# ON THE MOMENTS OF THE DENSITY OF ZEROS FOR THE RELATIVISTIC JACOBI POLYNOMIALS (*)

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SOMMARIO. - *In questo lavoro vengono rappresentati i momenti della densità degli zeri di nuovi sistemi polinomiali ortogonali, chiamati Polinomi Relativistici di Jacobi* $\left\{P_n^{(\alpha,\beta;N)}(x)\right\}_{n=0}^{\infty}$ *(brevemente RJP), per mezzo di un metodo dovuto a K. M. Case e di una formula di rappresentazione introdotta da P. E. Ricci, nella quale intervengono i polinomi generalizzati di Lucas del secondo tipo. Con l'utilizzo di un programma FORTRAN, vengono sviluppati esplicitamente calcoli numerici in qualche caso particolare.*

SUMMARY. - *In this paper the moments of the density of zeros of new orthogonal polynomial systems, called Relativistic Jacobi Polynomials* $\left\{P_n^{(\alpha,\beta;N)}(x)\right\}_{n=0}^{\infty}$ *(shortly RJP), are represented by means of a method due to K. M. Case and of a representation formula, introduced by P. E. Ricci, in terms of the generalized Lucas polynomials of the second kind. By using a FORTRAN program, numerical computations are explicitly developed in some particular case.*

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## 1. Introduction.

Recently, a particular attention has been devoted to the study of new polynomial systems, called the relativistic polynomials, that are solutions of a second order linear homogeneous differential equation of generalized hypergeometric-type, i. e. a differential equation of the form:

$$\sigma(x)y_n'' + \tau(x;n)y_n' + \lambda_n y_n = 0\,, \tag{1.1}$$

where $\sigma$ and $\tau$ are polynomials of degree not greater than 2 and 1 respectively and $\lambda_n$ is a constant depending on $n$ such that $\lambda_n = -n\tau' - (n(n-1)/2)\sigma''$.

We note that (1.1) represents a generalized form of the classical second order differential equation of hypergeometric type considered in [15]:

$$\sigma(x)y_n'' + \tau(x)y_n' + \lambda_n y_n = 0\,, \tag{1.2}$$

where $\sigma$ and $\tau$ are polynomials of the same degree as in (1.1) and the constant $\lambda_n$ satisfies the same conditions as before.

We can summarize the principal properties of these polynomial systems in the following two points:

i) each family of relativistic polynomials depends on a parameter $N$, which is related to the speed of light, and they constitute a generalization of the classical orthogonal polynomials, because they reduce to them in the non-relativistic limit $N \to \infty$,

ii) these polynomials are orthogonal with respect to a varying measure (depending on the degree $n$ of the generic polynomial), whose support is depending on $N$.

The first example of relativistic polynomials is given by the Relativistic Hermite Polynomials $\left\{H_n^{(N)}(x)\right\}_{n=0}^{\infty}$ (RHP). These polynomials have been found by V. Aldaya et al. [2] in order to express the wave functions of the quantum relativistic harmonic oscillator in configuration space. As a matter of fact, there exists (see [2]), a relation between the wave functions $\psi_n(t,x,p;N)$ of the quantum relativistic harmonic oscillator and the relativistic Hermite polynomials (RHP) $\left\{H_n^{(N)}(x)\right\}_{n=0}^{\infty}$.

If we use the notation

$$N = \frac{mc^2}{\hbar\omega}\,, \qquad\qquad x = \frac{\omega}{c}\sqrt{N}\,\xi\,,$$

$$\alpha(x;N) = \left(1 + \frac{x^2}{N}\right)^{1/2}, \quad P^0(x,p;N) = \sqrt{p^2 + m^2c^2\,\alpha(x;N)^2}\,,$$

$$f(x,p;N) = \frac{2mc^2}{\omega}\arctan\left(\frac{\sqrt{N}\,(P^0 - p + mc)}{mcx}\right)\,,$$

these wave functions can be written as follows:

$$\Psi_n(t,x,p;N) = \mathrm{e}^{if/\hbar}\mathrm{e}^{-in\omega t}2^{-n/2}\alpha^{-(n+N)}H_n^{(N)}(x).$$

Here, $n$ is the principal quantum number, $\omega$ represents the frequency of the oscillator and $p$ is the momentum.

Subsequently, J. S. Dehesa, A. Martinez, J. Torres, A. Zarzo have studied the basic properties of the RHP ([19], [20]).

Later on, other examples of relativistic polynomial systems, and the relative properties, have been given by M. X. He, P. Natalini, S. Noschese, P. E. Ricci ([8], [9], [11], [13], [14]). These polynomials correspond to the Relativistic Laguerre Polynomials $\left\{L_n^{(\alpha,N)}(x)\right\}_{n=0}^{\infty}$ (RLP), the Relativistic Szegö Polynomials, and the more recently introduced Relativistic Jacobi Polynomials $\left\{P_n^{(\alpha,\beta;N)}(x)\right\}_{n=0}^{\infty}$ (RJP), see Section 2. Actually, we are also studying all particular cases of the RJP, e. g. the relativistic type polynomials related to the classical Legendre, Gegenbauer and Chebyshev, and, moreover, we are considering the Relativistic Bessel Polynomials.

In this paper we are interested in studying, in a similar way to the RHP and RLP case (see [14]), the zero's distribution of the RJP in some particular cases. For this purpose, as well as in [14], we will use Case's method [5] (see Section 3) and a representation formula of P. E. Ricci [16] (see Section 4) in order to evaluate the moments about the origin $\mu_r'^{(n)}$ of the discrete normalized density distribution $\rho^{(n)}(x)$ of zeros $x_l\,(l = 1,\ldots,n)$ of the polynomial solutions of (1.1) or (1.2).

By assuming

$$y_r = \sum_{l=1}^{n}(x_l)^r, \qquad (r = 0,1,\ldots), \tag{1.3}$$

we have:

$$\mu_r'^{(n)} = \frac{1}{n} y_r \qquad (r = 0, 1, \ldots), \tag{1.4}$$

and

$$\rho^{(n)}(x) = \frac{1}{n} \sum_{k=1}^{n} \delta(x - x_{k,n}), \qquad \delta(x - x_{k,n}) = \text{Dirac delta}. \tag{1.5}$$

## 2. The RJP and their properties.

The RJP (see [8]) are defined by the following explicit formula:

$$\begin{aligned} P_n^{(\alpha,\beta;N)}(x) \quad = \quad & \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \prod_{i=n-k}^{n-1} \Big(\alpha + \beta + 2n + \\ & - i + \frac{2i+1}{2N}\beta\Big) \frac{1}{k!} \left(\frac{x-1}{2}\right)^2, \end{aligned} \tag{2.1}$$

where $\alpha, \beta \in \mathbb{R}$ with $\alpha > -1$ and $\beta > -1$, and moreover we assume, by definition,

$$\prod_{i=n}^{n-1} \left(\alpha + \beta + 2n - i + \frac{2i+1}{2N}\beta\right) = 1.$$

From (2.1), by means of simple transformations, we obtain for the RJP the following representation:

$$\begin{aligned} & P_n^{(\alpha,\beta;N)}(x) \\ & = \frac{(\alpha+1)_n}{n!} \sum_{k=0}^{n} \frac{(-n)_k \left[\frac{(\alpha+\beta+n+1)N+(n-1/2)\beta}{N-\beta}\right]_k}{(\alpha+1)_k} \times \\ & \times \left[\frac{(\beta-N)(x-1)}{2N}\right]^k \frac{1}{k!} \\ & = \frac{(\alpha+1)_n}{n!} F\Big[-n, \frac{(\alpha+\beta+n+1)N + (n-1/2)\beta}{N-\beta}; \\ & \qquad \alpha+1; \frac{(\beta-N)(x-1)}{2N}\Big] \end{aligned} \tag{2.2}$$

where $F = {}_2F_1$ denotes the Gauss hypergeometric function (see [1]), and $(a)_s := a(a+1)(a+2)\cdots(a+s-2)(a+s-1)$ is the Pochhammer symbol.

By using the Gauss hypergeometric functions theory (see [1]) it is shown, in [8], the following differential equation, of the type (1.1), satisfied by the RJP:

Proposition 1. *For every $n \in \mathbb{N}$ $(n \geq 1)$, the RJP $y_n(x;N) \equiv P_n^{(\alpha,\beta;N)}(x)$ satisfy the following second order differential equation of generalized hypergeometric type:*

$$\begin{aligned} &\left[\frac{N+\beta}{N-\beta} - \frac{2\beta}{N-\beta}x - x^2\right]\frac{d^2y_n}{dx^2} + \\ &+ \left[\frac{N(\beta-\alpha)+\beta(2n-3/2)}{N-\beta} + \right. \\ &\left. - \frac{N(\alpha+\beta+2)+\beta(2n-3/2)}{N-\beta}x\right]\frac{dy_n}{dx} + \\ &+ \frac{n}{N-\beta}\left[N(\alpha+\beta+n+1)+\beta(n-1/2)\right]y_n = 0\,. \end{aligned} \tag{2.3}$$

Moreover, by extending some Nikiforov and Uvarov ideas (see [15]), it is shown, in [8], the following varying orthogonality property for the RJP:

Proposition 2. *For every $N > \beta$, the polynomial system*

$$\left\{P_n^{(\alpha,\beta;N)}(x)\right\}_{n=0}^{\infty}$$

*satisfies the varying orthogonality relation, in $\left(-\frac{N+\beta}{N-\beta}, 1\right)$, with respect to the varying measure*

$$\rho_n(x;N)\,dx = (1-x)^{\alpha}\left(x + \frac{N+\beta}{N-\beta}\right)^{\frac{\beta(N+\alpha+2n+1/2)}{N-\beta}} dx.$$

Further properties, as the Rodrigues' formula, the integral representation and the three term recurrence relation for RJP have been shown in [8]. It is possible to note that all these properties reduce to the classical ones in the non relativistic limit $(N \to \infty)$.

## 3. The Case's method.

Originally, this method allows us to give sum rules for the powers of zeros of polynomial solutions of (1.2), i. e. allows us to evaluate the sum $y_r$, see (1.3), and consequently all the moments of the density $\mu_r'^{(n)}$, see (1.4). Nevertheless, *we can also extend* (see [14]) *the Case's method in order to investigate the distribution of the zeros of polynomial solutions of* (1.1).

Let be $x_l$ $(l = 1, \dots, n)$ the simple zeros of polynomial solutions of (1.1), and $a_j^{(2)}$ $(j = 0, 1, 2)$, $a_i^{(1)}(n)$ $(i = 0, 1)$ the coefficients of the polynomials $\sigma$ and $\tau$, respectively, of (1.1) (we note that the coefficients of $\tau$ are depending on the degree $n$).

From (1.1) we can obtain, by means of some transformations (see [5]), the following recurrence relation formula:

$$2 \sum_{j=0}^{2} a_j^{(2)} J_{r+j}^{2} = -a_0^{(1)}(n)\, y_r - a_1^{(1)}(n)\, y_{r+l} \quad (r \geq 0), \tag{3.1}$$

where

$$J_r^{(2)} = \sum_{\neq} \frac{(x_{l_1})^r}{(x_{l_1} - x_{l_2})}\,; \tag{3.2}$$

(the symbol $\sum_{\neq}$ means that the sum runs over all $l_j$ $(j = 1, \dots, n)$, subject to the condition that all $l$ are different).

Throughout (3.1) we can compute the sum $y_r$, provided that the sum rules (3.2) are known. Nevertheless, we note that evaluation of these sum rules is very difficult. A first method which allows one to calculate the (3.2) has been given by J. S. Dehesa et al. ([4], [7]), but the resulting expressions are highly non-linear, specially when $r$ is increasing. An alternative method of evaluation of (3.2) has been given by P. E. Ricci by using the generalized Lucas polynomials of the second kind (see [3]).

## 4. The representation formula for the sum rules $J_r^{(i)}$.

By running over the proof of this result (see [16]), we can assert, as well as for the Case's method, that the representation formula *is still valid in the case of polynomial solutions of generalized hypergeometric-type* (1.1).

As a matter of fact, assume for any fixed $n$, the following form for a polynomial solution of (1.1):

$$y_n(x) = x^n - u_1 x^{n-1} + u_2 x^{n-2} - \ldots + (-1)^n u_n\,, \tag{4.1}$$

and consider the elementary symmetric functions of zeros of $y_n$:

$$\begin{aligned} u_0 &= 1\,, & u_1 &= \sum_{i=1}^{n} x_i\,, \\ u_2 &= \sum_{i<j}^{1,n} x_i x_j\,, \cdots\,, & u_n &= x_1 x_2 \cdots x_n\,. \end{aligned}$$

By Newton's formulas it is possible to express $y_1, \ldots, y_n$ (see (1.3)) in terms of $u_1, \ldots, u_n$:

$$\left\{ \begin{array}{lcl} y_1 &=& u_1 \\ y_2 &=& u_1 y_1 - 2u_2 = u_1^2 - 2u_2 \\ y_3 &=& u_1 y_2 - u_2 y_1 + 3u_3 = u_1^3 - 3u_1 u_2 + 3u_3 \\ \multicolumn{3}{c}{\dotfill} \\ y_n &=& u_1 y_{n-1} - u_2 y_{n-2} + \cdots + (-1)^{n-1} n u_n \end{array} \right. \tag{4.2}$$

and vice versa:

$$\left\{ \begin{array}{lcl} u_1 &=& y_1 \\ u_2 &=& \dfrac{1}{2}\Big(u_1 y_1 - y_2\Big) = \dfrac{1}{2}\Big(y_1^2 - y_2\Big) \\ u_3 &=& \dfrac{1}{3}\Big(-u_1 y_2 + u_2 y_1 + y_3\Big) = \dfrac{1}{6}\Big(y_1^3 - 3y_1 y_2 + 2y_3\Big) \\ \multicolumn{3}{c}{\dotfill} \\ u_n &=& \dfrac{1}{n}\Big\{\Big[(-1)^n u_1 y_{n-1} + (-1)^{n-1} u_2 y_{n-2} + \cdots + \\ && \quad + u_{n-1} y_1\Big] + (-1)^{n-1} y_n\Big\}. \end{array} \right. \tag{4.3}$$

Consider furthermore the Lucas polynomials of the second kind in $n$ variables, $\Phi_k(u_1, \cdots, u_n)$, defined by the recurrence formula (see [3])

$$\begin{cases} \Phi_{-1}(u_1, \cdots, u_n) &= \Phi_0(u_1, \cdots, u_n) = \cdots \\ &= \Phi_{n-3}(u_1, \cdots, u_n) = 0 \\ \Phi_{n-2}(u_1 \cdots, u_n) &= 1 \\ \Phi_k(u_1, \cdots, u_n) &= u_1 \Phi_{k-1}(u_1, \cdots, u_n) + \\ & \quad + u_2 \Phi_{k-2}(u_1, \cdots, u_n) + \cdots + \\ & \quad + (-1)^{n-1} u_n \Phi_{k-n}(u_1, \cdots, u_n) \\ & \quad (k \geq n-1). \end{cases} \tag{4.4}$$

Then, even under the more general assumptions considered here (i. e. avoiding condition ii)), the following result holds true (see [16]):

PROPOSITION 3. *For any $n \in \mathbb{N}\,(n \geq 2)$, $r \in \mathbb{N}_0$, $i \in \mathbb{N}$ and such that $2 \leq i \leq n$ the sum rules $J_r^{(i)}$ can be represented by the formula:*

$$J_r^{(i)} = (i-1)! \sum_{k=0}^{n-i} (-1)^k \binom{n-k}{i} u_k \, \Phi_{n+r-i-k-1}(u_1, \cdots, u_n). \tag{4.5}$$

*In the particular case $i = 2$ we have:*

$$J_r^{(2)} = \sum_{k=0}^{n-2} (-1)^k \binom{n-k}{2} u_k \Phi_{n+r-k-3}(u_1, \cdots, u_n). \tag{4.6}$$

## 5. Applications.

In this last Section we consider some applications about the calculation of the moments of the density distribution (1.5) of zeros of the RJP in some particular case. By using the recurrence relation (3.1) and the representation formula (4.6), we have determined for each of the considered example some of the moments for these orthogonal polynomials, by varying the parameter $N$ and the degree $n$, and besides the most important statistical parameters, i. e. the mean value $M$, the variance $\sigma^2$, the Fischer coefficient $\gamma_1$ and the Pearson index $\gamma_2$ (or kurtosis). A plot of $\sigma^2$ and $\gamma_2$ as functions of increasing values of $N$ and $n$ are also shown.

The results have been obtained by using a special FORTRAN program named "Momenti", which is available. It is worth to be noticed that by assuming large values for $N$, the moments and the statistical parameters tend to be the corresponding values generated by the classical Chebyshev polynomials of first and second kind.

In the applications 3, 4, 5 and 6, we assume $\alpha = \beta = \lambda - \frac{1}{2}$, $(\lambda > -\frac{1}{2})$, and so we can write $P_n^{(\alpha,\beta;N)}(x) = P_n^{(\lambda;N)}(x)$ (these polynomials correspond to the case of the relativistic Gegenbauer polynomials).

APPLICATION 1. Relativistic Jacobi Polynomials: $P_n^{(0;100)}(x)$ $\left(\alpha = 2,\, \beta = -\frac{1}{2}\right)$.

Figure 1: Variance (a) and kurtosis (b) of the zero's distribution of the relativistic Jabobi polynomials ($\alpha = 2, \beta = -1/2$) of degree 100 in terms of the relativistic parameter N (varying from 10 to 1000).

| | $N = 10$ | $N = 20$ | $N = 200$ |
|---|---|---|---|
| $\mu_1'$ | − 0.125325724$D$–01 | − 0.124697561$D$–01 | − 0.124132285$D$–01 |
| $\mu_2'$ | 0.468709395$D$+00 | 0.481280373$D$+00 | 0.492587551$D$+00 |
| $\mu_3'$ | 0.114739704$D$–01 | − 0.136158012$D$–03 | − 0.111098052$D$–01 |
| $\mu_4'$ | 0.326038739$D$+00 | 0.346017770$D$+00 | 0.365135708$D$+00 |
| $\mu_5'$ | 0.266116414$D$–01 | 0.836599693$D$–02 | − 0.101398030$D$–01 |
| $\mu_6'$ | 0.251842381$D$+00 | 0.276164198$D$+00 | 0.301116746$D$+00 |
| $\mu_7'$ | 0.368399628$D$–01 | 0.147788352$D$–01 | − 0.933879502$D$–02 |
| $\mu_8'$ | 0.204781012$D$+00 | 0.231463624$D$+00 | 0.260926020$D$+00 |
| $M$ | − 0.125325724$D$–01 | − 0.124697561$D$–01 | − 0.124132285$D$–01 |
| $\sigma^2$ | 0.468552329$D$+00 | 0.481124878$D$+00 | 0.492433462$D$+00 |
| $\gamma_1$ | 0.907074476$D$–01 | 0.535302831$D$–01 | 0.209231686$D$–01 |
| $\gamma_2$ | − 0.151027762$D$+01 | − 0.150329097$D$+01 | − 0.149462519$D$+01 |

| | $N = 500$ | $N = 1000$ |
|---|---|---|
| $\mu'_1$ | $-$ 0.124094601$D$–01 | $-$ 0.124082041$D$–01 |
| $\mu'_2$ | 0.493341137$D$+00 | 0.493592326$D$+00 |
| $\mu'_3$ | $-$ 0.118590259$D$–01 | $-$ 0.121092562$D$–01 |
| $\mu'_4$ | 0.366449561$D$+00 | 0.366888619$D$+00 |
| $\mu'_5$ | $-$ 0.114472262$D$–01 | $-$ 0.118851309$D$–01 |
| $\mu'_6$ | 0.302891463$D$+00 | 0.303486240$D$+00 |
| $\mu'_7$ | $-$ 0.111051149$D$–01 | $-$ 0.116985097$D$–01 |
| $\mu'_8$ | 0.263098201$D$+00 | 0.263828407$D$+00 |
| $M$ | $-$ 0.124094601$D$–01 | $-$ 0.124082041$D$–01 |
| $\sigma^2$ | 0.493187142$D$+00 | 0.493438362$D$+00 |
| $\gamma_1$ | 0.187769804$D$–01 | 0.180623371$D$–01 |
| $\gamma_2$ | $-$ 0.149397152$D$+01 | $-$ 0.149375157$D$+01 |

APPLICATION 2. Relativistic Chebyshev Polynomials of I kind: $P_{100}^{(0;N)}(x)$.

Figure 2: Variance (a) and kurtosis (b) of the zero's distribution of the relativistic Chebyshev polynomials of first kind of degree 100 in terms of the relativistic parameter N (varying from 10 to 1000).

| | $N = 10$ | $N = 20$ | $N = 200$ |
|---|---|---|---|
| $\mu'_1$ | $-$ 0.125644047$D$–03 | $-$ 0.628179186$D$–04 | $-$ 0.628153318$D$–05 |
| $\mu'_2$ | 0.475173715$D$+00 | 0.487590778$D$+00 | 0.498759431$D$+00 |
| $\mu'_3$ | 0.235363918$D$–01 | 0.120710987$D$–01 | 0.123433450$D$–02 |
| $\mu'_4$ | 0.334216508$D$+00 | 0.353950889$D$+00 | 0.372834385$D$+00 |
| $\mu'_5$ | 0.384510426$D$–01 | 0.204325489$D$–01 | 0.215709463$D$–02 |
| $\mu'_6$ | 0.260899847$D$+00 | 0.284924047$D$+00 | 0.309570540$D$+00 |
| $\mu'_7$ | 0.485229475$D$–01 | 0.267363219$D$–01 | 0.291867078$D$–02 |
| $\mu'_8$ | 0.214388973$D$+00 | 0.240745424$D$+00 | 0.269846477$D$+00 |
| $M$ | $-$ 0.125644047$D$–03 | $-$ 0.628179186$D$–04 | $-$ 0.628153318$D$–05 |
| $\sigma^2$ | 0.475173700$D$+00 | 0.487590774$D$+00 | 0.498759431$D$+00 |
| $\gamma_1$ | 0.724025219$D$–01 | 0.357237486$D$–01 | 0.353094233$D$–02 |
| $\gamma_2$ | $-$ 0.151973819$D$+01 | $-$ 0.151120197$D$+01 | $-$ 0.150123427$D$+01 |

| | $N = 500$ | $N = 1000$ |
|---|---|---|
| $\mu'_1$ | $-$ 0.251256661$D$–05 | $-$ 0.125640836$D$–05 |
| $\mu'_2$ | 0.499503782$D$+00 | 0.499751892$D$+00 |
| $\mu'_3$ | 0.494459357$D$–03 | 0.247350400$D$–03 |
| $\mu'_4$ | 0.374132114$D$+00 | 0.374565784$D$+00 |
| $\mu'_5$ | 0.865945126$D$–03 | 0.433491197$D$–03 |
| $\mu'_6$ | 0.311323465$D$+00 | 0.311910938$D$+00 |
| $\mu'_7$ | 0.117432072$D$–02 | 0.588306680$D$–03 |
| $\mu'_8$ | 0.271991975$D$+00 | 0.272713211$D$+00 |
| $M$ | $-$ 0.251256661$D$–05 | $-$ 0.125640836$D$–05 |
| $\sigma^2$ | 0.499503782$D$+00 | 0.499751892$D$+00 |
| $\gamma_1$ | 0.141129200$D$–02 | 0.705465453$D$–03 |
| $\gamma_2$ | $-$ 0.150049667$D$+01 | $-$ 0.150024883$D$+01 |

Application 3. Relativistic Chebyshev Polynomials of I kind: $P_n^{(0;100)}(x)$.

Figure 3: Variance (a) and kurtosis (b) of the zero's distribution of the relativistic Chebyshev polynomials of first kind with relativistic parameter N=100 in terms of the quantum principal number n (varying from 110 to 190).

| | $n = 110$ | $n = 130$ | $n = 150$ |
|---|---|---|---|
| $\mu'_1$ | $-$ 0.114155352$D$–04 | $-$ 0.965248180$D$–05 | $-$ 0.836115228$D$–05 |
| $\mu'_2$ | 0.497517059$D$+00 | 0.497514414$D$+00 | 0.497512479$D$+00 |
| $\mu'_3$ | 0.246490805$D$–02 | 0.246841436$D$–02 | 0.247097852$D$–02 |
| $\mu'_4$ | 0.370679701$D$+00 | 0.370675556$D$+00 | 0.370672527$D$+00 |
| $\mu'_5$ | 0.429150626$D$–02 | 0.429627394$D$–02 | 0.429975743$D$–02 |
| $\mu'_6$ | 0.306676993$D$+00 | 0.306671715$D$+00 | 0.306667861$D$+00 |
| $\mu'_7$ | 0.578446127$D$–02 | 0.579023718$D$–02 | 0.579445440$D$–02 |
| $\mu'_8$ | 0.266323599$D$+00 | 0.266320396$D$+00 | 0.266315869$D$+00 |
| $M$ | $-$ 0.114155352$D$–04 | $-$ 0.965248180$D$–05 | $-$ 0.836115228$D$–05 |
| $\sigma^2$ | 0.497517059$D$+00 | 0.497514414$D$+00 | 0.497512479$D$+00 |
| $\gamma_1$ | 0.707262153$D$–02 | 0.707517077$D$–02 | 0.707702655$D$–02 |
| $\gamma_2$ | $-$ 0.150244431$D$+01 | $-$ 0.150244520$D$+01 | $-$ 0.150244584$D$+01 |

| | $n = 170$ | $n = 190$ |
|---|---|---|
| $\mu'_1$ | $-\,0.737456354D{-}05$ | $-\,0.659622710D{-}05$ |
| $\mu'_2$ | $0.497511002D{+}00$ | $0.497509838D{+}00$ |
| $\mu'_3$ | $0.247293529D{-}02$ | $0.247447763D{-}02$ |
| $\mu'_4$ | $0.370670216D{+}00$ | $0.370668396D{+}00$ |
| $\mu'_5$ | $0.430241400D{-}02$ | $0.430450687D{-}02$ |
| $\mu'_6$ | $0.306664922D{+}00$ | $0.306662607D{+}00$ |
| $\mu'_7$ | $0.306664922D{-}02$ | $0.580020025D{-}02$ |
| $\mu'_8$ | $0.266312419D{+}00$ | $0.266309703D{+}00$ |
| $M$ | $-\,0.737456354D{-}05$ | $-\,0.659622710D{-}05$ |
| $\sigma^2$ | $0.497511002D{+}00$ | $0.497509838D{+}00$ |
| $\gamma_1$ | $0.707843793D{-}02$ | $0.707954746D{-}02$ |
| $\gamma_2$ | $-\,0.150244632D{+}01$ | $-\,0.150244670D{+}01$ |

APPLICATION 4. Relativistic Chebyshev Polynomials of II kind: $P_{100}^{(1;N)}(x)$.

Figure 4: Variance (a) and kurtosis (b) of the zero's distribution of the relativistic Chebyshev polynomials of second kind of degree 100 in terms of the relativistic parameter N (varying from 10 to 1000).

| | $N = 10$ | $N = 20$ | $N = 200$ |
|---|---|---|---|
| $\mu'_1$ | $0.124362587D{-}03$ | $0.621853242D{-}04$ | $0.621902644D{-}05$ |
| $\mu'_2$ | $0.519549021D{+}00$ | $0.507278348D{+}00$ | $0.496228180D{+}00$ |
| $\mu'_3$ | $-\,0.254363502D{-}01$ | $-\,0.124224991D{-}01$ | $-\,0.121560360D{-}02$ |
| $\mu'_4$ | $0.414132255D{+}00$ | $0.390748058D{+}00$ | $0.370888914D{+}00$ |
| $\mu'_5$ | $-\,0.475834748D{-}01$ | $-\,0.224363351D{-}01$ | $-\,0.212697420D{-}02$ |
| $\mu'_6$ | $0.371461972D{+}00$ | $0.336390778D{+}00$ | $0.308520840D{+}00$ |
| $\mu'_7$ | $-\,0.694195383D{-}01$ | $-\,0.314929603D{-}01$ | $-\,0.288532208D{-}02$ |
| $\mu'_8$ | $0.353366328D{+}00$ | $0.305316940D{+}00$ | $0.269730221D{+}00$ |
| $M$ | $0.124362587D{-}03$ | $0.621853242D{-}04$ | $0.621902644D{-}05$ |
| $\sigma^2$ | $0.519549006D{+}00$ | $0.507278344D{+}00$ | $0.496228180D{+}00$ |
| $\gamma_1$ | $-\,0.684403162D{-}01$ | $-\,0.346445919D{-}01$ | $-\,0.350400683D{-}02$ |
| $\gamma_2$ | $-\,0.146573867D{+}01$ | $-\,0.148152506D{+}01$ | $-\,0.149380556D{+}01$ |

| | $n = 500$ | $n = 1000$ |
|---|---|---|
| $\mu'_1$ | $0.248755848D{-}05$ | $0.124390355D{-}05$ |
| $\mu'_2$ | $0.495491281D{+}00$ | $0.495245642D{+}00$ |
| $\mu'_3$ | $-0.485530495D{-}03$ | $-0.242646551D{-}03$ |
| $\mu'_4$ | $0.369603962D{+}00$ | $0.369176714D{+}00$ |
| $\mu'_5$ | $-0.847749416D{-}03$ | $-0.423368721D{-}03$ |
| $\mu'_6$ | $0.306778685D{+}00$ | $0.306201070D{+}00$ |
| $\mu'_7$ | $-0.114741548D{-}02$ | $-0.572592963D{-}03$ |
| $\mu'_8$ | $0.267586273D{+}00$ | $0.266877591D{+}00$ |
| $M$ | $0.248755848D{-}05$ | $0.124390355D{-}05$ |
| $\sigma^2$ | $0.495491281D{+}00$ | $0.495245642D{+}00$ |
| $\gamma_1$ | $-0.140267625D{-}02$ | $-0.701517310D{-}03$ |
| $\gamma_2$ | $-0.149455605D{+}01$ | $-0.149480428D{+}01$ |

Application 5. Relativistic Chebyshev Polynomials of II kind: $P_n^{(0;100)}(x)$.

Figure 5: Variance (a) and kurtosis (b) of the zero's distribution of the relativistic Chebyshev polynomials of second kind with relativistic parameter N=100 in terms of the quantum principal number n (varying from 110 to 190).

| | $n = 110$ | $n = 130$ | $n = 150$ |
|---|---|---|---|
| $\mu'_1$ | $0.113122161D{-}04$ | $0.957857883D{-}05$ | $0.830570509D{-}05$ |
| $\mu'_2$ | $0.497914788D{+}00$ | $0.498620185D{+}00$ | $0.499137480D{+}00$ |
| $\mu'_3$ | $-0.244395969D{-}02$ | $-0.245447302D{-}02$ | $-0.246219005D{-}02$ |
| $\mu'_4$ | $0.373619547D{+}00$ | $0.374509295D{+}00$ | $0.375161787D{+}00$ |
| $\mu'_5$ | $-0.429288387D{-}02$ | $-0.431367283D{-}02$ | $-0.432893161D{-}02$ |
| $\mu'_6$ | $0.312097765D{+}00$ | $0.313085631D{+}00$ | $0.313810084D{+}00$ |
| $\mu'_7$ | $-0.584777004D{-}02$ | $-0.587970006D{-}02$ | $-0.590313570D{-}02$ |
| $\mu'_8$ | $0.274048872D{+}00$ | $0.275102829D{+}00$ | $0.275875755D{+}00$ |
| $M$ | $0.113122161D{-}04$ | $0.957857883D{-}05$ | $0.830570509D{-}05$ |
| $\sigma^2$ | $0.497914788D{+}00$ | $0.498620184D{+}00$ | $0.499137480D{+}00$ |
| $\gamma_1$ | $-0.700412494D{-}02$ | $-0.701182945D{-}02$ | $-0.701745270D{-}02$ |
| $\gamma_2$ | $-0.149297773D{+}01$ | $-0.149366003D{+}01$ | $-0.149416175D{+}01$ |

| | $n = 170$ | $n = 190$ |
|---|---|---|
| $\mu'_1$ | $0.733145056D{-}05$ | $0.656176257D{-}05$ |
| $\mu'_2$ | $0.499533062D{+}00$ | $0.499845365D{+}00$ |
| $\mu'_3$ | $-\,0.246809544D{-}02$ | $-\,0.247276012D{-}02$ |
| $\mu'_4$ | $0.375660757D{+}00$ | $0.376054685D{+}00$ |
| $\mu'_5$ | $-\,0.434060779D{-}02$ | $-\,0.434983055D{-}02$ |
| $\mu'_6$ | $0.314364087D{+}00$ | $0.314801464D{+}00$ |
| $\mu'_7$ | $-\,0.592106862D{-}02$ | $-\,0.593523324D{-}02$ |
| $\mu'_8$ | $0.276466830D{+}00$ | $0.276933476D{+}00$ |
| $M$ | $0.733145056D{-}05$ | $0.656176257D{-}05$ |
| $\sigma^2$ | $0.499533062D{+}00$ | $0.499845365D{+}00$ |
| $\gamma_1$ | $-\,0.702173760D{-}02$ | $-\,0.702511112D{-}02$ |
| $\gamma_2$ | $-\,0.149454618D{+}01$ | $-\,0.149485015D{+}01$ |

APPLICATION 6. Relativistic Chebyshev Polynomials of III kind: $P_{100}^{(\alpha,\beta;N)}(x) \quad (\alpha = -1/2, \beta = 1/2)$.

Figure 6: Variance (a) and kurtosis (b) of the zero's distribution of the relativistic Chebyshev polynomials of third kind of degree 100 in terms of the relativistic parameter N (varying from 10 to 1000).

| | $N = 10$ | $N = 20$ | $N = 200$ |
|---|---|---|---|
| $\mu'_1$ | $0.512435941D{-}02$ | $0.506218375D{-}02$ | $0.500621887D{-}02$ |
| $\mu'_2$ | $0.521924032D{+}00$ | $0.509715853D{+}00$ | $0.498721931D{+}00$ |
| $\mu'_3$ | $-\,0.203051188D{-}01$ | $-\,0.735844480D{-}02$ | $0.379066131D{-}02$ |
| $\mu'_4$ | $0.417024081D{+}00$ | $0.393760131D{+}00$ | $0.374002942D{+}00$ |
| $\mu'_5$ | $-\,0.423368491D{-}01$ | $-\,0.173201684D{-}01$ | $0.288402855D{-}02$ |
| $\mu'_6$ | $0.374558780D{+}00$ | $0.339669308D{+}00$ | $0.311943395D{+}00$ |
| $\mu'_7$ | $-\,0.640578455D{-}01$ | $-\,0.263291371D{-}01$ | $0.212966693D{-}02$ |
| $\mu'_8$ | $0.356545113D{+}00$ | $0.308746357D{+}00$ | $0.273344578D{+}00$ |
| $M$ | $0.512435941D{-}02$ | $0.506218375D{-}02$ | $0.500621887D{-}02$ |
| $\sigma^2$ | $0.521897773D{+}00$ | $0.509690227D{+}00$ | $0.498696868D{+}00$ |
| $\gamma_1$ | $-\,0.751353510D{-}01$ | $-\,0.414943580D{-}01$ | $-\,0.105040151D{-}01$ |
| $\gamma_2$ | $-\,0.146711699D{+}01$ | $-\,0.148340427D{+}01$ | $-\,0.149616332D{+}01$ |

| | $N = 500$ | $N = 1000$ |
|---|---|---|
| $\mu'_1$ | 0.500248750D–02 | 0.500124387D–02 |
| $\mu'_2$ | 0.497988782D+00 | 0.497744392D+00 |
| $\mu'_3$ | 0.451697172D–02 | 0.475860393D–02 |
| $\mu'_4$ | 0.372724582D+00 | 0.372299525D+00 |
| $\mu'_5$ | 0.415663570D–02 | 0.457882118D–02 |
| $\mu'_6$ | 0.310210232D+00 | 0.309635597D+00 |
| $\mu'_7$ | 0.385854473D–02 | 0.443038127D–02 |
| $\mu'_8$ | 0.271211751D+00 | 0.270506744D+00 |
| $M$ | 0.500248750D–02 | 0.500124387D–02 |
| $\sigma^2$ | 0.497963757D+00 | 0.497719380D+00 |
| $\gamma_1$ | − 0.841309322D–02 | − 0.771541508D–02 |
| $\gamma_2$ | − 0.149694673D+01 | − 0.149720595D+01 |

APPLICATION 7. Relativistic Chebyshev Polynomials of IV kind: $P_n^{(\alpha,\beta;100)}(x) \quad (\alpha = 1/2,\ \beta = -1/2)$.

Figure 7: Variance (a) and kurtosis (b) of the zero's distribution of the relativistic Chebyshev polynomials of fourth kind with relativistic parameter N=100 in terms of the quantum principal number n (varying from 110 to 190).

| | $n = 110$ | $n = 130$ | $n = 150$ |
|---|---|---|---|
| $\mu'_1$ | − 0.455686984D–02 | − 0.385580618D–02 | − 0.334169439D–02 |
| $\mu'_2$ | 0.495232968D+00 | 0.495581722D+00 | 0.495837479D+00 |
| $\mu'_3$ | − 0.206923861D–02 | − 0.136817154D–02 | − 0.854062733D–03 |
| $\mu'_4$ | 0.367819031D+00 | 0.368254990D+00 | 0.368574703D+00 |
| $\mu'_5$ | − 0.234300280D–03 | 0.466744860D–03 | 0.980831973D–03 |
| $\mu'_6$ | 0.303525369D+00 | 0.304004957D+00 | 0.304356670D+00 |
| $\mu'_7$ | − 0.126547080D–02 | 0.196647535D–02 | 0.248052712D–02 |
| $\mu'_8$ | 0.262991536D+00 | 0.263498421D+00 | 0.263870158D+00 |
| $M$ | − 0.455686984D–02 | − 0.385580618D–02 | − 0.334169439D–02 |
| $\sigma^2$ | 0.495212203D+00 | 0.495566854D+00 | 0.495826312D+00 |
| $\gamma_1$ | 0.134888936D–01 | 0.125101557D–01 | 0.117910420D–01 |
| $\gamma_2$ | − 0.150003945D+01 | − 0.150041397D+01 | − 0.150068779D+01 |

| | $n = 170$ | $n = 190$ |
|---|---|---|
| $\mu'_1$ | $-\ 0.294855097D{-}02$ | $-\ 0.263817513D{-}02$ |
| $\mu'_2$ | $0.496033061D{+}00$ | $0.496187470D{+}00$ |
| $\mu'_3$ | $-\ 0.460924720D{-}03$ | $-\ 0.150555057D{-}03$ |
| $\mu'_4$ | $0.368819195D{+}00$ | $0.369012219D{+}00$ |
| $\mu'_5$ | $0.137395022D{-}02$ | $0.168430234D{-}02$ |
| $\mu'_6$ | $0.304625636D{+}00$ | $0.304837984D{+}00$ |
| $\mu'_7$ | $0.287361518D{-}02$ | $0.318394155D{-}02$ |
| $\mu'_8$ | $0.264154439D{+}00$ | $0.264378879D{+}00$ |
| $M$ | $-\ 0.294855097D{-}02$ | $-\ 0.263817513D{-}02$ |
| $\sigma^2$ | $0.496024367D{+}00$ | $0.496180510D{+}00$ |
| $\gamma_1$ | $0.112403510D{-}01$ | $0.108051187D{-}01$ |
| $\gamma_2$ | $-\ 0.150089670D{+}01$ | $-\ 0.150106134D{+}01$ |

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## REFERENCES

[1] ABRAMOVITZ M. and STEGUN I. A., *Handbook of mathematical function*, U. S. Department of Commerce, National Bureau of Standards, Appl. Math. Series **55** (1964).

[2] ALDAYA V., BISQUERT J. and NAVARRO-SALAS J., *The quantum relativistic harmonic oscillator: The relativistic Hermite polynomials*, J. Phys. Lett. **A 156** (1991), 381-385.

[3] BRUSCHI M. and RICCI P. E., *I polinomi di Lucas e di Tchebycheff in più variabili*, Rend. Mat. **VI**, Vol. **13** (1980).

[4] BUENDIA E., DEHESA J. K. and SANCHEZ-BUENDIA M. A., *On the zeros of eigenfunctions of polynomial differential operators*, J. Math. Phys. **26** (1985), 2729.

[5] CASE K. M., *Sum rules for zeros of polynomials (I) and (II)*, J. Math. Phys. **21** (1980), 702.

[6] CHIHARA T. S., *An introduction to orthogonal polynomials*, Gordon and Breach, New York (1978).

[7] DEHESA J. K., BUENDIA E. and SANCHEZ-BUENDIA M. A., *On the polynomial solutions of ordinary differential equations of the fourth order*, J. Math. Phys. **26** (1985), 1547.

[8] HE M. X. and NATALINI P., *The Relativistic Jacobi Polynomials*, (sub-

mitted).

[9] HE M. X., NOSCHESE S. and RICCI P. E., *The Relativistic Szegö Polynomials*, Integral Transforms and Special Functions (to appear).

[10] NAGEL B., *The relativistic Hermite polynomial is a Gegenbauer polynomial*, J. Math. Phys. **35** (1994), 1549.

[11] NATALINI P., *The Relativistic Laguerre Polynomials*, Rend. Mat. Appl. (to appear).

[12] NATALINI P., *Sul calcolo dei momenti della densità degli zeri dei polinomi ortogonali classici e semiclassici*, Calcolo (to appear).

[13] NATALINI P. and NOSCHESE S., *Some properties of the Relativistic Laguerre Polynomials*, Atti Sem. Mat. Fis. Univ. Modena (to appear).

[14] NATALINI P. and RICCI P. E., *On the moments of the density of zeros for the relativistic Hermite and Laguerre Polynomials*, Computers Math. Applic. **31**, no. 6 (1995), 87-96.

[15] NIKIFOROV A. F. and UVAROV V. B., *Special Functions of Mathematical Physics*, (Birkhauser, Basel, 1988).

[16] RICCI P. E., *Sum rules for zeros of polynomials and generalized Lucas polynomials*, J. Math. Phys. **34** (1993), 4884.

[17] RIORDAN J., *An introduction to combinatorial analysis*, Princeton U. P., Princeton (1980).

[18] SZEGÖ G., *Orthogonal polynomials*, Amer. Math. Sec. Colloq. Publ. **23**, Amer. Math. Soc., New York (1939).

[19] ZARZO A., DEHESA J. S. and TORRES J., *On a new set of polynomials representing the wave functions of the quantum relativistic harmonic oscillator*, J. Phys. A: Math. Gen.

[20] ZARZO A. and MARTINEZ A., *The quantum relativistic harmonic oscillator: Spectrum of zeros of its wave functions*, J. Math. Phys. **34** (7) (1993), 2926-2935.