GEOMETRICAL STRUCTURES ON DIFFERENTIABLE MANIFOLDS (*)

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Sommario. - *Si studiano le (X, G)-varietà e si danno alcuni esempi: quando il modello geometrico è la coppia $(G/H, H)$, si danno condizioni necessarie e sufficienti affinchè ad una riduzione del fibrato degli r-getti su una varietà differenziabile M corrisponda una (X, G)-struttura sopra M.*

Summary. - *We study (X, G)-manifolds and we give examples: when the geometric model is the couple $(G/H, H)$, we give necessary and sufficient conditions ensuring that a reduction of the r-frames bundle on a differentiable manifold M gives rise to a (X, G)-structure on M.*

1. Introduction.

The study of further structures on a differentiable manifold appears as one of the general frameworks in geometry.

Clearly, a very interesting situation is represented by those structures for which uniformization theorems are available. This is the case of (X, G)-manifolds, i. e. those manifolds locally modelled on geometric spaces (see [9]). Typical examples are *locally conformally flat* manifolds (see [8], [12]), *spherical* manifolds (see [4]), *quaternionic coordinate* manifolds (see [13]) and Riemannian manifolds *locally modelled* on homogeneous space (see [2]).

In the present paper we investigate (X, G)-structures and discuss several basic examples; moreover, when the model space is a homogeneous manifold, we describe (X, G)-structures as special reductions of the bundle of r-frames (see Propositions 3.1 and 3.2).

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We recall some facts about (X, G)-structures. Let X be a differentiable manifold and G be a *formally analytic* subgroup of Diff(X), i. e. such that if $g \in G$ coincides with id_X in some open subset of X, then $g = id_X$. The couple (X, G) is the *geometric model*. A (X, G)-*structure* on a differentiable manifold is given by an open covering $\{U_\alpha\}_{\alpha \in A}$ of M and diffeomorphisms $\varphi_\alpha : U_\alpha \to X$ onto open sets of X such that, for every pair (α, β) with $U_\alpha \cap U_\beta \neq \emptyset$, the change of coordinates map $\varphi_\alpha \circ \varphi_\beta^{-1}$ is the restriction of an element of G. A map $f : M \longrightarrow N$ between two (X, G)-manifolds is a (X, G)-*map* if for every $p \in M$ there exist a local chart (U, φ) around p and a local chart (V, ψ) around $f(p)$ for the (X, G)-geometries of M and N respectively such that $\psi \circ f \circ \varphi^{-1} : \varphi(U) \longrightarrow \psi(V)$ is the restriction of an element of G. A (X, G)-map is a local diffeomorphism. Let M be a simply connected (X, G)-manifold and $p_0 \in \tilde{M}$ and (U_0, φ_0) be a (X, G)-chart around p_0: we set $\Phi = \varphi_0 \quad \text{on}\, U_0$. Then we can analitically continue Φ on every curve for p_0 and since M is simply connected, we get a (X, G)-map $\Phi : M \to X$, that is unique up to left composition with elements of G. Φ is the *developing* map of the (X, G)-structure. If M is not simply connected, then we take the universal covering $\tilde{M}$ of M, that is still a (X, G)-manifold: the developing map $\Phi : \tilde{M} \longrightarrow X$ induces a homomorphism $\rho : \pi_1(M) \longrightarrow G$, such that

$$\Phi \circ [\gamma] = \rho([\gamma]) \circ \Phi \,, \qquad (*)$$

where $\pi_1(M)$ is viewed as the group of the deck transformations of $\tilde{M}$. The homomorphism ρ is called *holonomy representation* of the (X, G)-structure.

Vice versa, (X, G)-structures on M are determined by a homomorphism $\rho : \pi_1(M) \longrightarrow G$ and an equivariant immersion $\Phi : \tilde{M} \longrightarrow X$ (i. e. such that the $(*)$ holds). A (X, G)-structure on M is said to be *complete* if the developing map is a covering map on its image; it is said to be *uniformizable* if the developing map is injective. Note that in the latter case ρ is injective and $M = \Phi(\tilde{M})/\rho(\pi_1(M))$.

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2. Examples.

In this section we give a list of examples of (X, G)-manifolds.

1) LOCALLY CONFORMALLY FLAT MANIFOLDS. Let $X = S^n$ be the unit sphere in $\mathbb{R}^{n+1}$ and $G = C_n$ be the conformal group of S^n; we recall that an n-dimensional manifold M is called *locally conformally flat* if there exists an atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ such that, for every $\alpha \in A$, $\varphi_\alpha : U_\alpha \longrightarrow S^n$ is an open diffeomorphism on the image, and if $U_\alpha \cap U_\beta \neq \emptyset$ then the change of coordinates map is a conformal diffeomorphism (see [8], [12]). If $n > 2$, by Liouville's Theorem it follows that $\varphi_\alpha \circ \varphi_\beta^{-1}$ is the restriction of an element of C_n. Thus a locally conformally flat manifold is a (S^n, C_n)-manifold.

If M is compact and the conformal invariant $d(M)$ (see [12] for the definition) is less than $\frac{(n-2)^2}{2}$, then by a Theorem of [12] it follows that the developing map Φ is injective.

2) Take $X = \mathbb{C}$ and $G = \text{Aut } \mathbf{Hol}(\mathbb{C}) = \{f(z) = az + b \,|a, b \in \mathbb{C},\ a \neq 0\}$; the compact (X, G)-manifolds are the complex tori. In fact, let $\mathbb{C}/\Gamma$ be a complex torus and $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ be the complex atlas which defines the complex structure on $\mathbb{C}/\Gamma$; if $U_\alpha \cap U_\beta \neq \emptyset$, then $\varphi_\alpha \circ \varphi_\beta^{-1}$ is a translation. The converse is a consequence of the following

THEOREM. ([3]) *If M is compact and it is not a torus, then M cannot be covered by any system (x_α^1, x_β^2) of local coordinates such that $|\dfrac{\partial x_\alpha^i}{\partial x_\beta^j}|$ is constant on $U_\alpha \cap U_\beta$, for each pair of indeces (α, β).*

Note that in this case the (X, G)-structure is uniformizable and complete.

3) Fix $X = \mathbb{R}^n$ and $G = \text{Aff}(\mathbb{R}^n) = \mathbb{R}^n \rtimes \text{GL}(n, \mathbb{R})$, the affine transformations of $\mathbb{R}^n$. In such a case the (X, G)-manifolds are the locally flat manifolds (i. e., such that there exists a linear torsion free connection whose curvature vanishes).

4) Let X be a differentiable manifold and $G = \{e\}$ be the trivial subgroup of $\mathrm{Diff}(X)$. If M is a (X, G)-manifold it is possible to define a global map $\psi : M \longrightarrow X$ in the following way: for every $p \in M$ we take a (X, G)-chart $(U_\alpha, \varphi_\alpha)$ around p and we set $\psi(p) = \varphi_\alpha(p)$. Since $G = \{e\}$, the map ψ is well defined.

If M is compact, then $\psi : M \longrightarrow X$ is a covering projection. In such a case the $(X\,,\,e)$-structure is complete but not necessarily uniformizable.

Vice versa, a covering space (M, ψ) of X is a (X, G)-manifold: this is immediate because ψ is an equivariant immersion of M in X.

5) SPHERICAL MANIFOLDS. A connected real hypersurface M in the complex manifold N of complex dimension $(n+1)$ is said to be *spherical* if, at every point $p \in M$, there exists a local holomorphic coordinate system $(z_1, \ldots, z_{n+1})$ of N such that M is defined by

$$|z_1|^2 + \ldots + |z_{n+1}|^2 = 1$$

(see [4]). For example, the unit sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$ is a spherical manifold. Let B_{n+1} be the unit ball in $\mathbb{C}^{n+1}$; we recall that the group $\mathrm{SU}(n+1,\,1)$ acts transitively on B_{n+1} and on S^{2n+1} by the fractional linear transformations

$$z \longmapsto \frac{Az + B}{Cz + D}\,,$$

where $A \in M_{n,n}(\mathbb{C})$, $B \in M_{n,1}(\mathbb{C})$, $C \in M_{1,n}(\mathbb{C})$, $D \in \mathbb{C}$ satisfy the following identities:

$${}^t\bar{A}A -{}^t \bar{C}C = I_n, \quad {}^t\bar{A}B ={}^t \bar{C}D, \quad \bar{D}D -{}^t \bar{B}B = 1.$$

Further the automorphisms group of B_{n+1}, $\mathrm{Aut}(B_{n+1})$ and the CR-automorphisms group of S^{2n+1}, $\mathrm{Aut}_{\mathrm{CR}}(B_{n+1})$ are given by the quotient

$$\mathrm{SU}(n+1,\,1)/\text{ center}\,.$$

We have the following

THEOREM. ([1]) *Let f be a biholomorphic map from a connected neighbourhood U of $p \in S^{2n+1}$. If $f(U \cap S^{2n+1}) \subset S^{2n+1}$, then f is the restriction to U of a fractional linear transformation.*

Let M be a spherical manifold and $\mathcal{U} = (U_\alpha\,,\, \varphi_\alpha)_{\alpha\in A}$ be a spherical atlas: if $U_\alpha \cap U_\beta \neq \emptyset$, then the change coordinate map

$$\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \longrightarrow \varphi_\alpha(U_\alpha \cap U_\beta)$$

is a local biholomorphism from an open set in $\mathbb{C}^{n+1}$ intersecting S^{2n+1} to S^{2n+1}. By the previous Theorem it follows that $\varphi_\alpha \circ \varphi_\beta^{-1}$ is the restriction of a linear fractional transformation. Therefore spherical manifolds are $(S^{2n+1}, \mathrm{Aut}_{\mathrm{CR}}(S^{2n+1}))$-manifolds.

6) Let $X = S^{2n+1} = \{z \in \mathbb{C}^{n+1} : |z| = 1\}$ and $G = \mathbb{Z}_m$ be the m-cyclic group generated by $g = e^{2\pi i m}$, acting on S^{2n+1} by scalar multiplication; then the *lens space* is defined as $\mathrm{L}_{(m)}^{2n+1} = S^{2n+1}/\mathbb{Z}_m$. Set $\rho = id_{\mathbb{Z}_m}$ and $\Phi = id_{S^{2n+1}}$ $\mathbb{Z}_m$ being isomorphic to $\pi_1(\mathrm{L}_{(m)}^{2n+1})$;therefore it follows that $\mathrm{L}_{(m)}^{2n+1})$ is a (X, G)-manifold and, by definition, is both uniformizable and complete.

7) Coordinate quaternionic manifolds. We recall the definition of quaternionic structure in the sense of Sommese (see [13]). A *quaternionic manifold* is a differentiable manifold with an open cover $\{U_i\}$ of M and diffeomorphisms $\varphi_i : U_i \to \mathbb{R}^{4n}$ such that $\varphi_i \circ \varphi_j^{-1}$ is a quaternionic map with respect to the standard right quaternionic structure on $\mathbb{R}^{4n} \simeq \mathbf{H}^n$. By Proposition I of [13] it follows that the change of coordinates map is the restriction of a quaternionic affine map and therefore coordinate quaternionic manifolds are $(\mathbf{H}^n, \mathrm{Aff}\,(\mathbf{H}^n))$-manifolds.

By a result of [5] it follows that the compact $(\mathbf{H}, \mathrm{Aff}\,(\mathbf{H}))$-manifolds are uniformizable.

8) Let $X = S^n = O(n+1)/O(n)$ be the unit sphere in $\mathbb{R}^{n+1}$ and $H = O(n)$ be the orthogonal group as a subgroup of $O(n+1)$; let $\mathbb{Z}_2$ be the cyclic group of order two generated by a and $\mathbf{P}^n(\mathbb{R}) = S^n/\mathbb{Z}_2$ be the real projective space. $\mathbf{P}^n(\mathbb{R})$ is a $(S^n,\, O(n))$-manifold. It is sufficient to give the holonomy representation $\rho : \pi_1(\mathbf{P}^n(\mathbb{R})) \longrightarrow O(n)$ and the equivariant immersion $\Phi : S^n \longrightarrow S^n$, S^n being the universal covering of $\mathbf{P}^n(\mathbb{R})$. Since $\pi_1(\mathbf{P}^n(\mathbb{R}))$ is isomorphic to $\mathbb{Z}_2$, we set

$$\rho(e) = I \quad , \quad \rho(a) = -I$$

and $\Phi = id_{S^n}$, where I is the identity in $O(n)$.

9) Let $X = S^6 = \{x \in \mathcal{I}m\,\mathbf{Cay} \mid \;\| x \|= 1\}$ and G_2 =Aut ($\mathbf{Cay}$). We recall that

$$G_2 = \{g \in \mathrm{O}(7) \,:\, g^*(\omega) = \omega\}\,,$$

where $\omega \in \otimes^3(\mathrm{Im\ Cay})^*$ is given by

$$\omega(x, y, z) = \langle x, yz\rangle\,.$$

REMARK 2.1. If $\Gamma \subset \mathrm{O}(7)$ is a group acting freely on S^6, then $\Gamma \simeq \mathbb{Z}_2$. In fact, let $g \in \Gamma$; g has at least one real eigenvalue λ that is 1 or -1. If $\lambda = 1$ $(= -1)$ then g (respectively g^2) has fixed points and consequently $g = I$ $(g^2 = I)$. Therefore, if $g \neq I$ then all the eigenvalues of g are -1 and since g is diagonalizable, $g = -I$.

Since $\Gamma \not\subset G_2$, the previous remark implies that the only compact (S^6, G_2)-manifold is S^6.

3. (X, G)-structures as special reductions.

Let G be a Lie group and H be a closed subgroup. In this section we consider the (X, G)-manifolds whose geometric model is given by an n-dimensional homogeneous space $X = G/H$ and by the subgroup H. Let us denote by o the origin of X, (i. e. the coset H) and fix a linear frame $u_o \in L(X)_o$; we assume that the *linear isotropy representation of* H, $\alpha : H \longrightarrow \mathrm{GL}\,(n, \mathbb{R})$ defined by

$$\alpha(h) = u_o^{-1} \circ h_* \circ u_o$$

h_* being the differential of h in o, is faithful.

REMARK 3.1. If the subgroup H is compact, then this hypothesis is satisfied. In such a case the Lie algebra g of G admits an ad(H)-invariant scalar product which corresponds to a G-invariant metric on the homogeneous space $X = G/H$. If $h \in$Ker(α), then we have $h_*[o] = id_{T_oX}$ and $h(o) = o$, h being in H. Therefore h fixes the geodesics starting from $o \in X$. Let N be a normal neighbourhood of o in X and $U = \{x \in N \,:\, h(x) = x\} \neq \emptyset$: U is open and closed in X and consequently $h = e$, i. e. α is faithful.

Vice versa: if the linear isotropy representation of H is faithful and G admits a bi-invariant Riemannian metric, then H is compact. This fact is a consequence of the following

THEOREM. ([10]) *Let G be a connected Lie group; G has a bi-invariant metric if and only if*

$$G = \mathbb{R}^s \times K\,,$$

where K is a compact Lie group.

In particular we have that

$$H = \mathbb{R}^p \times K'\,.$$

By the faithfulness of the linear isotropy representation, the factor $\mathbb{R}^p$ cannot occur in the last decomposition.

Let V and V' be two neighbourhoods of o and

$$f : V \longrightarrow M\ ,\ f' : V' \longrightarrow M$$

be two diffeomorphisms onto their images such that $f(o) = f'(o) = p$; f and f' define the same r-jet at p if the have the same partial derivatives up to the order r at o. The equivalence class of f is called an *r-frame* at p and is denoted by $j^r_p(f)$. We set

$$\begin{aligned}
G^r(n) &= \{r - \text{frames at}\, o \in X\}\\
\Gamma^r_H &= \{j^r_o(f)\ :\ f \in\ H\}\\
L^r(M)_p &= \{r - \text{frames at}\, p \in M\}\\
L^r_G(M)_p &= \{j^r_p(f)\ \in L^r(M)_p\ :\ f^{-1}\,\text{is a}\,(X,G) - \text{chart}\\
&\qquad\qquad \text{around}\, p \in M\}\\
L^r(M) &= \bigcup_{p\in M} L^r(M)_p\\
L^r_G(M) &= \bigcup_{p\in M} L^r_G(M)_p.
\end{aligned}$$

The set $G^r(n)$ is a group with the product given by $j^r_o(f)\,j^r_o(g) = j^r_o(f \circ g)$. It acts on $L^r(M)$ on the right in the following way: if $u = j^r_p(f)\ \in L^r(M)$ and $a = j^r_o(g)\ \in G^r(n)$, then $ua = j^r_p(f \circ g)$. Let

$\pi : L^r(M) \longrightarrow M$ be the projection defined by $\pi(j^r_p(f)) = p$; then $(L^r(M), \pi, G^r(n))$ is a principal $G^r(n)$-bundle, called the *bundle of r-frames.* If $n = 1$, then $L^1(M)$ is the bundle of linear frames. We remark that $L^r_G(X) = G$ and that the subgroup Γ^r_H is isomorphic to H.

A H-reduction $P \subset L^r(M)$ is said to be *integrable* if for every $p \in M$ there exists a neighbourhood U of p and a diffeomorphism $\varphi : U \longrightarrow X$ onto its image such that

$$\varphi_* : P|_U \longrightarrow G|_{\varphi(U)},$$

where $\varphi_*(j^r_q(f)) = j^r_{\varphi(q)}(\varphi \circ f)$. We have the following

PROPOSITION 3.1. *Let M be a (X, H)-manifold; then $L^1_G(M)$ is an integrable H-reduction of $L^1(M)$.*

Proof. The subgroup H acts on $L^1_G(M)$ on the right in the following way: for $u = j^1_p(f) \in L^1_G(M)$ and $a = j^1_o(h) \in H$, then

$$ua = j^1_p(f \circ h).$$

Since an element $h \in G$ belongs to H if and only if $h(o) = o$, then $(f \circ h)^{-1}$ is a (X, H)-chart such that $(f \circ h)(o) = p$. Let $j^1_p(f)$, $j^1_p(f')$ be in $\pi^{-1}(p)$; by the definition of (X, H)-manifold it follows that

$$(f^{-1} \circ f')|_{f'^{-1}(U \cap U')} = h|_{f'^{-1}(U \cap U')}$$

where $h \in H$. Thus $j^1_p(f') = j^1_p(f \circ h)$, i.e. H is transitive on the fibre $\pi^{-1}(p)$. Therefore $L^1_G(M)$ is a subbundle of $L(M)$ whose structural group is H.

Let p be a point of M, $(U_\alpha, \varphi_\alpha, V_\alpha)$ be a local (X, H)-chart around p and $j^1_q(f) \in L^1_G(M)|_{U_\alpha}$; set

$$\varphi_{\alpha *}(j^1_q(f)) = j^1_{f^{-1}(q)}(\varphi_\alpha \circ f).$$

This definition does not depend on the local coordinates: if $(U_\beta, \varphi_\beta, V_\beta)$ is another local (X, H)-chart around p, we have

$$\varphi_\alpha \circ \varphi_\beta^{-1}|_{\varphi_\beta(U_\alpha \cap U_\beta)} = h_{\alpha\beta}|_{\varphi_\beta(U_\alpha \cap U_\beta)},$$

$h_{\alpha\beta} \in H$. Therefore, if $q \in U_\alpha \cap U_\beta$, we get

$$\begin{aligned} \varphi_{\alpha *}(j^1_q(f)) &= j^1_{f^{-1}(q)}(\varphi_\alpha \circ f) = j^1_{f^{-1}(q)}(\varphi_\alpha \circ \varphi_\beta^{-1} \circ \varphi_\beta \circ f) = \\ &= j^1_{f^{-1}(q)}(h_{\alpha\beta} \circ \varphi_\beta \circ f) = j^1_{f^{-1}(q)}(\varphi_\beta \circ f) \\ &= \varphi_{\beta *}(j^1_q(f)). \end{aligned}$$

This shows that $L_G(M)$ is integrable. ◊

PROPOSITION 3.2. *Let P be an integrable H-reduction of $L^2(M)$, the bundle of the 2-frames over M; then M is a (X, H)-manifold.*

Proof. We shall construct an atlas of (X, H)-geometry. Since P is integrable, for every $p \in M$ there exists a neighbourhood U_α and a diffeomorphism $\varphi_\alpha : U_\alpha \longrightarrow V_\alpha \subset X$, such that

$$\varphi_{\alpha *} : P|_{U_\alpha} \longrightarrow L^2_G(X)|_{V_\alpha} = G|_{V_\alpha} .$$

Then if $(U_\beta,\, \varphi_\beta,\, V_\beta)$ is another diffeomorphism, for $q \in U_\alpha \cap U_\beta$, we obtain

$$\varphi_{\alpha *}(j^2_q(\varphi_\beta^{-1})) = j^2_x(\varphi_\alpha \circ \varphi_\beta^{-1}) = j^2_x(h^x_{\alpha\beta}),$$

where $x = \varphi_\beta(q)$, $h^x_{\alpha\beta} \in H$. We shall prove that $h^x_{\alpha\beta}$ does not depend on x. The last relation implies that for every $x \in \varphi_\beta(U_\alpha \cap U_\beta)$ the change coordinate map $\varphi_\alpha \circ \varphi_\beta^{-1}$ and the linear transformation $h^x_{\alpha\beta}$ have the same partial derivatives up to the order 2; thus if $(U,\, \psi,\, V)$ is a local chart around o the diffeomorphism $\varphi_\alpha \circ \varphi_\beta^{-1}$ is linear and consequently $h^x_{\alpha\beta} = h_{\alpha\beta}$. Therefore

$$\varphi_\alpha \circ \varphi_\beta^{-1}|_{\varphi_\beta(U_\alpha \cap U_\beta)} = h_{\alpha\beta}|_{\varphi_\beta(U_\alpha \cap U_\beta)} ,$$

$\mathcal{A} = (U_\alpha,\, \varphi_\alpha,\, V_\alpha)$ is an atlas of (X, H)-geometry and M is a (X, H)-manifold. ◊

If we consider as the model space the couple $(G/H,\, H)$ such that the subgroup H can be embedded into the group $G^r(n)$ via the r-representation of isotropy, (i. e. the elements of H are known when we give the partial derivatives up to the order r at the point o) , then the previous Propositions can be generalized in the following way:

PROPOSITION 3.3. *If M is a (X, H)-manifold, then $L^r_G(M)$ is an integrable H-reduction of $L^r(M)$.*

As for the case $r = 1$ an integrable H-reduction of L^{r+1} determines a (X, H)-structure on M. We have the following

PROPOSITION 3.4. *If P is an integrable H-reduction of the bundle of* (r+1)*-jets L^{r+1}, then M is a (X, H)-manifold.*

To finish this Section, we give a description of the group $G^2(n)$. We may suppose $X = \mathbb{R}^n$. By definition

$$G^2(n) = \{j_0^2(f) \mid f : U \longrightarrow \mathbb{R}^n \text{ is a diffeomorphism } f(0) = 0\}$$

and the group operation is defined by $j_0^2(f)\, j_0^2(f') = j_0^2(f \circ f')$. Every 2-frame $u = J_0^2(f)$ has a unique polynomial representation given by

$$g(x) = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} u_j^i x^j + \sum_{j,k=1}^{n} u_{jk}^i x^j x^k \right) e_i$$

$\{e_1, \ldots, e_n\}$ being the canonical basis of $\mathbb{R}^n$, $x = \sum_{i=1}^{n} x^i e_i$ and $u_{jk}^i = u_{kj}^i$. The $(u_j^i,\, u_{jk}^i)$ define a coordinate system in $G^2(n)$. Therefore, we may identify every 2-jet $u = j_0^2(f)$ with the couple $(A,\, \alpha)$, where A is the Jacobian matrix (u_j^i) and α is the Hessian matrix, i. e. α is a bilinear form on $\mathbb{R}^n \times \mathbb{R}^n$ taking its values in $\mathbb{R}^n$. Thus, the product expression has the following form

$$(A,\, \alpha)\,(B,\, \beta) = (AB,\, \gamma)$$

where AB denotes the matrices product and γ is defined by $\gamma(x, y) = \alpha(Bx, By) + A\beta(x, y)$. The identity element is the couple $(I, 0)$ and the inverse of $(A,\, \alpha)$ has the following representation

$$(A,\, \alpha)^{-1} = (A^{-1},\, \beta),$$

β being defined by $\beta(x, y) = -A^{-1}\alpha(A^{-1}x, A^{-1}y)$.

4. The Riemannian case.

In this section we take as the model space a simply connected Riemannian homogeneous space $(X\,,\, k)$, k being an invariant metric on X. We recall the well known

THEOREM. ([11]) *The group* Iso(M) *of isometries of a Riemannian manifold* M *is a Lie transformation group with respect to the compact-open topology. For each* $x \in M$, *the isotropy subgroup* $\mathrm{Iso}_x(M)$ *is compact. If* M *is compact,* Iso(M) *is also compact.*

Therefore $X = G/H$, where G=Iso(X) and H is the isotropy group at the origin o of X; moreover, the *linear isotropy representation of* H

$$\alpha : H \longrightarrow \mathrm{GL}\,(n, \mathbb{R})$$

is faithful, H being compact and $\alpha(H) \subset \mathrm{O}\,(n, \mathbb{R})$.

Let M be a $(X\,,\,H)$-manifold; by Proposition 3.1 it follows that the bundle $L^1(M) = L(M)$ reduces to $H \subset \mathrm{O}\,(n, \mathbb{R})$ and this gives a Riemannian structure on M. Since the reduction is integrable, the $(X\,,\,H)$-manifold M is *locally isometric* to the model space X. In particular M is *locally homogeneous.*

Let us consider now a Riemannian manifold $(M\,,\,g)$ locally isometric to the model space $(X\,,\,k)$. We recall the following result

THEOREM. *Let* M *and* M' *be connected and simply connected, complete analytic Riemannian manifolds. Then every isometry between connected open subsets of* M *and* M' *can be uniquely extended to an isometry between* M *and* M' (see [7]).

Since a Riemannian homogeneous space is analytic and complete, the previous Theorem implies that if $f : V \longrightarrow X\,,\ f' : U' \longrightarrow X$ are two local isometries onto their images, with $U \cap U' \neq \emptyset$, then the local isometry of X

$$(f' \circ f^{-1})|_{f(U \cap U')} : f(U \cap U') \to f'(U \cap U')$$

can be extended to a global isometry. Thus we have the following

PROPOSITION 4.1. *If* $(M\,,\,g)$ *is* locally isometric *to a simply connected Riemannian homogeneous space* $(X = G/H\,,\,k)$, *then* M *is a* $(X\,,\,G)$*-manifold.*

We recall that if $(M\,,\,g)$ is a connected Riemannian manifold, then any isometry $f : M \to M$ is determined by the value which f and its differential df take in $p \in M$. Therefore in the case of a Riemannian homogeneous model, Propositions 3.1 and 3.2 can be collected in the following

PROPOSITION 4.2. *Let* $(X = G/H, k)$ *be a homogeneous Riemannian manifold;* M *is a* (X, H)*-manifold if and only if there exists an integrable* H*-reduction of the bundle of the linear frames on* M.

REFERENCES

[1] ALEXANDER H., *Holomorphic mappings from the ball and polydisc*, Math. Ann. **209** (1974), 249-256.

[2] BENOIST LABOURIE, *Sur le espace homogènes modèles de variétè compact*, Hautes Etudes Scientifique Publ. Math. **76** (1992), 99-109.

[3] BENZECRÍ J. P., *Variétés localment affines*, Seminaire Ehresmann (May 1959).

[4] BURNS D. JR. and SHNIDER S., *Spherical hypersurfaces in complex manifolds*, Invent. Math. **33** (1976), 223-246.

[5] KATO M., *Compact differentiable 4-folds with quaternionic structures*, Math. Ann. **248** (1980), 79-96.

[6] KOBAYASHI S., *Transformation Groups in Differential Geometry*, Springer-Verlag Berlin Heidelberg New York, 1972.

[7] KOBAYASHI S. and NOMIZU N., *Foundations of Differential Geometry*, Vol. **I** and **II**, Interscience Publishers, New York, 1963, 1969.

[8] KUIPER N., *On conformally flat manifolds in the large*, Ann. Math. **50** (1949), 916-924.

[9] KULKARNI R. and PINKALL U., *Uniformization of geometric structures with application to conformal geometry*, Springer Lecture Notes in Math. **1209**.

[10] MILNOR J., *Curvature of left invariant metrics on Lie groups*, Adv. in Math. **21** (1976), 293-329.

[11] MYERS S. B. and STEENROD N., *The group of isometries of a Riemannian manifold*, Ann. Math. **40** (1939), 400-416.

[12] SCHOEN R. and YAU S.-T., *Conformally flat manifolds, Kleinian groups and scalar curvature*, Invent. Math. **92** (1988), 47-71.

[13] SOMMESE A., *Quaternionic manifolds*, Math. Ann. **212** (1975), 191-214.