A Regularity Result for a Class of Anisotropic Systems

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Sommario. - *Si prova la regolarità parziale dei minimi del funzionale* $I(u) = \int_\Omega G(Du)$, *con* G *integrando convesso a crescita anisotropa. Non si fanno ipotesi speciali sulla struttura di* G.

Summary. - *We prove partial regularity of minimizers of the functional* $I(u) = \int_\Omega G(Du)$, *where* G *is a convex integrand satisfying anisotropic growth condition. No special structure assumption is needed on* G.

1. Introduction

In this paper we study the partial regularity of minimizers of integral functionals of the type

$$I(u) = \int_\Omega G(Du(x))dx \tag{1.1}$$

$u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$, $N \geq 1$, where G is a C^2 convex integrand satisfying the growth condition:

$$C|\xi|^q \leq G(\xi) \leq L(1 + |\xi|^p) \tag{1.2}$$

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with $p > q$.

Few years ago it was observed that even in the scalar case, i.e. $N = 1$, minimizers of (1.1) may fail to be regular (see [M2], [G2]), when p is too large with respect to q. On the other hand, one can prove regularity of scalar minimizers of (1.1) if p is not too far away from q (see e.g. [M3], [FS] and the references given in [M3]). More precisely, in [M3] it is shown that if one writes down the Euler equation for the functional I, under suitable assumptions on p and q, the Moser iteration argument still works, thus leading to a *sup* estimate for the gradient Du of the minimizer.

Clearly this approach can not be carried on in the vector valued case, i.e. when $N > 1$. As far as we know, the only regularity results for systems are proved under special structure assumptions (see [AF2], [M4]).

Namely, the model case covered in [AF2] is the functional

$$\int_\Omega |Du|^p + \sum_{\alpha=1}^{k} |D_\alpha u|^{p_\alpha}$$

with $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$, $N \geq 1$, $1 \leq k \leq n$, $2 \leq p < p_\alpha$, and p_α not too far from p, while in [M4], it is proved everywhere regularity of minimizers of (1.1) when $G(\xi) = f(|\xi|)$.

In this paper we prove that if G satisfies (1.2) and the strong ellipticity assumption

$$\langle D^2G(\xi)\eta, \eta\rangle \geq \gamma(1 + |\xi|^2)^{\frac{q-2}{2}}|\eta|^2$$

and

$$2 \leq q < p < \min\left\{q + 1, \frac{qn}{n-1}\right\}, \tag{1.3}$$

a minimizer $u \in W^{1,q}(\Omega; \mathbb{R}^N)$ of functional (1.1) is $C^{1,\alpha}$ for all $\alpha < 1$ in an open set $\Omega_0 \subset \Omega$ such that $\text{meas}(\Omega \setminus \Omega_0) = 0$.

We point out that a part from condition (1.3), no special structure assumption is needed on G.

The proof of our result goes through a more or less standard blow-up argument aimed to establish a decay estimate on the excess function for the gradient

$$U(x_0, r) = \fint_{B_r(x_0)} |Du - (Du)_{x_0,r}|^2 + |Du - (Du)_{x_0,r}|^q dx.$$

The essential tool in the case we consider, is a lemma due to Fonseca and Malý (see [FM] and also Lemma 2.3 below) which makes possible to connect in the annulus $B_r \setminus B_s$ two $W^{1,q}$ functions v and w with a function $z \in W^{1,p}(B_r \setminus B_s)$ if $q < p < \frac{qn}{n-1}$.

2. Statements and preliminary Lemmas

Let us consider the functional

$$I(u) = \int_\Omega G(Du(x))dx$$

where Ω is a bounded open set of $\mathbb{R}^n$, $n \geq 2$. Let $G : \mathbb{R}^{nN} \to \mathbb{R}$, $N \geq 2$, satisfy the following assumptions:

$$G \in C^2 \tag{$H1$}$$

$$C|\xi|^q \leq G(\xi) \leq L(1+|\xi|^p) \tag{$H2$}$$

$$\langle D^2G(\xi)\eta, \eta\rangle \geq \gamma(1+|\xi|^2)^{\frac{q-2}{2}}|\eta|^2 \tag{$H3$}$$

$$\text{where} \qquad 2 \leq q < p < \min\left\{q+1, \frac{qn}{n-1}\right\}$$

It is well known that

$$|DG(\xi)| \leq c(1+|\xi|^{p-1}). \tag{$H4$}$$

We say that $u \in W^{1,q}(\Omega, \mathbb{R}^N)$ is a minimizer of I if

$$I(u) \leq I(u+v)$$

for any $v \in u + W_0^{1,q}(\Omega; \mathbb{R}^N)$.

Remark 1. If u is a local minimizer of I and $\phi \in C_0^1(\Omega; \mathbb{R}^N)$ from the minimality condition one has for any $\varepsilon > 0$

$$\begin{aligned} 0 &\leq \int_\Omega [G(Du+\varepsilon D\phi) - G(Du)]dx \\ &= \varepsilon \int_\Omega dx \int_0^1 \frac{\partial G}{\partial \xi_\alpha^i}(Du + \varepsilon t D\phi) D_\alpha \phi^i dt \end{aligned}$$

Dividing this inequality by ε, and letting ε go to zero, from (H4) and the assumption $p \leq q+1$ we get

$$\int_\Omega \frac{\partial G}{\partial \xi_\alpha^i}(Du) D_\alpha \phi^i dx \geq 0$$

and therefore by the arbitrariness of ϕ the usual Euler-Lagrange system holds:

$$\int_\Omega \frac{\partial G}{\partial \xi_\alpha^i}(Du) D_\alpha \phi^i dx = 0 \qquad \forall \phi \in C_0^1(\Omega; \mathbb{R}^N)$$

We prove the following

THEOREM 2.1. *Let G be as above and let $u \in W^{1,q}(\Omega; \mathbb{R}^N)$ be a minimizer of I. Then there exists an open subset Ω_0 of Ω such that*

$$\text{meas}(\Omega \setminus \Omega_0) = 0$$

and

$$u \in C^{1,\alpha}(\Omega_0, \mathbb{R}^N) \qquad \text{for} \quad \text{all} \quad \alpha < 1\,.$$

In the following, we will denote by u a $W^{1,q}(\Omega; \mathbb{R}^N)$ minimizer of $\int_\Omega G(Du)dx$ and assume that G satisfies (H1), (H2), (H3). We set for every $B_r(x_0) \subset \Omega$

$$U(x_0, r) = \fint_{B_r(x_0)} |Du - (Du)_{x_0,r}|^2 + |Du - (Du)_{x_0,r}|^q dx,$$

where

$$\fint_{B_r(x_0)} g = (g)_{x_0,r} = \frac{1}{\text{meas}(B_r(x_0))} \int_{B_r(x_0)} g.$$

The next Lemma can be found in [FM], (Lemma 2.2), in a slightly different form.

LEMMA 2.1. *Let $v \in W^{1,q}(B_1(0))$ and $0 < s < r < 1$. There exists a linear operator $T : W^{1,q}(B_1(0)) \to W^{1,q}(B_1(0))$ such that*

$$Tv = v \qquad \text{on} \quad (B_1 \setminus B_r) \cup B_s$$

and for all $\mu > 0$, for all $p < q\frac{n}{n-1}$

$$\begin{aligned}
&||Tv||_{W^{1,2}(B_r \setminus B_s)} + \mu ||Tv||_{W^{1,p}(B_r \setminus B_s)} \\
&\leq C \Big\{ (r-s)^\sigma \Big[\sup_{t \in (s,r)} (t-s)^{-\frac{1}{2}} ||v||_{W^{1,2}(B_t \setminus B_s)} + \\
&\quad + \sup_{t \in (s,r)} (r-t)^{-\frac{1}{2}} ||v||_{W^{1,2}(B_r \setminus B_t)} \Big] + \\
&\quad + \mu (r-s)^\tau \Big[\sup_{t \in (s,r)} (t-s)^{-\frac{1}{q}} ||v||_{W^{1,q}(B_t \setminus B_s)} + \\
&\quad + \sup_{t \in (s,r)} (r-t)^{-\frac{1}{q}} ||v||_{W^{1,q}(B_r \setminus B_t)} \Big] \Big\}
\end{aligned}$$

where $C = C(n,p,q) > 0$, $\sigma = \sigma(n) > 0$ and $\tau = \tau(n,p,q) > 0$.

Let us recall an elementary lemma also proved in [FM].

LEMMA 2.2. *Let ψ be a continuous nondecreasing function on an interval $[a,b]$, $a<b$. There exist $a' \in [a, a+\frac{1}{3}(b-a)]$, $b' \in [b-\frac{1}{3}(b-a), b]$ such that $a \le a' < b' \le b$ and*

$$\begin{aligned} \frac{\psi(t)-\psi(a')}{t-a'} &\le 3\frac{\psi(b)-\psi(a)}{b-a} \\ \frac{\psi(b')-\psi(t)}{b'-t} &\le 3\frac{\psi(b)-\psi(a)}{b-a} \end{aligned} \tag{2.3}$$

for all $t \in (a',b')$.

Finally the next result is a straightforward generalisation to our case of Lemma 2.4 in [FM]. We give the proof here for completeness.

LEMMA 2.3. *Let v, $w \in W^{1,q}(B_1(0))$ and $\frac{1}{4} < s < r < 1$. Fix $q < p < \frac{nq}{n-1}$, for all $\mu > 0$ and $m \in IN$ there exist a function $z \in W^{1,q}(B_1(0))$ and $\frac{1}{4} < s < s' < r' < r < 1$ with r', s' depending on v, w and μ, such that*

$$z = v \quad \text{on} \quad B_{s'}\,, \quad z = w \quad \text{on} \quad B_1 \setminus B_{r'}, \tag{2.4}$$

$$\frac{r-s}{m} \ge r'-s' \ge \frac{r-s}{3m}$$

and

$$\begin{aligned} &||z||_{W^{1,2}(B_{r'}\setminus B_{s'})} + \mu ||z||_{W^{1,p}(B_{r'}\setminus B_{s'})} \\ &\le C\frac{(r-s)^\rho}{m^\rho}\Big[⨍_{B_r\setminus B_s} \Big(1+|Dv|^2+|Dw|^2+|v|^2+|w|^2+ \\ &\qquad\qquad + m^2\frac{|v-w|^2}{(r-s)^2}\Big) + \\ &\quad + \mu^q ⨍_{B_r\setminus B_s}\Big(1+|Dv|^q+|Dw|^q+|v|^q+|w|^q+ \\ &\qquad\qquad + m^q\frac{|v-w|^q}{(r-s)^q}\Big)\Big]^{\frac{1}{2}} \end{aligned} \tag{2.5}$$

where $C = C(n,p,q) > 0$ and $\rho = \rho(p,q,n) > 0$.

Proof. As in Lemma 2.4 in [FM], choose $m \in IN$ and set

$$\begin{aligned} f \,=\, & 1+|Dv|^2+|Dw|^2+|v|^2+|w|^2+m^2\frac{|v-w|^2}{(r-s)^2} + \\ & + \mu^q\Big(1+|Dv|^q+|Dw|^q+|v|^q+|w|^q+m^q\frac{|v-w|^q}{(r-s)^q}\Big)\,. \end{aligned}$$

We may find $k \in \{1, ..., m\}$ such that

$$\int_{B_{s+\frac{k(r-s)}{m}} \setminus B_{s+\frac{(k-1)(r-s)}{m}}} f dx \le \frac{1}{m} \int_{B_r \setminus B_s} f dx \ ,$$

Set, for $t \in [s + \frac{(k-1)(r-s)}{m}, s + \frac{k(r-s)}{m}]$,

$$\psi(t) = \int_{B_t \setminus B_s} f dx$$

which is a continuous nondecreasing function. By Lemma 2.2, there exists $[s', r'] \subset [s + \frac{(k-1)(r-s)}{m}, s + \frac{k(r-s)}{m}]$ such that

$$\frac{r-s}{m} \ge r' - s' \ge \frac{r-s}{3m}$$

and

$$\begin{aligned} \int_{B_t \setminus B_{s'}} f dx &\le 3 \frac{(t-s')m}{r-s} \int_{B_{s+\frac{k(r-s)}{m}} \setminus B_{s+\frac{(k-1)(r-s)}{m}}} f dx \\ &\le 3 \frac{t-s'}{r-s} \int_{B_r \setminus B_s} f dx, \end{aligned} \tag{2.6}$$

$$\int_{B_{r'} \setminus B_t} f dx \le 3 \frac{r'-t}{r-s} \int_{B_r \setminus B_s} f dx \tag{2.7}$$

for all $t \in (s', r')$. Set

$$u = \begin{cases} v(x) & \text{if} \quad x \in B_{s'} \\ \frac{(r'-|x|)v(x) + (|x|-s')w(x)}{r'-s'} & \text{if} \quad x \in B_{r'} \setminus B_{s'} \\ w(x) & \text{if} \quad x \in B_1 \setminus B_{r'}. \end{cases}$$

A direct computation shows that

$$|u|^2 + |Du|^2 + \mu^q(|u|^q + |Du|^q) \le Cf \ .$$

If we apply Lemma 2.1 to the function u, we then find $z \in W^{1,q}(B_1)$ satisfying (2.4). Moreover, from (2.6) and (2.7) one readily cheks that

$$
\begin{aligned}
&||z||_{W^{1,2}(B_{r'}\setminus B_{s'})} + \mu||z||_{W^{1,p}(B_{r'}\setminus B_{s'})} \\
&\leq c\Bigg\{ \frac{(r'-s')^\sigma}{(r'-s')^{\frac{1}{2}}} |B_{r'} \setminus B_{s'}|^{\frac{1}{2}} \left(\fint_{B_r\setminus B_s} f \right)^{\frac{1}{2}} + \\
&\quad + \frac{(r'-s')^\tau}{(r'-s')^{\frac{1}{q}}} |B_{r'} \setminus B_{s'}|^{\frac{1}{q}} \left(\fint_{B_r\setminus B_s} f \right)^{\frac{1}{q}} \Bigg\} \\
&\leq c\Bigg\{ (r'-s')^\sigma \left(\fint_{B_r\setminus B_s} f \right)^{\frac{1}{2}} + (r'-s')^\tau \left(\fint_{B_r\setminus B_s} f \right)^{\frac{1}{q}} \Bigg\},
\end{aligned}
$$

from which (2.5) follows choosing $\rho = \min\{\sigma, \tau\}$.

◊

3. Proof of Theorem 1

As usual, to get the partial regularity result stated in Theorem 1, we need a decay estimate for the excess function $U(x_0, r)$ defined in section 2.

PROPOSITION 3.1. *Fix $M > 0$. There exists a constant $C_M > 0$ such that for every $0 < \tau < \frac{1}{4}$, there exists $\epsilon = \epsilon(\tau, M)$ such that, if*

$$|(Du)_{x_0,r}| \leq M \qquad \textit{and} \qquad U(x_0, r) \leq \epsilon$$

then

$$U(x_0, \tau r) \leq C_M \tau^2 U(x_0, r)\,.$$

Proof. Fix M and τ. We shall determine C_M later.

We argue by contradiction. We assume that there exists a sequence $B_{r_h}(x_h)$ satisfying

$$B_{r_h}(x_h) \subset \Omega, \qquad |(Du)_{x_h,r_h}| \leq M, \qquad \lim_h U(x_h, r_h) = 0,$$

but

$$U(x_h, \tau r_h) > C_M \tau^2 U(x_h, r_h)\,. \tag{3.1}$$

Set

$$a_h = (u)_{x_h,r_h} \qquad A_h = (Du)_{x_h,r_h} \qquad \lambda_h^2 = U(x_h, r_h)\,.$$

Step 1. [Blow up.] We rescale the function u in each $B_{r_h}(x_h)$ to obtain a sequence of functions on $B_1(0)$. Set

$$v_h(y) = \frac{1}{\lambda_h r_h}[u(x_h + r_h y) - a_h - r_h A_h y],$$

then

$$Dv_h(y) = \frac{1}{\lambda_h}[Du(x_h + r_h y) - A_h]\,.$$

Clearly we have

$$(v_h)_{0,1} = 0 \qquad (Dv_h)_{0,1} = 0\,.$$

Moreover,

$$⨍_{B_1(0)} (1 + \lambda_h^{q-2}|Dv_h|^{q-2})|Dv_h|^2 dy = 1\,. \tag{3.2}$$

Passing possibly to a subsequence we may suppose that

$$v_h \rightharpoonup v \qquad \text{weakly} \quad \text{in} \quad W^{1,2}(B_1; \mathbb{R}^N) \tag{3.3}$$

and, since $\forall h \quad |A_h| \le M$,

$$A_h \to A\,. \tag{3.4}$$

Step 2. Now we show that

$$\int_{B_1(0)} \frac{\partial^2 G}{\partial \xi^i_\alpha \partial \xi^j_\beta}(A) D_\beta v^j D_\alpha \phi^i dy = 0 \qquad \forall \phi \in C^1_0(B_1; \mathbb{R}^N)\,. \tag{3.5}$$

Since we assume $p - 1 \le q$ we can write the usual Euler-Lagrange system for u (see Remark 1). Then, rescaling in each $B_{r_h}(x_h)$, we get for any $\phi \in C^1_0(B_1; \mathbb{R}^N)$ and any $1 \le i \le N$

$$\int_{B_1(0)} \frac{\partial G}{\partial \xi^i_\alpha}(A_h + \lambda_h Dv_h) D_\alpha \phi^i dy = 0\,.$$

Then

$$\frac{1}{\lambda_h}\int_{B_1(0)} [\frac{\partial G}{\partial \xi^i_\alpha}(A_h + \lambda_h Dv_h) - \frac{\partial G}{\partial \xi^i_\alpha}(A_h)] D_\alpha \phi^i dy = 0\,. \tag{3.6}$$

Let us split

$$\begin{aligned} B_1 &= E_h^+ \cup E_h^- \\ &= \{y \in B_1 : \lambda_h |Dv_h(y)| > 1\} \cup \{y \in B_1 : \lambda_h |Dv_h(y)| \le 1\}, \end{aligned}$$

then by (3.2) we get

$$|E_h^+| \le \int_{E_h^+} \lambda_h^2 |Dv_h|^2 dy \le \lambda_h^2 \int_{B_1(0)} |Dv_h|^2 dy \le c\lambda_h^2. \tag{3.7}$$

Now, by (H4) and Hölder inequality, we observe that

$$\begin{aligned} &\frac{1}{\lambda_h} \left| \int_{E_h^+} [DG(A_h + \lambda_h Dv_h) - DG(A_h)] D\phi dy \right| \\ &\le \frac{c}{\lambda_h} |E_h^+| + c\lambda_h^{p-2} \int_{E_h^+} |Dv_h|^{p-1} dy \\ &\le c\lambda_h + c \left(\int_{E_h^+} \lambda_h^{q-2} |Dv_h|^q dy \right)^{\frac{p-1}{q}} \lambda_h^{\frac{2p-q-2}{q}} |E_h^+|^{\frac{q-p+1}{q}} \le c\lambda_h \end{aligned}$$

where we used the assumption $p - 1 \le q$.
From this it follows that

$$\lim_h \frac{1}{\lambda_h} \int_{E_h^+} [DG(A_h + \lambda_h Dv_h) - DG(A_h)] D\phi dy = 0. \tag{3.8}$$

On E_h^- we have

$$\begin{aligned} &\frac{1}{\lambda_h} \int_{E_h^-} [DG(A_h + \lambda_h Dv_h) - DG(A_h)] D\phi dy \\ &= \int_{E_h^-} \int_0^1 D^2 G(A_h + s\lambda_h Dv_h) Dv_h D\phi ds dy \\ &= \int_{E_h^-} \int_0^1 [D^2 G(A_h + s\lambda_h Dv_h) - D^2 G(A_h)] Dv_h D\phi ds dy + \\ &\quad + \int_{E_h^-} D^2 G(A_h) Dv_h D\phi dy \,. \end{aligned}$$

Note that (3.7) ensures that $\chi_{E_h^-} \to \chi_{B_1}$ in $L^r(B_1)$ for all $r < \infty$ and by (3.2) we have, passing possibly to a subsequence,

$$\lambda_h Dv_h(y) \to 0 \qquad \text{a.e.} \qquad \text{in} \quad B_1 \,.$$

Then, by (3.3), (3.4) and the uniform continuity of D^2G on bounded sets, we get

$$\lim_h \frac{1}{\lambda_h}\int_{E_h^-}[DG(A_h+\lambda_h Dv_h)-DG(A_h)]D\phi dy$$
$$=\int_{B_1} D^2G(A)DvD\phi dy\,.$$

By (3.6), (3.8) and the above equality, we obtain that v satisfies equation (3.5), which is elliptic by (H3). We have for any $0<\tau<1$

$$\fint_{B_\tau}|Dv-(Dv)_\tau|^2dy\le c\tau^2\fint_{B_1}|Dv-(Dv)_1|^2dy\le c\tau^2. \qquad (3.9)$$

Moreover we have

$$v\in C^\infty(B_1;\mathbb{R}^N)\,. \qquad (3.10)$$

and

$$\lambda_h^{\frac{q-2}{q}}(v_h-v)\rightharpoonup 0 \quad \text{weakly in } W^{1,q}_{\text{loc}}(B_1;\mathbb{R}^N)$$

Step 3. [UPPER BOUND.] We set

$$G_h(\xi)=\frac{1}{\lambda_h^2}[G(A_h+\lambda_h\xi)-G(A_h)-\lambda_h DG(A_h)\xi]$$

and for every $r<1$

$$I_{h,r}(w)=\int_{B_r}G_h(Dw)dy\,.$$

Note that by the strong ellipticity assumption (H3) it follows that $G_h(\xi)\ge 0$, for any ξ. Fix $\frac{1}{4}<s<1$. Passing to a subsequence we may always assume that

$$\lim_h[I_{h,s}(v_h)-I_{h,s}(v)]$$

exists. We shall prove that

$$\lim_h[I_{h,s}(v_h)-I_{h,s}(v)]\le 0\,. \qquad (3.11)$$

Consider $r>s$ and fix $m\in IN$. Observe that , since $v\in W^{1,q}(B_1)$ and $v_h\in W^{1,q}(B_1)$, Lemma 2.3, with $\mu=\lambda_h^{\frac{p-2}{p}}$, implies that there exist $z_h\in W^{1,q}(B_1)$ and $\frac{1}{4}<s<s_h<r_h<r<1$ such that

$$z_h=v \quad\text{on}\quad B_{s_h} \qquad z_h=v_h \quad\text{on}\quad B_1\setminus B_{r_h}$$

and

$$
\begin{aligned}
&||z_h||_{W^{1,2}(B_{r_h}\setminus B_{s_h})} + \lambda_h^{\frac{p-2}{p}} ||z_h||_{W^{1,p}(B_{r_h}\setminus B_{s_h})} \\
&\quad \leq C\frac{(r-s)^\rho}{m^\rho}\Big[\fint_{B_r\setminus B_s} (1+|Dv|^2+|Dv_h|^2+|v|^2+|v_h|^2+ \\
&\qquad\qquad + m^2\frac{|v-v_h|^2}{(r-s)^2})+ \\
&\quad + \lambda_h^{\frac{p-2}{p}q}\fint_{B_r\setminus B_s}(1+|Dv|^q+|Dv_h|^q+|v|^q+|v_h|^q+ \\
&\qquad\qquad + m^q\frac{|v-v_h|^q}{(r-s)^q})\Big]^{\frac{1}{2}}
\end{aligned}
\tag{3.12}
$$

Since by (3.10), Dv is locally bounded on B_1 we get

$$
\begin{aligned}
&I_{h,s}(v_h) - I_{h,s}(v) \\
&\quad \leq I_{h,r_h}(v_h) - I_{h,r_h}(v) + I_{h,r_h}(v) - I_{h,s}(v) \\
&\quad = I_{h,r_h}(v_h) - I_{h,r_h}(v) + \int_{B_{r_h}\setminus B_s} G_h(Dv) \\
&\quad \leq I_{h,r_h}(z_h) - I_{h,r_h}(v) + c(r-s) \\
&\quad \leq c\int_{B_{r_h}\setminus B_{s_h}} [G_h(Dz_h) - G_h(Dv)] + c(r-s)\,.
\end{aligned}
\tag{3.13}
$$

where we used the minimality of v_h . As $|G_h(\xi)| \leq c(|\xi|^2 + \lambda_h^{p-2}|\xi|^p)$ (see [AF], Lemma II.3), we get by (3.12)

$$
\begin{aligned}
&I_{h,r_h}(z_h) - I_{h,r_h}(v) \\
&\leq c\int_{B_{r_h}\setminus B_{s_h}} |Dz_h|^2 + \lambda_h^{p-2}|Dz_h|^p \\
&\leq C\frac{(r-s)^{2\rho}}{m^{2\rho}}\Big[\fint_{B_r\setminus B_s}(1+|Dv|^2+|Dv_h|^2+|v|^2+|v_h|^2+ \\
&\qquad\qquad + m^2\frac{|v-v_h|^2}{(r-s)^2})\Big]^{\frac{p}{2}}+ \\
&\quad + C\frac{(r-s)^{2\rho}}{m^{2\rho}}\Big[\lambda_h^{\frac{p-2}{p}q}\fint_{B_r\setminus B_s}(1+|Dv|^q+|Dv_h|^q+|v|^q+|v_h|^q+ \\
&\qquad\qquad + m^q\frac{|v-v_h|^q}{(r-s)^q})\Big]^{\frac{p}{2}} \\
&= J_{h,1} + J_{h,2}\,.
\end{aligned}
$$

Since $v_h \to v$ in $L^2(B_1;\mathbb{R}^N)$ we have, using (3.2)

$$
\limsup_{h\to\infty} J_{h,1} \leq Cm^{-2\rho}\,.
$$

Moreover, since

$$\lambda_h^{\frac{q(p-2)}{p}} \int_{B_1} |Dv_h|^q = \lambda_h^{\frac{2(p-q)}{p}} \lambda_h^{q-2} \int_{B_1} |Dv_h|^q \leq C\lambda_h^{\frac{2(p-q)}{p}}$$

and

$$\lambda_h^{\frac{q(p-2)}{p}} \int_{B_1} |v_h - v|^q \leq c\lambda_h^{\frac{q(p-2)}{p}} \int_{B_1} |Dv_h|^q \leq c\lambda_h^{\frac{2(p-q)}{p}}$$

we have

$$\lim_h J_{h,2} = 0.$$

Hence we conclude letting first $m \to \infty$ and then $r \to s$ in (3.13).

Step 4. [Lower bound.] We shall prove that, for a.e. $\frac{1}{4} < r < \frac{1}{2}$, if $t < r$ then

$$\limsup_h \int_{B_t} |Dv - Dv_h|^2 (1 + \lambda_h^{q-2} |Dv - Dv_h|^{q-2})$$

$$\leq \lim_h \left[I_{h,r}(v_h) - I_{h,r}(v) \right].$$

For any Borel set $A \subset B_1$, let us define

$$\mu_h(A) = \int_A (|v_h|^2 + |Dv_h|^2) dx \ .$$

Passing possibly to a subsequence, since $\mu_h(B_1) \leq c$, we may suppose

$$\mu_h \rightharpoonup \mu \quad \text{weakly} * \text{ in the sense of measures,}$$

where μ is a Borel measure over B_1. Then for a.e. $r < 1$

$$\mu(\partial B_r) = 0$$

and let us choose such a radius r. Consider $\frac{1}{4} < t < s < r$, also such that $\mu(\partial B_s) = 0$, and fix $m \in IN$. Observe that , as $v_h \in W^{1,q}(B_1)$ Lemmas 2.3 implies that there exist $z_h \in W^{1,q}(B_1)$ and $\frac{1}{4} < s < s_h < r_h < r < 1$ such that

$$z_h = v_h \quad \text{on} \quad B_{s_h} \qquad z_h = v_h \quad \text{on} \quad B_1 \setminus B_{r_h}$$

$$r_h - s_h \geq \frac{r - s}{3m}$$

and

$$\begin{aligned}||z_h||_{W^{1,2}(B_{r_h}\setminus B_{s_h})} &+ \lambda_h^{\frac{p-2}{p}}||z_h||_{W^{1,p}(B_{r_h}\setminus B_{s_h})} \\ &\leq C\frac{(r-s)^\rho}{m^\rho}\Big[\fint_{B_r\setminus B_s}(1+|Dv_h|^2+|v_h|^2)+ \\ &+\lambda_h^{\frac{(p-2)q}{p}}\fint_{B_r\setminus B_s}(1+|Dv_h|^q+|v_h|^q)\Big]^{\frac{1}{2}}\end{aligned} \tag{3.14}$$

Passing possibly to a subsequence, we may suppose that

$$z_h \rightharpoonup v_{r,s} \qquad \text{weakly} \quad \text{in} \quad W^{1,2}(B_1)\,.$$

and

$$v_{r,s} = v \quad \text{in} \quad (B_1\setminus B_r)\cup B_s$$

Moreover from (3.14) it is clear that

$$\lambda_h^{q-2}\int_{B_1}|Dz_h|^q \leq c \tag{3.15}$$

Consider $\zeta_h \in C_0^\infty(B_{r_h})$ such that $0\leq\zeta_h\leq 1$, $\zeta_h = 1$ on B_{s_h} and $|D\zeta_h| \leq \frac{C}{r_h - s_h}$ and set

$$\psi_h^\epsilon = \zeta_h(z_h - v_{r,s}^\epsilon)\ ,$$

where $v_{r,s}^\epsilon = \rho_\epsilon \star v_{r,s}$, and ρ_ϵ is the usual sequence of mollifiers. Now, setting $v^\epsilon = \rho_\epsilon\star v$, we observe that

$$\begin{aligned}&I_{h,r_h}(v_h) - I_{h,r_h}(v^\epsilon) \\ &= I_{h,r_h}(v_h) - I_{h,r_h}(z_h) + I_{h,r_h}(z_h) - I_{h,r_h}(v_{r,s}^\epsilon + \psi_h^\epsilon)+ \\ &\quad + I_{h,r_h}(\psi_h^\epsilon + v_{r,s}^\epsilon) - I_{h,r_h}(v_{r,s}^\epsilon) - I_{h,r_h}(\psi_h^\epsilon)+ \\ &\quad I_{h,r_h}(v_{r,s}^\epsilon) - I_{h,r_h}(v^\epsilon) + I_{h,r_h}(\psi_h^\epsilon) \\ &= R_{h,1}+R_{h,2}+R_{h,3}+R_{h,4}+R_{h,5}\end{aligned} \tag{3.16}$$

To bound $R_{h,1}$ we observe that

$$\begin{aligned}I_{h,r_h}(v_h) - I_{h,r_h}(z_h) &= \int_{B_{r_h}\setminus B_{s_h}} G_h(Dv_h) - \int_{B_{r_h}\setminus B_{s_h}} G_h(Dz_h) + \\ &\geq\ -\int_{B_{r_h}\setminus B_{s_h}} G_h(Dz_h)\end{aligned}$$

on the other hand we have

$$\begin{aligned}\int_{B_{r_h}\setminus B_{s_h}} G_h(Dz_h) \le& \int_{B_{r_h}\setminus B_{s_h}} |Dz_h|^2+\lambda_h^{p-2}|Dz_h|^p\\ \le& \, cm^{-2\rho}\Big[-\!\!\!\!\!\!\int_{B_r\setminus B_s} 1+|Dv_h|^2+|v_h|^2+\\ &+\lambda_h^{\frac{p-2}{p}q} -\!\!\!\!\!\!\int_{B_r\setminus B_s} 1+|Dv_h|^q+|v_h|^q\Big]^{\frac{p}{2}}\end{aligned}$$

and then arguing as we did in Step 3 to bound $J_{h,1}$ we get

$$\limsup_h \int_{B_{r_h}\setminus B_{s_h}} G_h(Dz_h) \le Cm^{-2\rho}$$

hence, letting $h\to\infty$ we get

$$\liminf_h R_{h,1} \ge -Cm^{-2\rho} \tag{3.17}$$

We obtain that

$$\begin{aligned} R_{h,2} =& \int_{B_{r_h}\setminus B_{s_h}} G_h(Dz_h)-G_h(D\psi_h^\epsilon+Dv_{r,s}^\epsilon)\\ \ge& -c\int_{B_{r_h}\setminus B_{s_h}} |D\psi_h^\epsilon+Dv_{r,s}^\epsilon|^2+\lambda_h^{p-2}|D\psi_h^\epsilon+Dv_{r,s}^\epsilon|^p\\ \ge& -c\int_{B_{r_h}\setminus B_{s_h}} \Big(|Dz_h|^2+\lambda_h^{p-2}|Dz_h|^p+|Dv_{r,s}^\epsilon|^2+\\ &+\lambda_h^{p-2}|Dv_{r,s}^\epsilon|^p\Big)-c\int_{B_{r_h}\setminus B_{s_h}}\Bigg(m^2\frac{|z_h-v_{r,s}^\epsilon|^2}{(r-s)^2}+\\ &+m^p\lambda_h^{p-2}\frac{|z_h-v_{r,s}^\epsilon|^p}{(r-s)^p}\Bigg)\\ =& -S_{h,1}-S_{h,2}\end{aligned} \tag{3.18}$$

where we used the bound $r_h-s_h\ge\frac{r-s}{3m}$. By (3.15), since $p<q^*$, we get

$$\begin{aligned}\int_{B_1}\lambda_h^{p-2}|z_h|^p \le& \, c\lambda_h^{p-2}\left\{\int_{B_1}|z_h-(z_h)_{0,1}|^p+|(z_h)_{0,1}|^p\right\}\\ \le& \, c\lambda_h^{p-2}\left\{\left(\int_{B_1}|z_h-(z_h)_{0,1}|^{q^*}\right)^{\frac{p}{q^*}}+\left(\int_{B_1}|z_h|\right)^p\right\}\end{aligned}$$

$$\leq c\lambda_h^{p-2}\left\{\left(\int_{B_1}|Dz_h|^q\right)^{\frac{p}{q}}+\left(\int_{B_1}|z_h|^2\right)^{\frac{p}{2}}\right\}$$
$$\leq c\lambda_h^{\frac{2(p-q)}{q}}\left(\lambda_h^{q-2}\int_{B_1}|Dz_h|^q\right)^{\frac{p}{q}}+c\lambda_h^{p-2}.$$

where we used (3.14) to bound $\left(\int_{B_1}|z_h|^2\right)^{\frac{1}{2}}$.Therefore

$$\limsup_{h\to\infty} S_{h,2}\leq c\frac{m^2}{(r-s)^2}\int_{B_{\frac{1}{2}}}|v_{r,s}-v_{r,s}^{\epsilon}|^2\ .$$

To bound $S_{h,1}$, observe that for every h

$$\begin{aligned}&\int_{B_{r_h}\setminus B_{s_h}}|Dv_{r,s}^{\epsilon}|^2\\
&\leq c\int_{B_r\setminus B_s}|Dv_{r,s}|^2+c\int_{B_{\frac{1}{2}}}|Dv_{r,s}-Dv_{r,s}^{\epsilon}|^2\\
&\leq \liminf_j c\int_{B_r\setminus B_s}|Dz_j|^2+c\int_{B_{\frac{1}{2}}}|Dv_{r,s}-Dv_{r,s}^{\epsilon}|^2\\
&= c\liminf_j\int_{(B_r\setminus B_s)\setminus(B_{r_j}\setminus B_{s_j})}|Dv_j|^2+\\
&\quad + c\limsup_j\int_{B_{r_j}\setminus B_{s_j}}|Dz_j|^2+c\int_{B_{\frac{1}{2}}}|Dv_{r,s}-Dv_{r,s}^{\epsilon}|^2\end{aligned}$$

We control the second integral as usual using Lemma 2.3, while the first is less or equal than $c\mu(B_r\setminus B_s)$.
Moreover we can estimate

$$\int_{B_{r_h}\setminus B_{s_h}}|Dz_h|^2+\lambda_h^{p-2}|Dz_h|^p$$

as we did in Step 3 to bound $J_{h,1}$. Hence

$$\begin{aligned}\liminf_h R_{h,2}\geq\ & -cm^{-2\rho}-c\mu(B_r\setminus B_s)+\\
& -c\int_{B_{\frac{1}{2}}}|Dv_{r,s}-Dv_{r,s}^{\epsilon}|^2+\\
& -\frac{cm^2}{(r-s)^2}\int_{B_{\frac{1}{2}}}|v_{r,s}-v_{r,s}^{\epsilon}|^2\end{aligned}\qquad(3.19)$$

To bound $R_{h,3}$ we observe that

$$G_h(A+B) - G_h(A) - G_h(B) = \int_0^1 \int_0^1 D^2 G_h(sA+tB)ABdsdt$$

and

$$D^2 G_h(sDv^\epsilon_{r,s} + tD\psi^\epsilon_h) = D^2 G(A_h + s\lambda_h Dv^\epsilon_{r,s} + t\lambda_h D\psi^\epsilon_h)$$

is bounded and converges to $D^2G(A)$ a.e.. Since

$$R_{h,3} = \int_{B_{r_h}} dx \int_{[0,1]\times[0,1]} D^2 G(A_h + s\lambda_h Dv^\epsilon_{r,s} + \\ + t\lambda_h D\psi^\epsilon_h) Dv^\epsilon_{r,s} D\psi^\epsilon_h dsdt$$

and we may suppose that $\psi^\epsilon_h \rightharpoonup \psi^\epsilon$ weakly in $W^{1,2}(B_1)$, where

$$\int_{B_1} |D\psi^\epsilon|^2 \le c\frac{m^2}{(r-s)^2}\int_{B_{\frac{1}{2}}} |v_{r,s} - v^\epsilon_{r,s}|^2 + \\ + c\int_{B_{\frac{1}{2}}} |Dv_{r,s} - Dv^\epsilon_{r,s}|^2 \tag{3.20}$$

we get easily

$$\limsup_h |R_{h,3}| \le c(M)||Dv^\epsilon_{r,s}||_{L^2(B_{\frac{1}{2}})} ||D\psi^\epsilon||_{L^2(B_{\frac{1}{2}})} \,. \tag{3.21}$$

To bound $R_{h,4}$ we observe that

$$\begin{aligned} nR_{h,4} &= \int_{B_{r_h}\setminus B_{s_h}} [G_h(Dv^\epsilon_{r,s}) - G_h(Dv^\epsilon)] \\ &\ge -\int_{B_{r_h}\setminus B_{s-\epsilon}} G_h(Dv^\epsilon) \\ &\ge -c|B_r \setminus B_{s-\epsilon}| \,. \end{aligned}$$

Then

$$\liminf_h R_{h,4} \ge -c|B_r \setminus B_{s-\epsilon}| \,. \tag{3.22}$$

Moreover (H3) implies

$$\begin{aligned} |R_{h,5}| &= I_{h,r_h}(\psi^\epsilon_h) \\ &= \int_{B_{r_h}} G_h(D\psi^\epsilon_h) \\ &\ge \gamma\int_{B_t} (1+\lambda_h^{q-2}|Dv^\epsilon - Dv_h|^{q-2})|Dv^\epsilon - Dv_h|^2 \end{aligned} \tag{3.23}$$

for ϵ small enough.

Passing to a subsequence we may suppose that

$$\limsup_h R_{h,5} = \lim_h R_{h,5} .$$

Therefore returning to the (3.16), from (3.17), (3.19), (3.21), (3.22) and (3.23) we get

$$\begin{aligned}
&\liminf_h [I_{h,r}(v_h) - I_{h,r}(v^\epsilon)] \\
&\geq \gamma \limsup_h \int_{B_s} (1 + \lambda_h^{q-2}|Dv^\epsilon - Dv_h|^{q-2})|Dv^\epsilon - Dv_h|^2 + \\
&\quad - c|B_r \setminus B_{s-\epsilon}| - c\mu(B_r \setminus B_s) - c||Dv^\epsilon_{r,s}||_{L^2(B_{\frac{1}{2}})}||D\psi^\epsilon||_{L^2(B_{\frac{1}{2}})} + \\
&\quad - cm^{-2\rho} - \int_{B_{\frac{1}{2}}} |Dv_{r,s} - Dv^\epsilon_{r,s}|^2 - c\frac{m^2}{(r-s)^2}\int_{B_{\frac{1}{2}}} |v_{r,s} - v^\epsilon_{r,s}|^2 .
\end{aligned}$$

Passing to the limit as $\epsilon \to 0^+$ we get easily

$$\begin{aligned}
&\liminf_h [I_{h,r}(v_h) - I_{h,r}(v)] \\
&\geq \gamma \limsup_h \int_{B_s} (1 + \lambda_h^{q-2}|Dv - Dv_h|^{q-2})|Dv - Dv_h|^2 + \\
&\quad - c|B_r \setminus B_s| - c\mu(B_r \setminus B_s) - cm^{-2\rho}
\end{aligned}$$

then passing to the limit as $m \to \infty$ and $s \to r$ we get

$$\limsup_h \int_{B_r} |Dv - Dv_h|^2(1 + \lambda^{q-2}|Dv - Dv_h|^q) \leq \lim_h [I_{h,r}(v_h) - I_{h,r}(v)].$$

Step 5. [Conclusion.] From the two previous steps we conclude that, for any B_τ, with $0 < \tau < \frac{1}{4}$

$$\lim_h \int_{B_\tau} |Dv - Dv_h|^2(1 + \lambda^{q-2}|Dv - Dv_h|^q) = 0 .$$

Now, from this equality and by (3.9) we get

$$\begin{aligned}
&\lim_h \frac{U(x_h, \tau r_h)}{\lambda_h^2} \\
&= \lim_h \frac{1}{\lambda_h^2} \fint_{B_{\tau r_h}(x_h)} (|Du - (Du)_{\tau r_h}|^2 + |Du - (Du)_{\tau r_h}|^q)dx
\end{aligned}$$

$$
\begin{aligned}
&= \lim_h \fint_{B_\tau} (|Du - (Du)_\tau|^2 + \lambda_h^{q-2}|Du - (Du)_\tau|^q)dy \\
&= \fint_{B_\tau} (|Dv - (Dv)_\tau|^2)dy \\
&\le C_M^* \tau^2
\end{aligned}
$$

which contradicts (3.1) if we choose $C_M = 2C_M^*$.

$\diamondsuit$

The proof of Theorem 1 follows by proposition 3.1 by a standard iteration argument, see [G1].

REMARK 2. Notice that the proof of Proposition 3.1 and of Theorem 1 still works if, beside assuming $p < \frac{nq}{n-1}$, we have $p \le q + 1$.

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