Localization of the Solution of a One-Dimension One-Phase Stefan Problem

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Sommario. - *Studiamo la localizzazione, l'insieme dei punti di blow up ed alcuni aspetti della velocità di propagazione della frontiera libera di soluzioni di un problema di Stefan unidimensionale ad una fase.*

Summary. - *We study localization, the set of blow up points and some aspects of the speed of the free boundary of solutions of a one-dimensional, one-phase Stefan problem.*

1. Introduction

Consider the one-phase Stefan problem

$$\begin{array}{rcll} u_t & = & u_{xx} & \text{in} \quad 0 < x < s(t),\ 0 < t < T \\ -u_x(0,t) & = & h(t) & \text{on} \quad 0 < t < T \\ u(x,0) & = & u_0(x) & \text{on} \quad 0 \le x \le b \\ u_x(s(t),t) & = & 0 & \text{on} \quad 0 < t < T \\ -u_x(s(t),t) & = & \dot{s}(t) & \text{on} \quad 0 < t < T \\ s(0) & = & b & \end{array} \tag{1.1}$$

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Partially supported by FONDECYT under Grant 1940705.

We will assume throughout this note that u_0 is a nonnegative smooth function supported in $[0, b)$ and that h is a nonnegative, continuous function which is bounded on any interval $[0, T-\varepsilon)$ for $0 < \varepsilon < T$. We will also assume that $T < +\infty$.

Problem (1.1) has been studied by several authors. A proof of existence, uniqueness and regularity properties of the solution $(u(x,t), s(t))$ can be found in [CP].

It is well known that, under our conditions, the free boundary $s(t)$ of Problem (1.1) is a nondecreasing function of t. As such we can define $s(T) = \lim s(t)$ as $t \to T$. We say that the solution u is localized if $s(T) < \infty$.

In this note we study localization, the set of blow up points and some aspects of the speed of the free boundary of solutions of (1.1).

We need now some definitions. For any $h : [0, T) \to R^+ \cup \{0\}$ we define the function $\xi : [0, T) \to R^+ \cup \{0\}$ by

$$\int_0^t h(s)\, ds = \exp\left(\frac{\xi^2(t)}{4(T-t)}\right) - 1\,. \tag{1.2}$$

We also set

$$K = \limsup_{t \to T} \xi(t)\,. \tag{1.3}$$

For $w : [0, +\infty) \times [0, T) \to R$ we define the "blow up set " as

$$B(w) = \overline{\{x \in [0, +\infty) / w(x,t) \text{ is not bounded in } [0,T)\}} - \{0\}\,. \tag{1.4}$$

The reason for taking the point $x = 0$ off the set where w is not bounded will become clear in Section 4 where we deal with the case when the only possible point where u, the solution of (1.1), is unbounded is $x = 0$.

The rest of this paper is organized as follows. In Section 2 we give an exact characterization of the set of blow up points of v, where v is the solution of the classical linear problem

$$\begin{array}{rcll} v_t &=& v_{xx} & \text{in } [0,+\infty) \times [0,T) \\ -v_x(0,t) &=& h(t) & \text{on } [0,T) \\ v(x,0) &=& 0 & \text{on } [0,+\infty)\,. \end{array} \tag{1.5}$$

Actually it is proved that $B(v) = (0, K]$.
Section 3 is devoted to show that if u is the solution of (1.1) and v is the solution of (1.5), then the sets $B(u)$ and $B(v)$ coincide.

In Section 4 we deal with the special case when $K = 0$.
Finally in Section 5 we prove the localization result, which is an easy consequence of the results of Sections 2 and 3, and we study the speed of the free boundary of the solution of Problem (1.1).

2. The linear problem

In this section we characterize the set of blow up points of the solution of the classical linear problem (1.5).

The following proposition, that can be of interest in itself, will be needed later. Related but partial results in this direction can be found in [SGKM].

PROPOSITION 2.1. *Let v be the solution of problem (1.5) and recall that $K = \limsup \xi(t)$ as $t \to T$. Then $B(v) = (0, K]$.*

Proof. We define $V(x,t) = \int_x^{+\infty} v(y,t)\,dy$ and we observe that V satisfies

$$\begin{array}{rcll} V_t & = & V_{xx} & \text{in} \quad [0,+\infty) \times [0,T) \\ V(0,t) & = & \displaystyle\int_0^t h(s)\,ds & \text{on} \quad [0,T) \\ V(x,0) & = & 0 & \text{on} \quad [0,+\infty) \end{array} \tag{2.1}$$

It is well known, see [DM], that

$$\begin{aligned} 0 & \leq V(x,t) \\ & = \int_0^t V(0,s) \frac{\partial}{\partial x} P(x,t-s)\,ds + \int_0^{+\infty} V(y,0) P(x-y)\,dy \end{aligned} \tag{2.2}$$

where $P(x,t) = \frac{1}{(2\pi t)^{1/2}} \exp\left(-\frac{x^2}{4t}\right)$ is the familiar Gauss kernel.
We claim that $B(v) = B(V)$. Indeed, since under our assumptions $v(x,t)$ is non increasing as a function of x for a fixed t, it follows that $B(v) \subset B(V)$. As for the converse if $x_0 \notin B(v)$, then there exists $M > 0$ so that $v(x,t) \leq M$ for any $(x,t) \in [x_0,+\infty) \times [0,T)$. Let w be the solution of $w_t = w_{xx}$ in $[x_0,+\infty) \times (0,T)$, $w(x_0,T) = M$ on $[0,T)$ and $w(x,0) \equiv 0$ on $[x_0,+\infty)$. By the maximum principle $v \leq w$ and since, from the classical formulae, $\int_{x_0}^{+\infty} w(y,t)\,dy$ remains bounded

$\forall t \in [0,T)$, it follows that $x_0 \notin \mathrm{int}B(V)$. Hence $B(V) \subset B(v)$ and the claim is proved.
Our task now is to show that $B(V) = (0,K]$. In order to see this we first take $x > K$ and let $a = \frac{x-K}{2}$. There exists $t_0 > 0$ so that $\int_{t_0}^t h(x)\,dx \le \exp\left(\frac{(K+a)^2}{4(T-t)}\right)$ $\forall t \in [t_0,T)$. An application of formula (2.2) gives

$$\begin{aligned} 0 &\le V(x,t) \\ &\le \frac{1}{(2^3\pi)^{1/2}} \int_{t_0}^t \frac{\exp\left(\frac{(K+a)^2-x^2}{4(t-s)}\right)}{(t-s)^{3/2}}\,ds + \\ &+ \int_0^{+\infty} V(y,t_0)P(x-y,t-t_0)\,dy \end{aligned}$$

$\forall t \in [t_0,T)$. This implies that $B(V) \subset (0,K]$.
We prove the converse now. Take $x < K$ and let $\{t_n\}$ be an increasing sequence so that $t_n \to T$ as $n \to +\infty$ and $\xi(t_n) \to K$ as $n \to +\infty$. Let $\varepsilon > 0$ and fix $n \in N$. Set

$$U(x,t) = \int_x^{+\infty} \exp\left(\frac{\xi^2(t_n)}{4(T-t_n)}\right) P(y+\varepsilon, t-t_n)\,dy\,.$$

The function $U(x,t)$ satisfies $U_t = U_{xx}$ in $D_n = [0,+\infty) \times [t_n,T)$. Moreover $0 = U(x,t_n) \le V(x,t_n)$ $\forall x \in [0,+\infty)$ and $U(0,t) \le \exp\left(\frac{\xi^2(t_n)}{4(T-t_n)}\right) \le \int_0^t h(s)\,ds = V(0,t)$, $\forall t \in [t_n,T)$. By the maximum principle we obtain $U(x,t) \le V(x,t)$, $\forall (x,t) \in D_n$. Since ε is arbitrary it follows that

$$\frac{1}{(2\pi)^{1/2}} \int_x^{+\infty} \frac{\exp\left(\frac{\xi^2(t_n)-y^2}{4(T-t_n)}\right)}{(T-t_n)^{1/2}}\,dy \le \limsup_{t\to T} V(x,t) \quad \forall n \in N\,.$$

Consequently $(0,K] \subset B(V)$ and the proposition is proved. ◊

3. The set of blow up points

In this section we use the results of the previous one to characterize the set of blow up points of solutions of the Stefan problem. We first show the following result.

PROPOSITION 3.1. *If u is a solution of (1.1) then the set $B(u)$ is independent of the initial condition u_0.*

Proof. By pointwise comparison it is enough to prove that if u is the solution of (1.1) with a nondecreasing initial condition u_0 and $\bar{u}$ is the solution with $\bar{u}_0 = 0$ and $\bar{s}(0) = 0$, then $B(u) = B(\bar{u})$. Also by pointwise comparison we have $B(\bar{u}) \subset B(u)$. In order to prove the reverse inclusion we recall that

$$\int_0^t h(s)\, ds = \int_0^\infty u(y,t)\, dy + s(t) - \int_0^{+\infty} u_0(y)\, dy\,.$$

Hence

$$\int_0^{+\infty} (u - \bar{u})(y,t) dy = [\bar{s}(t) - s(t)] + \int_0^{+\infty} u_0(y)\, dy\,.$$

Since under our conditions $\bar{s}(t) \leq s(t)$ for any $t \in [0,T)$ we have that the right hand side of the above equation is bounded. Take $x_1 \in B(u)$ and $0 < x_0 < x_1$. Since $\bar{u} \leq u$ the last inequality implies the existence of $C > 0$ so that $\int_{x_0}^{x_1} (u - \bar{u})(x,t)\, dx \leq C$ for $t \in [0,T)$. Now as u and $\bar{u}$ are non increasing in x a moment of reflection shows that $x_0 \in B(\bar{u})$. Hence $B(u) \subset B(\bar{u})$ and the proposition is proved. $\diamondsuit$

REMARK. We note that the same argument of the above proof shows that the closure of the set $\{x \in (0,+\infty)/u(x,t) \to +\infty \text{ as } t \to T\}$ is also independent of the initial condition.

The following result states essentially that if u is a solution of (1.1) and $x \in (0,+\infty)$ then either $u(x,t) \to +\infty$ as $t \to T$ or $u(x,t)$ is bounded for $t \in [0,T)$.

PROPOSITION 3.2. *If u is the solution of problem (1.1) then*

$$B(u) = \overline{\{x \in (0,+\infty)/u(x,t) \to +\infty \ as\ t \to T\}} - \{0\}\,.$$

Proof. One inclusion is obvious. As for the other we observe that in view of Proposition 3.1 and the above remark we can assume that u_0 is non increasing. Since $h \geq 0$, by the maximum principle, $u(x,t)$ is non increasing in x for each fixed t. Now, let $x_0 > 0$ be such that $u(x_0,t)$ is not bounded and let $x_1 < x_0$. Our task is to prove that $u(x_1,t) \to +\infty$ as $t \to T$. To do this, let $M > 0$ and let $p(x,t)$ be the solution of $p_t = p_{xx}$ in $[0,x_0] \times [0,+\infty)$, $-p_x(0,t) = 0$, $p(x_0,t) = 0$ and $p(x,0) = 2M$. There exists t_1 such that $p(x_1,t) > M$, $\forall t \in [0,t_1)$. Since $u(x_0,t)$ is not bounded, we can find t_2 so that $T - t_2 < t_1$ and $u(x_0,t_2) > 2M$. Hence $u(x,t_2) > 2M$, $\forall x \in [0,x_0)$. A comparison argument with the function $p(x,t-t_2)$ in the region $[0,x_0] \times [t_2,T)$ ends the proof. ◊

PROPOSITION 3.3. *If u is a solution of problem (1.1) with $u_0 = 0$ and v a solution of (1.5) then $B(u) = B(v)$.*

Proof. It is well known that $u \le v$ therefore $B(u) \subset B(v)$. In order to prove the opposite inclusion we observe that

$$\int_0^t h(s)\,ds = \int_0^\infty v(y,t)\,dy$$

and

$$\int_0^t h(s)\,ds = \int_0^\infty u(y,t)\,dy + s(t)\,.$$

Consequently

$$\int_0^\infty [v(y,t) - u(y,t)]\,dy = s(t) \tag{3.1}$$

Now we distinguish two cases. If u is not localized, then $B(u) = (0,\infty)$. Indeed, if $u(x_0,t)$ were bounded for some x_0, then a comparison argument with a suitably chosen solution of the Stefan problem with constant Dirichlet boundary data, would prove that it is localized. Therefore in this case $B(u) = B(v) = (0,+\infty)$. If u is localized take $x_0 \notin B(u)$. There exist $a > 0$ and $M > 0$ so that $u(x_0 - a,t) \le M$ for any $t \in [0,T)$. From (3.1) and the fact that $v(x,t)$ is non increasing as a function of x we obtain

$$a(v(x_0,t) - M) \le \int_{x_0-a}^{x_0} [v(y,t) - M]\,dy \le s(t)\,.$$

As the right hand side in the above inequality is bounded, we get that $B(v) \subset [0,x_0]$ for any $x_0 \notin B(u)$ and hence $B(v) \subset B(u)$. ◊

We state now the main result of this section which is an immediate consequence of the two propositions above and the result of the previous section.

THEOREM 3.1. *Let u be the solution of problem (1.1). Then* $B(u) = (0, K]$.

We end this section with the following remark.

REMARK. If $K > 0$, then $u(0,t) \to +\infty$ as $t \to T$.

4. The case $K = 0$

As we have seen in Theorem 3.1, if $K = 0$, then $B(u) = \emptyset$. This means that $u(x,t)$ is bounded for $t \in [0,T)$ for any fixed $x \in (0,+\infty)$. It is the purpose of this section to study the behaviour of $u(0,t)$ in the case that $K = 0$.

Let u be the solution of (1.1) with $u_0 \equiv 0$ and $b = 2$. Let v be a solution of (1.5) and let v_C be a solution of

$$\begin{array}{rcll} v_t & = & v_{xx} & \text{in} \quad [0,+\infty) \times [0,T) \\ -v_x(0,t) & = & h(t) & \text{on} \quad [0,T) \\ v(x,0) & = & -Cx & \text{on} \quad [0,2] \\ v(x,0) & = & 0 & \text{on} \quad [2,+\infty) \end{array} \tag{4.1}$$

It is possible to pick C large enough so that $v_C(1,t) \le u(1,t)$ $\forall t \in [0,T)$. On the region $(0,1) \times (0,T)$ the three functions u, v and v_C, satisfy the equation $w_t = w_{xx}$. Moreover $-(v_C)_x(0,t) = -u_x(0,t) = -v_x(0,t)$ for $t \in [0,T)$, $v_C(x,0) \le u(x,0) \le v(x,0)$ for $x \in [0,1]$ and $v_C(1,t) \le u(1,t) \le v(1,t)$ for $t \in [0,T]$. Therefore, by the maximum principle, $v_C(x,t) \le u(x,t) \le v(x,t)$ for all $(x,t) \in (0,1) \times (0,T)$.

Since, by superposition $v_C = v - g$ in $(0,+\infty) \times (0,T)$ where g is a bounded function, we have proved that the behaviour of $u(0,t)$ as $t \to T$ is the same as the behaviour of $v(0,t)$ as far as boundedness is concerned.

We recall now that

$$v(0,t) = d \int_0^t \frac{h(s)}{\sqrt{t-s}}\, ds \tag{4.2}$$

where d is a constant.

As opposite to the case of interior points, where either $u(x,t)$ is bounded or $\lim u(x,t) = +\infty$ as $t \to T$, in this case a third possibility occurs. In this case $u(0,t)$ can be unbounded but oscillating as illustrated by the following example.

Let $\{t_n\}_{n=0}^{+\infty}$ be an increasing sequence in $[0,T)$ such that $|t_{n+1} - t_n| < 1$ and $\lim t_n = T$ as $n \to \infty$. Consider the "function"

$$h(t) = \sum_{n=0}^{+\infty} (t_{n+1} - t_n)^2 \delta(t - t_n)\,,$$

where δ is the Dirac's delta function centered at 0.
According to (4.2) one has $v(0, t_n) = +\infty$ for all n. On the other hand setting $r_n = \frac{t_n + t_{n+1}}{2}$ one has

$$\begin{aligned} v(0, r_n) &= d \sum_{k=0}^{n} \frac{(t_{k+1} - t_k)^2}{\sqrt{r_k - t_k}} \leq d \sum_{k=0}^{n} \frac{(t_{k+1} - t_k)^2}{\sqrt{r_k - t_k}} \\ &= d \sum_{k=0}^{n} \sqrt{2} (t_{k+1} - t_k)^{\frac{3}{2}} \leq d \sum_{k=0}^{n} \sqrt{2} (t_{k+1} - t_k) \leq d\sqrt{2}T \end{aligned}$$

for all n.

The above example can be modified, by using properly chosen mollifiers, in order to have a smooth function as boundary datum instead of a distribution.

5. The free boundary

We can prove now our result about localization.

THEOREM 5.1. *The solution u of problem (1.1) is localized if and only if* $K = \limsup \xi(t) < +\infty$ *as* $t \to T$.

Proof. If $K = +\infty$ it follows from Theorem 3.1 that u is not localized. If $K < +\infty$, then there exist $A > 0$ and $M > 0$ such that $u(x,t) < M$, $\forall x \in [A, +\infty]$, $\forall t \in [0,T)$. Now by pointwise comparison with a solution of the Stefan problem with constant Dirichlet boundary data the theorem follows since $T < +\infty$. $\diamondsuit$

Using exactly the same arguments of [CP] we can prove the following theorem about the speed of the free boundary.

THEOREM 5.2. *If* $s(T) > K$, *then there exists a constant* $C > 0$ *such that* $0 \leq \dot{s}(t) \leq C$, $\forall t \in [0,T)$.

Proof. The results in [CP] prove that $\dot{s}(t)$ is bounded in any interval $[0, T-\varepsilon]$ for any $0 < \varepsilon < T$. We chose now A and B so that $K < A < B < s(T)$. There exists $M > 0$ so that $u(A,t) \leq M$, $\forall t \in [0,T)$. Fix t_0 such that $B < s(t_0)$. The function $g(x,t) = \frac{M}{s(t_0)-A}[s(t_0)-x] - u(x,t)$ satisfies $g_t = g_{xx}$ in the region $A \leq x \leq s(t)$, $0 \leq t \leq t_0$ and is nonnegative on its parabolic boundary. Therefore, by the maximum principle, $u(x,t) \leq \frac{M}{s(t_0)-A}[s(t_o)-x]$ in the region $A \leq x \leq s(t)$, $0 \leq t \leq t_0$. As $u(s(t_0),t_0) = 0$, we obtain $\dot{s}(t_0) = -u_x(s(t_0),t_0) \leq \frac{M}{s(t_0)-A} \leq \frac{M}{B-A}$ for any such t_0 and the result is proved. ♢

In order to prove our next result we need the following comparison lemma. Its proof is similar to the corresponding one for the porous medium equation and can be found in [CE].

LEMMA 5.1. *Let u and $\bar{u}$ be two solution of problem (1.1) with initial conditions u_0, $\bar{u}_0$ and boundary data $-u_x(0,t) = h(t)$, $-\bar{u}_x(0,t) = \bar{h}(t)$ respectively.*
Assume that

$$\int_x^{+\infty} \bar{u}_0(y)\,dy \leq \int_x^{+\infty} u_0(y)\,dy \quad \forall x \in [0,+\infty)$$

and

$$\int_0^{+\infty} \bar{u}_0(y)\,dy + \int_0^t \bar{h}(s)\,ds \leq \int_0^{+\infty} u_0(y)\,dy + \int_0^t h(s)\,ds \quad \forall t \in [0,T)\,.$$

Then

$$\int_x^{+\infty} \bar{u}(y,t)\,dy \leq \int_x^{+\infty} u(y,t)\,dy \quad \forall (x,t) \in [0,+\infty) \times [0,T)\,.$$

We also recall that the function

$$w(x,t) = b \int_{\frac{x}{\sqrt{t}}}^{a} \exp\left(\frac{z^2}{4}\right)\,dz$$

is a solution of

$$\begin{aligned} w_t &= w_{xx} && \text{in } 0 < x < \sigma(t) = a\sqrt{t},\ 0 < t < +\infty \\ -w_x(\sigma(t),t) &= \dot{\sigma}(t), && \text{for } 0 < t < +\infty \\ w(x,0) &= 0 \end{aligned}$$

provided that $2b = a\exp\left(\frac{a^2}{4}\right)$.

At this point, instead of using the function $\xi(t)$, it will be more convenient to work with the function $\rho(t)$ defined by

$$\int_0^t h(s)\,ds = \rho(t)\exp\left(\frac{\rho^2(t)}{4(T-t)}\right)$$

REMARK. It can be easily checked that

$$\limsup_{t\to T}\rho(t) = \limsup_{t\to T}\xi(t) = K\,.$$

Our next theorem gives a lower estimate for the penetration of the free boundary.

THEOREM 5.3. *If u is the solution of problem (1.1), then $\rho(t) \le s(T)$, $\forall t \in [0,T)$.*

Proof. There is no loss of generality in assuming $u_0 = 0$. Fix $t_0 \in [0,T)$. A direct computation gives

$$-\int_{t_0}^t w_x(0, s-t_0)\,ds = 2b\sqrt{t-t_0} = \sigma(t-t_0)\exp\left(\frac{\sigma^2(t-t_0)}{4(t-t_0)}\right)$$

Now we choose the constant b in w in such a way that

$$\begin{aligned}\int_0^{t_0} h(s)\,ds &= \rho(t_0)\exp\left(\frac{\rho^2(t_0)}{4(T-t_0)}\right)\\ &= \sigma(T-t_0)\exp\left(\frac{\sigma^2(T-t_0)}{4(T-t_0)}\right)\\ &= -\int_{t_0}^T w_x(0,s-t_0)\,ds\,.\end{aligned}$$

Therefore

$$-\int_{t_0}^t w_x(0,s-t_0)\,ds \le \int_0^t h(s)\,ds \quad \forall t \in [t_0, T)\,.$$

So, using Lemma 5.1 for u and $\bar{u}(x,t) = w(x, t - t_0)$ in the domain $[0, +\infty) \times [t_0, T)$, we obtain

$$\int_x^{+\infty} w(y,t)\,dy \le \int_x^{+\infty} u(y,t)\,dy \quad \forall (x,t) \in [0,+\infty) \times [t_0, T)$$

and, since the functions are nonnegative, $\rho(t_0) \le s(T)$. $\diamondsuit$

Remark. We note that, as a consequence of the previous results, if the function $\rho(t)$ satisfies $\limsup_{t\to T} \rho(t) < \sup_{t\in[0,T)} \rho(t) < +\infty$, then the solution u of (1.1) is localized and $\dot{s}(t)$ is uniformly bounded in $[0,T)$. The following example shows that there are solutions that are localized for which $\dot{s}(t)$ is not bounded.

Example. The formula

$$z(x,t) = \sum_{j=0}^{+\infty} \frac{\partial}{\partial t_j} \left\{ \frac{[x - s(t)]^{2j+1}}{(2j+1)!}\,\dot{s}(t) \right\}$$

provides a solution of

$$\begin{aligned} z_t &= z_{xx} \quad \text{for } 0 < x < s(t) \\ z(s(t),t) &= 0 \\ -z_x(s(t),t) &= \dot{s}(t) \end{aligned}$$

for smooth data $s(t)$. See [H].

Taking $s(t) = \sqrt{T} - \sqrt{T-t}$ in the above formula we obtain the desired example.

We note that in this example there exists $K > 0$ so that $u(x,t) \to +\infty$ if $x < K$ and $u(x,t) = 0$, $\forall t \in [0,T)$ if $x > K$.

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Pervenuto in Redazione l'8 Maggio 1996.