to an examination of the localization of the singularities of such objects, and, therefore, to the so-called microlocalization of the sheaves which define them (cf. [KKK], [SKK]).

It would therefore seem opportune, at this point, to provide a "survey" on the state of the art in this sector, and on the kind of problems which have to be confronted. In this work, we intend to propose a completely autonomous introduction, in the hope that this will stimulate the research in this sector, which seems to us to be promising.

We shall now describe the structure of this work: the second section is dedicated to a quick introduction to Clifford Algebras and to monogenic functions. The case of quaternions and of regular functions are treated as a special case. In the tird section the sheaf of regular functions will be studied while in the fourth the sheaf of monogenic functions will be studied. Section five contains some new and exciting results which point to a generalization of the theory to the case f several variables. Section six is dedicated to the microlocalization of the sheaves introduced in the two preceding chapters and to some applications to the study of ordinary differential equations. Finally, we conclude the paper with an analysis of some recent work of Sommen in a very related area [S1] [S2]. The author is grateful to the referee for pointing out some important references.

2. Introduction to monogenic functions on Clifford Algebras

In this paragraph we shall introduce monogenic functions first in the space of quaternions and later on in Clifford algebras. The standard reference is [BDS]. With $\mathcal{H}$ we denote the algebra of quaternions, with the standard base $\{1, i, j, k\}$ in which i, j, k are immaginary units ($i^2 = j^2 = k^2 = -1$) and $ij = k$, $jk = i$, $ki = j$, $ji = -k$, $kj = -i$, $ik = -j$.

We shall denote the generic quaternion $q \in \mathbb{H}$ by

$$q = x_0 + ix_1 + jx_2 + kx_3 .$$

In the following $\mathcal{H}$ will have the Euclidean topology of R^4. We define the right and left Cauchy-Feuter operators by:

$$\frac{\partial_l}{\partial \bar{q}} = \frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3}$$
$$\frac{\partial_r}{\partial \bar{q}} = \frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_1}i + \frac{\partial}{\partial x_2}j + \frac{\partial}{\partial x_3}k .$$

Definition 2.1. Let $U \subset \mathbb{H}$ an open set, $f : U \to \mathbb{H}$ a differentiable function in the real sense. We say that f is left regular if it satisfies

$$\frac{\partial_l f}{\partial \bar{q}} = 0 ,$$

and we say that f is right regular if

$$\frac{\partial_r f}{\partial \bar{q}} = 0 .$$

Remark 2.1. The simplest examples of regular functions are the regular polynomials such as

$$(x_0 - ix_1), \ (x_0 - ix_2) ,$$

and their powers. On the other hand, the space of regular functions enjoys some surprising properties, such as the fact that the product of two regular functions is not regular (see Remark 2.1), and so, for example, $f(q) = q^n$ is not regular as well.

We shall denote with $R_l(U)$ and $R_r(U)$ respectively the spaces (on which H acts by external multiplication) of left and right regular on U.

If U is an open set in H, we shall indicate with $\mathcal{E}^p(U)$ the vector space of the C^∞ p-forms on U with real values. The elements of $\mathcal{E}^P_H(U) := \mathcal{E}^P(U) \otimes_R H$ will be called C^∞ p-forms with values in H.

We shall introduce some forms which will be useful later:

$$\begin{aligned} dq &= dx_0 + idx_1 + jdx_2 + kdx_3 \\ d\bar{q} &= dx_0 - idx_1 - jdx_2 - kdx_3 \\ Dq &= dx_1 \wedge dx_2 \wedge dx_3 - idx_0 \wedge dx_2 \wedge dx_3 - jdx_0 \wedge dx_1 \wedge dx_3 + \\ &\quad - kdx_0 \wedge dx_1 \wedge dx_2 \,, \\ \vartheta &= dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3 \,. \end{aligned}$$

It can be noticed that $dq \in \mathcal{E}^1_H(H)$, $Dq \in \mathcal{E}^3_H(H)$, while $\vartheta \in \mathcal{E}^4_H(H)$ is the canonical volume form, which satisfies the following identity:

$$d\bar{q} \wedge Dq = -Dq \wedge dq = 4\vartheta \,.$$

REMARK 2.2. The product of two regular functions is not necessarily regular, indeed

$$\frac{\partial (fg)}{\partial \bar{q}} = \frac{\partial f}{\partial \bar{q}} g + f \frac{\partial g}{\partial x_0} + if \frac{\partial g}{\partial x_1} + if \frac{\partial g}{\partial x_2} + kf \frac{\partial g}{\partial x_3} \,.$$

The theory of regular functions in one variable has been developed in close analogy with the classical theory of one complex variable.

In particular, it is possible to prove results which are in the same spirit as the usual Cauchy formulas, where the customany integration over a closed path is replaced by integration over a closed 3-manifold.

THEOREM 2.1. (Cauchy-Feuter I) *Let $f, g : U \subset H \to H$ be two C^1 functions such that $f \in R_l(U)$, $g \in R_r(U)$. If V is an open set with piecewise smooth boundary, and $\overline{V} \subset U$, we have*

$$\int_{\partial V} gDqf = 0 \,.$$

Specifically, if a C^1-function $f : U \to H$ is left regular then

$$\int_{\partial V} Dqf = 0$$

for each open set V.

THEOREM 2.2. (Cauchy-Feuter II) *Let $f : U \subseteq H \to H$ be a C^1, left regular function, and let $V \subseteq U$ be as above. Then for any $q_0 \in V$, and for*

$$G(q) = \frac{q^{-1}}{|q|^2} = \frac{\overline{q}}{|q|^4}$$

we have

$$f(q_0) = \frac{1}{2\pi^2} \int_{\partial V} G(q - q_0) Dq f(q) \, .$$

THEOREM 2.3. (Morera) *Let $f : U \subseteq H \to H$ be a continuous function such that*

$$\int_{\partial V} Dqf = 0$$

for every domain V with piecewise smooth boundary and $\overline{V} \subset U$. Then $f \in R_l(U)$.

The other area in which the theory of regular functions mimiks the theory of holomorphic functions is the one concerned with expansion in power series. In this case, however, the difficulties due to the lack of commutativity make the computations and the results more unpleasant. We therefore need a bit of notations.

Let $m \geq 0$ be an integer and let σ_m be the set of all the triples $\nu = [m_1, m_2, m_3]$ of non negative integers so that $m_1 + m_2 + m_3 = m$; set $e_0 = 1$, $e_1 = i$, $e_2 = j$, $e_3 = k$, and define

$$P(q) = \frac{1}{m!} \sum (x_0 i_{\lambda_1} - x_{\lambda_1}) \cdots (x_0 i_{\lambda_m} - x_{\lambda_m})$$

where the summation is extended to all the m-tuples $(\lambda_1, \ldots, \lambda_m)$ such that $1 \leq \lambda_1, \ldots, \lambda_m \leq 3$ and so that the number of the λ_j which is equal to h is exactly m_h for $h = 1, 2, 3$.

We can observe that the polynomials P_ν are homogeneous of degree of m over R. Moreover, each P_ν left regular.

As a consequence of Theorem 2.2 one has

THEOREM 2.4. *Let $f : U \subseteq H \to H$ be a regular function. Let $p \in U$ and set $\delta = \text{dist}\,(p, \partial U)$.*
Then, if $|q_0 - p| < \delta$ we have

$$f(q_0) = \sum_{m=0}^{\infty} \sum_{\nu \in \sigma_m} P_\nu (q_0 - p) q_2 \, ,$$

where

$$P_\nu = \frac{1}{2\pi^2} \int_{|q-p|=\delta} G_\nu(q-p) Dq f(q) \,.$$

REMARK 2.3. Unfortunately it is not possible to obtain a "good" extension of Theorem 2.4 to the case of more variables and thus no expansion in powers series is known for regular functions on H^n, at least for the moment. It is also true however, that every regular function is harmonic and thus real analytic with quaternionic variables on $H^n = R^{4n}$; thus a powers series exists, but in the 4^n real variables not necessarity in the n quaternionic variables.

We shall now introduce monogenic functions with values in a Clifford's algebra.

Let $V_{n,s}(0 \leq s < n)$ be a real n-dimensional linear space with a base $(e_1, \ldots, e_n)$ and a bilinear form (V, W), for $< V, W >$ in $V_{n,s}$, such that

$$\begin{aligned} (e_i, e_j) &= \;\; 0 \;\text{ for } i \,\text{different from}\, j \,, \\ (e_i, e_i) &= \;\; 1 \;\text{ for } i = 1, \ldots, s \,, \\ (e_i, e_i) &= -1 \text{ for } i = s+1, \ldots, n \end{aligned}$$

Considering the 2^n dimensional linear space $C(V_{n,s})$ with basis given by $\{e_A : A = (h_1, \ldots, h_r),\, 1 \leq h_1 < \ldots < h_r \leq n\}$, we can define the product

$$e_a \cdot e_B = (-1)^{\#(A \cap B) \setminus S} (-1)^{p(A,B)} e_{A \Delta B}$$

where S is the set $\{1, 2, \ldots, n\}$, $\#(X)$ indicates the cardinality of a set X, and

$$p(A, B) = \sum_{j \in B} p(A, j) \,, \qquad p(A, j) = \#\{i \in A;\, i > j\} \,.$$

DEFINITION 2.2. The real associative (not commutative) algebra $C(V_{n,s})$ is called the Universal Clifford Algebra over $V(n, s)$. When $s = 0$, we obtain the so-called Clifford Algebra which is denoted by A_n or simply A when no confusion can arise.

Let C_p be the linear subspace formed by all the products e_A, with $\#A = p$, whose elements are called p-vectors.

We shall define the elementary operations of inversion and of transposition over $C(V_{n,s})$ as follows:

– for each vector v in C_p, we shall define its inversion by

$$v^* = (-1)^p v\,.$$

– for each product e_A, we shall define its transposition by

$$(e_A)^t = (-1)^{(\#A-1)\#A\backslash 2 e_A}\,.$$

DEFINITION 2.3. We shall define the involution, combination of the inversion and the transposition previously defined, as

$$e_A = (e_A)^{*t}$$

for each vector e_A in the base, then extend it linearly to every Clifford number, expressed as a combination of these products.

In a natural way, it is also possible to define the conjugation of a function $f : U \to A$ from an open set U of R^{m+1} with values in Clifford Algebra A.

DEFINITION 2.4. Let $x = (x_0, \ldots, x_m)$ be the generic point of R^{m+1}. The Cauchy-Riemann operator on the space of Clifford Algebra valued functions is defined as

$$D = e_0 \partial/\partial x_1 + \cdots + e_m \partial/\partial x_m\,.$$

REMARK 2.4. Since a Clifford Algebra is not commutative Df and fD are different functions. We can also define a conjugated operator $\overline{D}$, in an obvious way, substituting every e_i, with its conjugated form.

DEFINITION 2.5. A function f in $C^1(U, A)$ is called left monogenic in U if, on U, $Df = 0$.
A 1-form differential with values in A, which we will frequently use in the sequel is given by

$$ds = e_0 dx_0 - e_1 dx_1 + \cdots + (-1)^m e_m dx_m\,.$$

THEOREM 2.5. *Let f and g respectively, be right and left monogenic on the open set U. Then, for each $(m+1)$-chain c contained in U one has*

$$\int_{\partial c} f ds g = 0\,.$$

Precisely, let E be the so-called Cauchy kernel defined in $R^{m+1} \setminus (0)$ by

$$E(x)\frac{1}{\omega_{n+1}}\frac{\overline{x}}{|x|^{m+1}} ,$$

where

$$\omega_{n+1} = 2\pi^{(m+1)/2}\frac{1}{\Gamma(\frac{m+1}{2})}$$

indicates the area of the unitary sphere S^m in R^{m+1}. The function E is obviously both a right and a left fundamental solution for the operator D.

THEOREM 2.6. (Cauchy's integral formula) *Let f be a left-monogenic function over U and left c be a compact $(m+1)$ dimensional orientated differentiable variety contained in U.*

a) *se* $x \in int(c) \quad \int_{\partial c} E(y-x)dsf(y) = f(x)$

b) *se* $x \in U \setminus c \quad \int_{\partial c} E(y-x)dsf(y) = 0.$

THEOREM 2.7. (Morera's Theorem) *A function f is left monogenic in an open set U if and only if it is continuous in U and if it satisfies*

$$\int_{\partial I} dsf = 0$$

for every closed hypercube I in U.

3. The sheaf of regular functions and their boundary values

In this chapter we shall tackle the study of the sheaf $\mathcal{R}_l$ of left regular functions of a quaternionic variable. We begin with some notations: If U is an open set of H^n

- $\mathcal{R}_l(U)$ (respectively $\mathcal{R}_l(U)$ denotes the right-sided vectoral space (left) of the regular left-sided functions over U.
- $\mathcal{R}_l$ denotes the presheaf $\{\mathcal{R}_l(U)\}, U \subset H^n$ with the usual restriction applications

$$\rho_{UV} : \mathcal{R}_l(U) \to \mathcal{R}_l(V) , \quad V \subseteq U$$

– $H^j(U, \mathcal{R}_l)$ will indicate the j-th cohomology group of U with coefficients in the sheaf $\mathcal{R}_l$.

It is well-known that

$$H^0(U, \mathcal{R}_l) = \Gamma(U, \mathcal{R}_l) = \mathcal{R}_l(U)\,.$$

The result that follows is a generalization on the well-known Mittag-Leffler theorem, and is demonstrated analogously.

THEOREM 3.1. *Let U be an open set in H. Then*

$$H^1(U, \mathcal{R}_l) = 0\,.$$

REMARK 3.1. In general this theorem is false for open set sets in H^n, $n > 1$. So if $U \subseteq H^n$ and K is a compact set contained in U, it can be demonstrated that

$$H^1(U \setminus K, \mathcal{R}_l) \neq 0\,.$$

We will discuss this phenomenon in more detail in Section 5.

Now we shall focus our attention on a single quaternionic variable and define a sheaf of boundary values of regular functions.

To justify this approach we shall begin by giving a quaternionic version of Painlevé's theorem which is a simple consequence of the quaternionic version of Morera's theorem.

Let $H^+ = \{q \in H\ : x_0 > 0\}\,, H^- = \{q \in H\ : x_0 < 0\}$ and

$$\tilde{H} = \{q \in H\ : x_0 = 0\}\,.$$

THEOREM 3.2. *Let U be an open set H and let $U^+ = U \cap H^+$, $U^- = U \cap H^-$; if $F \in \mathcal{R}_l(U^+ \cup U^-)$, $F \subset C^0(\overline{U})$ then F is regular over U.*

DEFINITION 3.1. For U open set in $\tilde{H}$ and V open set in H so that U is relatively closed in V, we shall define the space of left H-hyperfunctions over U as

$$\mathcal{F}(U) = \frac{\mathcal{R}_l(V \setminus U)}{\mathcal{R}_l(V)}\,.$$

It can be observed that (in virtue of Theorem 3.2) the notion of H-hyperfunction exactly represents the difference of boundary values of a function $F \in \mathcal{R}_l(V \setminus U)$ along the boundary $\tilde{H}$.

The definition of $\mathcal{F}(U)$ does not depend on the choice of the open set V, as it can be shown using the Mittag-Leffler Theorem in the standard way (Theorem 3.1).

We can now consider the natural presheaf defined over $\tilde{H}$ by associating to each open set U in $\tilde{H}$ the space $\mathcal{F}(U)$. Using again the Mittag-Leffler Theorem we obtain the following result:

Theorem 3.3. *The presheaf* $\{\mathcal{F}(U)\}$ *is a flabby sheaf.*

If K is a compact set in $\tilde{H}$ and Ω is an open set in $\tilde{H}$ we shall denote with $\mathcal{F}_K(\Omega)$ the space of H-hyperfunctions over Ω whose support is contained in K.

Definition 3.2. Let $f \in \mathcal{R}_l(\Omega)$ and let $F \in \mathcal{R}_l(V \setminus K)$ be defining function for f. We shall define the integral of f over Ω as

$$\int_\Omega Dqf = \int_\Gamma DqF$$

where Γ is a closed smooth three-dimensional boundary variety of a quadridimensional set Σ diffeomorphic to a ball containing K and such that $\Gamma \subset V$.

The classical theory of hyperfunctions shows that the space B_K (R) can be naturally identified with the space of analytic functionals carried by K. It is now possible to prove a similar result for the case of H-hyperfunctions.

Definition 3.3. Let K be a compact set in H. We define the space of germs of right H-analytic functions on K by setting

$$G(K) = \operatorname*{ind\,lim}_{\substack{U \supset K \\ U\,\text{open}}} \mathcal{R}_r(U)$$

and endowing $G(K)$ with its natural topology of the inductive limit.

The classic Fantappié-Köthe-Martineau-Sato duality theorem can be generalized as follows:

THEOREM 3.4. *Let K be a compact set in H and let V be an open set containing K. Then*

$$((G(K))' \equiv \frac{\mathcal{R}_l(V \setminus K)}{\mathcal{R}_l(V)}$$

where $((G(K))'$ denotes the space of left continuous H-linear funciotnals over $G(K)$

COROLLARY 3.1. *If K is a compact set in $\tilde{H}$, then $(G(K))' \equiv \mathcal{F}_K(H)$.*

We note that this last theorem, together with his corollary, is a generalization of the classic treatment of analytic functionals to the case of regular functions of a quaternionic variable. Note moreover, that, as already happens for analytical functionals, the spaces $\{G'(R)\}$ do not define a sheaf, and therefore it is not possible to spreak properly of support of an H-functional.

4. The sheaf of monogenic functions and their boundary values

With the same approach, considering the monogenic functions with values in a Clifford algebra from the point of view of sheaf theory, we now introduce a sheaf of boundary values of monogenic functions with values in a Clifford algebra.

Let $U \subseteq \mathbb{R}^{n+1}$ be an open set, let A be a Clifford algebra and denote by $LM(U, A)$ the right-sided A-module of all left monogenic functions from U in A.

DEFINITION 4.1. We shall indicate by $\mathcal{LM}$ the presheaf $\{LM(U, A)\}$, for U in $\mathbb{R}^{m+1}$, with the usual restriction applications

$$\rho_{UV} : LM(U, A) \to LM(V, A) \qquad V \subseteq U\,;$$

$\mathcal{LM}$ defines a complete presheaf and, therefore, a sheaf on $\mathbb{R}^{m+1}$.

It can be noted that the stalk of $\mathcal{LM}$ ay the generic point x in $\mathbb{R}^{m+1}$ is not a ring but only an abelian group with respect to the usual sum of the functions.

For each open set U in $\mathbb{R}^{m+1}$ we shall indicate with $H^j(U, \mathcal{LM})$ the j-th cohomology group of U with coefficients in the sheaf $\mathcal{LM}$.

As it is well known the 0th-cohomology group is the group of the global sections of the sheaf, which coincides with the left A-module of the monogenic functions over $\mathbb{R}^{m+1}$; such a space is also known as the space of left entire functions. The following version of the Mittag-Leffler Theorem is valid in this case.

THEOREM 4.1. *Let U be an open set in $\mathbb{R}^{m+1}$. Then*

$$H^1(U, \mathcal{LM}) = 0\,.$$

We shall now try to characterize the boundary values of monogenic functions.

We can write the coordinates in the $(m+1)$-dimensional open set as $x = (x_0, \ldots, x_m)$ and set $U_0 = U \cap (x_0 = 0)$ for every open set U in $\mathbb{R}^{m+1}$.

DEFINITION 4.2. An open neighborhood U of U_0 in $\mathbb{R}^{m+1}$ is called 0-normal if for each x in U the intersection of U with the segment $(x + te_0)$ is connected and contains exactly one point of U_0.

Given a set U, we define, $U^+ = \{x \in U \ : x_0 > 0\}$ and $U^- = \{x \in U \ : x_0 < 0\}$.

The Clifford algebra version of Painlevé's Theorem is thus

THEOREM 4.2. *Let U be an open set in $\mathbb{R}^{m+1}$. If a function F belongs to $\mathcal{LM}(U_+ \cup U_-)$ and is continuous over $\overline{U}$, then F is monogenic to the left over U.*

DEFINITION 4.3. Let V be an open set in $(x_0 = 0)$, U a 0-normal neighborhood of V. We define the A-module of Clifford hyperfunctions by V

$$\mathcal{F}(V) = \frac{\mathcal{LM}(U \setminus V)}{\mathcal{LM}(U)}\,.$$

It can be easily shown that $\mathcal{F}(V)$ constitutes a flabby sheaf over the space $(x_0 = 0)$, which allows a definition of the notion of support for a Clifford hyperfunction.

Definition 4.4. If K is a compact set in $(x_0 = 0)$ and V is an open set in $(x_0 = 0)$, we shall denote with $\mathcal{F}_K(V)$ the space of the Clifford hyperfunctions over V whose support is contained in K.

Definition 4.5. Let $f \in \mathcal{F}_K(V)$ and let

$$F \in \mathcal{LM}(V \setminus K, A)\,.$$

Then the integral of f over V can be defined by

$$\int_V f(x)dx = \int_\Gamma dsF$$

where Γ is a smooth, closed m-dimensional variety diffeomorphic to a ball containing K and such that $\Gamma \subset U$.

Now we shall introduce a sheaf of "real monogenic" functions as the inductive limit over $(x_0 = 0)$ of monogenic functions over $\mathbb{R}^{m+1}$.

Definition 4.6. Let K be a compact set in $\mathbb{R}^{m+1}$. We define the space of the germs of right Clifford analytical functions over K by

$$G(K) = \operatorname*{ind\,lim}_{\substack{V \supseteq K \\ V\,\text{open}}} \mathcal{RM}(V, A)$$

and giving $G(K)$ the usual inductive limit topology.

The following result is a generalization of the Köethe duality theorem and of Theorem 3.4.

Theorem 4.3. *Let K be a compact set in $\mathbb{R}^{m+1}$ and let V be an open set containing K. Then*

$$((G(K))' = \frac{LM(V \setminus K, A)}{LM(V, A)}\,.$$

Corollary 4.1. *If K is a compact set in $(x_0 = 0)$ then $(G(K))' = \mathcal{F}_K(\mathbb{R}^m)$.*

It is known that the product of two monogenic functions is not necessarily monogenic because of the non-commutativity of A. However it is possible to define a different notion of product which makes A an algebra. With this aim, we shall introduce the so-called C-K product (the Cauchy-Kowalewski product).

Let x be a point in $(x_0 = 0)$ and let f be a monogenic function in an open set U which contains x, we denote the boundary values of f in x from above and from below with $f(x+0)$ and with $f(x-0)$ respectively. For each open set V in $(x_0 = 0)$ and for each function $f : V \to A$ there exists a 0-normal open neighborhood U of V and a unique function f^*, left monogenic in U, such that $f^*(x+0) = f(x)$; the proof of this general fact is given in [BDS].

Definition 4.7. Let V be an open set in $(x_0 = 0)$ and let f be an A-valued analytic function A. The maximal left monogenic extension f^* of f, described earlier, is called left C-K extension of f.

Let f and g be left entire functions (but the same can be done in the case where f and g are left monogenic functions over some open set which has a non-empty intersection with $x_0 = 0$); then the restrictions of f and of g to $(x_0 = 0)$ are A-valued analytical functions $(x_0 = 0)$ A and so their product.

Such a product therefore a left C-K entire extension: this extension, indicated by $f\#g$ is called C-K product of f and g.

In [BDS] it is shown that $LM(\mathbb{R}^{m+1}, A)$, supplied with the C-K product, is a real algebra.

This product coincides with the usual product of two holomorphic functions when we take $m = 1$ and when A is the algebra of the complex numbers.

Now we proceed to discuss ordinary differential linear equations with A-valued analytical coefficients.

Observe that by writing $f = [F]$ we indicate that f is a Clifford hyperfunction and that F is a left monogenic function which represents f (in a non-unique way) in the quotient

$$\mathcal{F}(V) = \frac{LM(U \setminus V)}{LM(U)}.$$

It is then obvious that Clifford hyperfunctions can be added by just adding the corresponding representatives.

More interesting is the fact that differential operators can be defined in an analogous way.

If x_i $(i = 0, \ldots, m)$ are real variables in $\mathbb{R}^{m+1}$, then it can be shown that the C-K extension of the later m coordinates are

$$z_i = x_i e_0 - x_0 e_i \qquad i = 1, \ldots, m\,,$$

and we can use this fact to extend the differential operators from the space $(x_0 = 0)$ to all $\mathbb{R}^{m+1}$. Precisely the operator $\frac{\partial}{\partial x_i}$ is extended in a natural way to the operator $\frac{\partial}{\partial z_i}$ defined by

$$\frac{\partial}{\partial z_i} = \left(\frac{\partial}{\partial x_i}\right) e_0 - \left(\frac{\partial}{\partial x_0}\right) e_i\,.$$

On the other hand, each analytic function f on an open set in $(x_0 = 0)$ extends (in an unique way) to its C-K extension which we have denoted with f^*. We can then consider linear operators which act on the sheaf of the Clifford hyperfunctions over an open set V as follows: if

$$P = P\left(\frac{d}{dx}, x\right) = \sum_{|I| \leq m} a_I(x) \frac{d^I}{dx^I}$$

where $\frac{d}{dx} = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m}\right)$ and $a_i(x)$ are A-valued analytic functions, then

$$P\left(\frac{d}{dx}, x\right) f = \left[P\left(\frac{d}{dz}, z\right) \# F\right]\,,$$

where $z = (z_1, \ldots, z_m)$, the symbol $\#$ denotes that we have considered the C-K product and $\frac{d}{dz} = \left(\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_m}\right)$.

We use this definition to show how some things become simple in this environment.

THEOREM 4.4. *Let V be an open subset in $(x_0 = 0)$. If f belongs to $\mathcal{F}(V)$ then there exists g in $\mathcal{F}(V)$ such that*

$$P\left(\frac{d}{dx}, x\right) g = f\,.$$

THEOREM 4.5. *Let W be an open subset of the open set V, contained in $(x_0 = 0)$. If f belongs to $\mathcal{LM}(V)$, then every solution h in $\mathcal{LM}(V)$ to the equation $P\left(\frac{d}{dx}, x\right) g = f$ can be extended to a solution g in $\mathcal{LM}(V)$ to the same equation.*

5. The Cauchy-Fueter complex

In a recent work [ABLSS], some new important have been obtained by studying regular functions of several quaternionic variables as solutions of a specific system, the so-called Cauchy-Fueter system.

The idea, which goes back to Ehrenpreis Malgrange and Palamodov, [E], [PA], but see also [S] and [K], consists in looking at a regular function as being a differentiable function of $4n$ real variables, which satisfies an overdetermined system of differential equations, i.e. the Cauchy-Fueter system.

In particular, a function $f : \mathbb{H}^n \to \mathbb{H}$ can be seen as a four-vector of infinitely differentiable functions $f = (f_0, f_1, f_2, f_3)$ which satisfies, for every variable q_t $(t = 1, \ldots, n)$ the system

$$\frac{\partial f}{\partial \overline{q}_t} = 0\,.$$

However, by writing explicitly the expression for the Cauchy-Fueter operator and for $f = f_0 + if_1 + jf_2 + kf_3$, we obtain that a regular function of several quaternionic variables is the solution of

$$\begin{bmatrix} \frac{\partial}{\partial x_0} & -\frac{\partial}{\partial x_1} & -\frac{\partial}{\partial x_2} & -\frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_0} & -\frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_0} & -\frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_3} & -\frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_0} \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

This fact gives us immediately an exact sequence of sheaves as given below:

$$0 \to \left(\mathcal{E}^4\right)^P \hookrightarrow (\mathcal{E})^4 \xrightarrow{P} (\mathcal{E})^{4n} \tag{5.1}$$

where P denotes the system of partial differential equations written above and where we note that we could choose the sheaf $\mathcal{E}$ of infinitely differentiable functions because of the ellipticity of the Cauchy-Fueter system (see e.g. [BDS]).

The interesting problem, therefore, is to see in which sense it is possible to resolve the complex written above. This is a classical problem in the algebraic theory of systems and the interested reader is referred to [K] for details. What the authors have done in [ABLSS] is therefore an algebraic translation of the complex (5.1) which gives the following complex in which S denotes the ring of polynomials in $4n$ complex variables (complex (5.2) is essentially the Fourier transform of complex (5.1)):

$$0 \to S^4 \xrightarrow{A} S^{4n}\,. \tag{5.2}$$

In this complex, if we fix e.s. $n = 2$, the matrix A is of course given by

$$A = \begin{pmatrix} x_{01} & -x_{11} & -x_{21} & -x_{31} \\ x_{11} & x_{01} & -x_{31} & x_{21} \\ x_{21} & x_{31} & x_{01} & -x_{11} \\ x_{31} & -x_{21} & x_{11} & x_{01} \\ x_{02} & -x_{12} & -x_{22} & -x_{32} \\ x_{12} & x_{02} & -x_{32} & x_{22} \\ x_{22} & x_{32} & x_{02} & -x_{12} \\ x_{32} & -x_{22} & x_{12} & x_{02} \end{pmatrix}.$$

It is well known (see e.g. [M-S]) that an exact resolution of (5.2) can always be done if and only if the matrix A has torsion free cokernel, but in general it is not too easy to verify whether A has torsion free cokernel.

In [ABLSS] the authors have however found an important criterion which we state here:

LEMMA 5.1. *Suppose A is an $m \times n$ matrix of maximal rank m ($m \leq n$). Then coker(A) is torsion free iff the $n \times m$ minors are relatively prime.*

The specific case we have to deal with, the matrix A has maximal rank (as it is immrediately seen) and because of its special shape, the god of its 4×4 minors is one, so that its cokernel is indeeed torsion free.

An immediate consequence of this fact is (see [M-S]) the following result on the removability of compact singularities:

THEOREM 5.1. *Let be K compact, $K \subseteq \mathbb{H}^n$, $\mathbb{H}^n \backslash K$ connected, $n \geq 2$. If $f \in R(\mathbb{H}^n \setminus K)$ then there exists $\tilde{f} \in R(\mathbb{H})^n$, s.t. $\tilde{f} = f$ on $\mathbb{H}^n \setminus K$.*

This result, in particular, extends to the case of Clifford algebras, thus showing the lack of compact singularities for all monogenic functions of more than one variable.

By general algebraic analysis, we now know the existence of a matrix B such that:

$$0 \to S^4 \xrightarrow{A} S^{4n} \xrightarrow{B}$$

is exact at S^{4n}.

It is however impossible to explicitely write the form of such a matrix, or even its range, unless we use some computer algebra package. In [ABLSS] CoCoA was used to find out specific form for B when $n = 2$ and 3. In the case of $n = 2$, in particular, it can be shown that the complex can be completely resolved and one has

$$0 \to S^4 \xrightarrow{A} S^8 \xrightarrow{B} S^8 \xrightarrow{C} S^4 \to 0$$

where C is again a Cauchy-Fueter type system. More interestingly the sequence is exact at all levels (except of course that the last map is not surjective) so that one can claim that for the Cauchy-Fueter system both the first and the second Ext groups vanish. This, in particular, allows us to show the following fundamental result:

THEOREM 5.2. *Let $K \subseteq \mathbb{H}^2$ be a compact convex set such that*

$$\dim H^j(K, \mathcal{E}^c) < \infty\,, \qquad j = 1, 2$$

then

$$H^3(\mathbb{H}^2, \mathbb{H}^2 \setminus K; R) \cong (R(K))\,.$$

This last result shows how our own duality theorem (which we proved for one quaternionic variable) can be generalized at least to the case of two quaternionic variables.

It is of course of great interest to investigate how to utilize the methods described in [ABLSS] to obtain similar characterization for more than two and three variables. Let us conclude this section by noticing the similarity between these results and the usual resolution of the Cauchy-Riemann system (where both the first and the last operators which appear are indeed Cauchy-Riemann operators). The difficulty in extending these ideas to the case of the Cauchy-Fueter system consists in the lack of anything which could replace the Dolbeaut sequence which exists for the case of the holomorphic forms.

6. Microlocalization

In this section we introduce the study of the sheaf of the singularity of the H-hyperfunctions and we construct the natural sheaf of the H-microfunctions.

These sheaves have been extensively studied in [F] and [FGS].

Let us begin by giving some concrete examples of $\mathbb{H}$-hyperfunctions.

Let $F \in \mathcal{R}_l(\mathbb{H}^+)$. The function

$$\tilde{F}^+ = \begin{cases} F & \text{su } \mathbb{H}^+ \\ 0 & \text{su } \mathbb{H}^- \end{cases}$$

clearly belongs to $R(\mathbb{H} \setminus \tilde{\mathbb{H}})$, and in this way it defines an element in $\mathcal{F}(\tilde{\mathbb{H}})$ which represents the boundary value of F.

Using a notation which is classic in the theory of hyperfunctions, we can write

$$b_+F = [\tilde{F}^+] = F(ix_1 + jx_2 + kx_3 + 0)\,.$$

In an analogous way we can define

$$b_-F = F(ix_1 + jx_2 + kx_3 - 0) = [\tilde{F}^-]$$

for $F \in \mathcal{R}(\mathbb{H}^-)$, where

$$\tilde{F}^- = \begin{cases} O & \text{su } \mathbb{H}^+ \\ F & \text{su } \mathbb{H}^- \end{cases}$$

and it is clear that if $F \in \mathcal{R}(\mathbb{H} \setminus \tilde{\mathbb{H}})$, then

$$[F] = b_+F - b_-F\,.$$

If $f \in \mathcal{G}_r(\tilde{\mathbb{H}})$ or $f \in \mathcal{G}_r(U)$ for an open set U in $\tilde{\mathbb{H}}$, again we see that in a natural way f defines a $\mathbb{H}$-huperfunction over $\tilde{\mathbb{H}}$, or over U, which is obtained by considering a regular extension F a suitable 0-normal neighbourhood of U in $\mathbb{H}$.

Then there is a natural injection of sheaves

$$0 \hookrightarrow \mathcal{G} \overset{i}{\hookrightarrow} \mathcal{F}$$

where the sheaves are defined over the topological space $\tilde{\mathbb{H}}$. With the aim of studying the singularity of the elements of the sheaf $\mathcal{F}$, we shall introduce over $\tilde{\mathbb{H}}$ the sheaf $\mathcal{F}/i\mathcal{G}$.

As it is well-known, $\mathcal{F}/i\mathcal{G}$ is not necessarily a sheaf (the quotient of the sheaves is not necessarily a sheaf). In this specific case it is possible, however, to show that $\mathcal{F}/i\mathcal{G}$ is actually a sheaf.

However, it is known from the real analytic case, that the sheaf $\mathcal{F} \setminus i\mathcal{G}$ does not provide any new information on the singularities of hyperfunctions. The reason is that only by looking at their microlocal behaviour (i.e. their behaviour on the cotangent bundle) it is possible to derive substantial information.

Our next step will be to prove that the microlocalization procedure (i.e. the lifting up to the cotangent bundle) can be successfully performed even in the quaternionic case. We now lift our objects from $\tilde{\mathbb{H}}$ to $S^*\tilde{\mathbb{H}}$, its spherical cotangent bundle.

We think of $S^*\tilde{\mathbb{H}}$ as the topological space $(\tilde{\mathbb{H}}, +\infty) \coprod (\tilde{\mathbb{H}}, -\infty)$ with the induced topology of the bases $(I, +\infty) \coprod (J, -\infty)$ for I, J open sets in $\tilde{\mathbb{H}}$.

We wish to define a sheaf of quaternionic microfunctions over $S^*\tilde{\mathbb{H}}$ whose objects are the singularities of hyperfunctions.

In this regard we provide the following definition:

Definition 6.1. Let $x_0 \in \tilde{\mathbb{H}}$. A hyperfunction f defined in a neighborhood of x_0 is called microregular in $(x_0 + i\infty)$ if, in a (possibly different) neighborhood of x_0, f is the boundary value from above of a regular function F.

Definition 6.2. Let $F \in \mathcal{F}$. We shall define its singularity support $ss(f)$ as the set of the points of $S^*\tilde{\mathbb{H}}$ in which f is microregular.

Definition 6.3. For each open set $U \subseteq S^*\tilde{\mathbb{H}}$ we shall define

$$\mathcal{C}(U) = \frac{\mathcal{F}(\tilde{\mathbb{H}})}{\{f \in \mathcal{F}(\tilde{\mathbb{H}}) \, : ss(f) \cap U = \emptyset\}} .$$

Definition 6.3 implies that the section of $\mathcal{C}$ are the (microlocal) singularities of sections $\mathcal{F}$. Moreover, if $f \in \mathcal{F}(\tilde{\mathbb{H}})$, then it induces an element $[f] \in \mathcal{C}(S^*\tilde{\mathbb{H}})$ such that $\operatorname{supp}([f]) = ss(f)$.

It is now possible to prove that $\{U, \mathcal{C}(U)\}$ for all the open sets U in $S^*\tilde{\mathbb{H}}$, gives a sheaf over $S^*\tilde{\mathbb{H}}$, which we call the sheaf of quaternionic microfunctions.

THEOREM 6.1. *The following short sequence of sheaves on* $\tilde{\mathbb{H}}$ *is exact:*

$$0 \to \mathcal{G} \hookrightarrow \mathcal{F} \to \pi_* \mathcal{C} \to 0\,,$$

where π *is the canonical projective* $\pi : S^*\tilde{\mathbb{H}} \to \tilde{\mathbb{H}}$.

THEOREM 6.2. *The sheaf* $\mathcal{C}$ *is flabby.*

It can be observed that the flabbyness of $\mathcal{C}$ is important for applications of the theory of differential equations while the exactness of the short sequence is analogous to Sato's theorem on the microfunctions sheaf. A complete discussion of the sheaf $\mathcal{C}$ is too lenghty and complex to be given here.

7. Sommen's monogenic differential Calculus

No survey on microlocal analysis of hypercomplex functions would be complete without a discussion of the related (but not overlapping) work of Sommen.

Sommen's work in this area dates back to the late seventies early eighties when, in [S1] and in [S2], he developed a theory of hyperfunctions and microfunctions with values in a Clifford Algebra.

In his theory, Clifford hyperfunctions on $\Omega \subseteq \mathbb{R}^{n+1}$ are defined as elements of $M(\mathbb{R}^{n+1}\backslash\overline{\Omega})/M(\mathbb{R}^{n+1}\backslash\partial\Omega)$ for M the space of monogenic functions, but unlike what we described in [S2] approach, Sommen concentrated his analysis [S2] on the study of singularities of such objects. This allows the author to construct a natural theory of microfunctions in [S3]. We could say that the mathematical object in both Sommen's and our work is the same, but quite different is the spirit which guides the analysis. In particular, as shown in Section 5, our approach naturally leads to the study of several Clifford variables, unlike what happens in Sommen'case. On the other hand, Sommen's approach demonstrates its great relevance as it essentially leads to a theory of monogenic differential analysis.

While we refer the reader to [S3], [S4], and the literature there in for more details, we would like to give some general ideas on Sommen's calculus as he developed it, since we believe that it could be a source of great interest especially as it way lead to a new theory of quaternionic Clifford residues. As we have shown in our previous sections, one of the main obstacles to the development of an algebraic theory of the solutions to the Cauchy-Fueter system (and, more generally, to the Dirac equation) lies in the back of a Dolbeault-like resolution for the sheaf of monogenic functions. Our approach to this general problem is given in Section 5 and 6 of this survey.

Sommen, on the other hand, develops a monogenic cohomology theory. Let $F : \mathbb{R}^{m+1} \to 4$ be a A-values differential k-form for A a Clifford algebra; if $\partial_x = \sum_{j=1}^{m} e_j \partial_{x_j}$ is the Dirac operator, we say that F is (left) monogenic if $\partial_x F = 0$. Sommen uses for example these differential forms to establish a Cauchy-Pompein type formula for differential forms which generalizes the usual Cauchy formula for monogenic functions. But most interesting is the De Rham complex which resolves the space M_l of a monogenic O-forms as follows

$$0 \to M_l^0 \overset{d_0}{\to} M_l' \to \ldots \to M_l^m \overset{d_m}{\to} 0\,,$$

where M_l^j is the space of monogenic l-forms and $d_j = d/M_l^j$. If one now defines, [S1], the monogenic cohomology

$$\mathcal{H}_l^j = \frac{\operatorname{Ker} d_j}{\operatorname{Im} d_j}\,,$$

it is possible to show that the spaces so obtained are <u>not</u> isomorphic to the De Rham cohomology spaces.

This understanding of this isomorphism leads Sommen to the construction of three types of left monogenic k-forms:

$$Mc_l^k \qquad \text{(closed forms)}$$

$$Me_l^k \qquad \text{(exact forms)}$$

and

$$ME_l^k = \{\sigma \in M_l^k \,:\, \sigma = d_x \wedge G,\ \text{for}\, G\, \text{monogenic}\}\,,$$

where this last space is what Sommen calls the space of strongly exact forms, Sommen's main result establishes the following isomorphisms:

$$H^k \cong MC_l^k / Me_l^k$$

$$H_l^k \cong H_{DeRham}^k \oplus \mathcal{H}_{DeRham}^{k-1}$$

It should be noted how Sommen successfully used these results to further develop a duality theory for the space (M_r^{m-k}), i.e. the space of left k-currents. One should note here that the case of $k = m$ corresponds to monogenic functionals, which, as special cases, give compactly supported hyperfunctions. The reader should compare this with our version of Köethe's duality Theorem 3.4.

In particular, Sommen also shows how the space of these functionals is isomorphic to a space of monogenic differential forms which vanish at infinity (in perfect analogy with the well known Martineau's lectures in Lisbon, 1963). Let us conclude by pointing out that Sommen has further pushed his approach towards a theory of Clifford-Radon transform, but a discussion of this aspect would lead us two or for. We just refer the readers to [S5], [S4] and the literature there in.

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