# ON $\mathbf{X} - \vartheta-$SPLITTING AND $\mathbf{X} - \vartheta-$JOINTLY CONTINUOUS TOPOLOGIES ON FUNCTION SPACES (*)

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SOMMARIO. - *In questo articolo definiamo una relazione su $\Theta(Y, Z)$, l'insieme di tutte le funzioni $\theta$-continue di uno spazio topologico $Y$ in uno spazio topologico $Z$. Studiamo inoltre la connessione di questa relazione con le nozioni di $\mathbf{X}$-$\theta$-splitting e di topologie $\mathbf{X}$-$\theta$-continue su questo insieme, in cui $X$ è lo spazio di Sierpinski oppure $X = \mathbf{D}$.*

SUMMARY. - *In this paper we define a relation on the set $\Theta(Y, Z)$ of all $\vartheta-$continuous functions of a topological space $Y$ into a topological space $Z$ and we study the connection of this relation with the notions of $\mathbf{X} - \vartheta-$splitting and $\mathbf{X} - \vartheta-$jointly continuous topologies on this set, where $X$ is the Sierpinski space or $X = \mathbf{D}$.*

## 1. Introduction.

Let $Y$, $Z$ be topological spaces and let $f$ be a map of $Y$ into $Z$. Then $f$ is *$\vartheta-$continuous at* $y \in Y$ if for every open neighbourhood $V$ of $f(y)$ there exists an open neighbourhood $U$ of $y$ such that $f(Cl(U)) \subseteq Cl(V)$. (Let $Y$ be a topological space, then by $Cl(A)$ we denote the closure of $A$ in $Y$). The map $f$ is *$\vartheta-$continuous* on $Y$ if it is $\vartheta-$continuous at each point of $Y$. (See for example [F], [I-F] and [J]). A continuous function $f : Y \to Z$ is $\vartheta-$continuous, but the converse is true when $Z$ is regular, that is the closed neighbourhoods of any point form a local base. In what follows by $\Theta(Y, Z)$ we denote the set of all $\vartheta-$continuous maps of $Y$ into $Z$. If $\tau$ is a topology on the set $\Theta(Y, Z)$, then the corresponding topological space is denoted

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by $\Theta_\tau(Y,Z)$. Let $(Y,\tau)$ be a topological space. Then, by $\mathcal{N}_Y(y)$, where $y \in Y$ we denote the family of all open neighbourhoods of $y$ in $Y$, that is $\mathcal{N}_Y(y) = \{U \in \tau : y \in U\}$.

A point $x$ in a a space is in the $\vartheta-$*closure* of a subset $A$ of the space $x \in Cl_\vartheta(A)$ if each open subset $V$ about $x$ satisfies $A \cap Cl(V) \neq \emptyset$. A is $\vartheta-$*closed* if $Cl_\vartheta(A) = A$. (See for example [J]).

Let $X$ be a space and $F : X \times Y \to Z$ be a $\vartheta-$continuous map. If $F$ has $\vartheta-$continuous restrictions to $\{x\} \times Y$ for any $x \in X$, then by $F_x$, where $x \in X$, we denote the $\vartheta-$continuous map of $Y$ into $Z$, for which $F_x(y) = F(x,y)$, for every $y \in Y$. By $\widehat{F}$ we denote the map of $X$ into the set $\Theta(Y,Z)$, for which $\widehat{F}(x) = F_x$, for every $x \in X$.

Let $G$ be a map of the space $X$ into the set $\Theta(Y,Z)$. By $\widetilde{G}$ we denote the map of the space $X \times Y$ into the space $Z$, for which $\widetilde{G}(x,y) = G(x)(y)$, for every $(x,y) \in X \times Y$.

By $\mathbf{S}$ we denote the Sierpinski space, that is, the set $\{0,1\}$ equipped with the topology $\tau(\mathbf{S}) \equiv \{\emptyset, \{0,1\}, \{1\}\}$, and by $\mathbf{D}$ the set $\{0,1\}$ with the trivial topology.

Let $\mathcal{A}$ be a family of spaces. A topology $\tau$ on the set $\Theta(Y,Z)$ is called $\mathcal{A}-\vartheta-$*splitting* (respectively, $\mathcal{A}-\vartheta-$*jointly continuous*) (see [G]) if and only if for every element $X$ of $\mathcal{A}$, the $\vartheta-$continuity of a map $F : X \times Y \to Z$ (respectively, a map $G : X \to \Theta_\tau(Y,Z)$) implies the $\vartheta-$continuity of the map $\widehat{F} : X \to \Theta_\tau(Y,Z)$ (respectively, of the map $\widetilde{G} : X \times Y \to Z$).

Obviously, if $\mathcal{A}$ is the family of all spaces, then the notions $\mathcal{A}-\vartheta-$splitting and $\mathcal{A}-\vartheta-$jointly continuous coincide with the notions $\vartheta-$splitting and $\vartheta-$jointly continuous, respectively. (See [$C_1$] and [$C_2$]). Also, these notions coincide with the notions $\mathcal{A}-$splitting and $\mathcal{A}-$jointly continuous topologies, respectively if $Z$ is a regular space. (See [G-I-$P_1$]).

If $\mathcal{A} = \{X\}$, then instead of "$\mathcal{A}-\vartheta-$splitting" and "$\mathcal{A}-\vartheta-$jointly continuous" we write "$X-\vartheta-$splitting" and "$X-\vartheta-$jointly continuous".

In the present paper we define a relation on the set $\Theta(Y,Z)$ of all $\vartheta-$continuous functions of a topological space $Y$ into a topological space $Z$ and we study the connection of this relation with the notions of $X-\vartheta-$splitting and $X-\vartheta-$jointly continuous topologies on this set, where $X$ is the Sierpinski space or $X = \mathbf{D}$.

## 2. The relation " $\prec$ " on $\Theta(Y,Z)$.

2.1. Definition and notations.

For every space $Y$ with a topology $\tau$ we define a relation $''\overset{\tau}{\prec}''$ on $Y$ as follows: if $x,y\in Y$, then we write $x\overset{\tau}{\prec}y$ if and only if $x\in \mathrm{Cl}_\vartheta(\{y\})$ and $y\in \mathrm{Cl}_\vartheta(\{x\})$, that is for every $U\in\mathcal{N}_Y(x)$ we have $y\in Cl(U)$ and for every $V\in\mathcal{N}_Y(y)$ we have $x\in Cl(V)$. Clearly this relation is reflexive and symmetric. Also, if the space $Y$ is regular, then the relation $''\overset{\tau}{\prec}''$ is an equivalence relation.

We define a relation $\prec$ on $\Theta(Y,Z)$ as follows: if $g,f\in\Theta(Y,Z)$, then we write $g\prec f$ if and only if $g(y)\overset{\tau}{\prec}f(y)$, for every $y\in Y$, where $\tau$ is the topology of the space $Z$.

2.2. Theorem.

*The following propositions are true:*

*(1) If a topology $\tau$ on $\Theta(Y,Z)$ is $\mathbf{S}-\vartheta-$splitting then from the condition $g\prec f$ it follows that $g\overset{\tau}{\prec}f$, where $f,g\in\Theta(Y,Z)$ and $Z$ regular.*

*(2) If from the condition $g\prec f$ it follows that $g\overset{\tau}{\prec}f$, then the topology $\tau$ on $\Theta(Y,Z)$ is $\mathbf{S}-\vartheta-$splitting.*

*Proof.* (1) Let $\tau$ be an $\mathbf{S}-\vartheta-$splitting topology on $\Theta(Y,Z)$ and let $g\prec f$, where $g,f\in\Theta(Y,Z)$. We prove that $g\overset{\tau}{\prec}f$.

Let $F:\mathbf{S}\times Y\to Z$ be a map for which $F(1,y)=f(y)$ and $F(0,y)=g(y)$, where $y\in Y$. We prove that $F$ is $\vartheta-$continuous.

Let $F(1,y)=f(y)$ and $U\in\mathcal{N}_Z(f(y))$. Since $f$ is $\vartheta-$continuous, there exists an open neighbourhood $V$ of $y$ in $Y$ such that $f(Cl(V))\subseteq Cl(U)$. We prove that $F(Cl(\mathbf{S}\times V))\subseteq Cl(U)$.

Indeed, if $(1,y_1)\in Cl(\mathbf{S}\times V)=\mathbf{S}\times Cl(V)$, then $F(1,y_1)=f(y_1)\in Cl(U)$. If $(0,y_1)\in Cl(\mathbf{S}\times V)$, then $F(0,y_1)=g(y_1)$. Since $g\prec f$ we have $g(y_1)\in Cl_\vartheta(\{f(y_1)\})$ and $f(y_1)\in Cl_\vartheta(\{g(y_1)\})$. Hence, $g(y_1)\in Cl(U)$.

Now, let $F(0,y)=g(y)$ and $U\in\mathcal{N}_Z(g(y))$. Since $g$ is $\vartheta-$continuous there exists an open neighbourhood $V$ of $y$ in $Y$ such that $g(Cl(V))\subseteq Cl(U)$. As the above we can prove that $F(Cl(\mathbf{S}\times V))\subseteq Cl(U)$. Thus, the map $F$ is $\vartheta-$continuous.

Since $\tau$ is $\mathbf{S}-\vartheta-$splitting the map $\widehat{F}:\mathbf{S}\to\Theta_\tau(Y,Z)$ is $\vartheta-$continuous. We have that $\widehat{F}(1)=f$ and $\widehat{F}(0)=g$.

Let $W$ be an open neighbourhood of $g$ in $\Theta_\tau(Y,Z)$. Since $\widehat{F}$ is $\vartheta-$continuous there exists an open neighbourhood $V$ of 0 in $\mathbf{S}$ such that $\widehat{F}(Cl(V))\subseteq Cl(W)$. Obviously, $Cl(V)=\mathbf{S}$. Hence we have $\widehat{F}(\mathbf{S})\subseteq Cl(W)$. Thus, $\widehat{F}(1)=f\in Cl(W)$ and $g\in Cl_\vartheta(\{f\})$. Similarly we can prove that $f\in Cl_\vartheta(\{g\})$. Hence $g\prec f$.

(2) Let $\tau$ be a topology on $\Theta(Y,Z)$ such that from the condition $g\prec f$ it follows that $g\overset{\tau}{\prec}f$. We prove that $\tau$ is $\mathbf{S}-\vartheta-$splitting.

Let $F:\mathbf{S}\times Y\to Z$ be a $\vartheta-$continuous map. Consider the map $\widehat{F}:\mathbf{S}\to\Theta_\tau(Y,Z)$. Let $\widehat{F}(1)=f$ and $\widehat{F}(0)=g$. We prove that $g\prec f$.

Indeed, let $y\in Y$ and let $U\in\mathcal{N}_Z(g(y))$. We must prove that $f(y)\in Cl(U)$. Since $F$ is $\vartheta-$continuous and $F(0,y)=g(y)$ there exists an open neighbourhood $W=O\times V$ of $(0,y)$ in $\mathbf{S}\times Y$ such that

$$F(Cl(O\times V))=F(\mathbf{S}\times Cl(V))\subseteq Cl(U).$$

Hence $F(1,y)=f(y)\in Cl(U)$. Similarly we can prove that if $U$ is an open neighbourhood of $f(y)$ in $Z$, then $g(y)\in Cl(U)$. Thus, $g\prec f$.

By assumption $g\overset{\tau}{\prec}f$. Let $U$ be an open neighbourhood of g in $\Theta_\tau(Y,Z)$. Since $g\underset{\tau}{\prec}f$ we have that $g\in Cl_\vartheta(\{f\})$ and $f\in Cl_\vartheta(\{g\})$. Thus $f\in Cl(U)$. Hence

$$\widehat{F}(Cl(\mathbf{S}))=\widehat{F}(\mathbf{S})\subseteq Cl(U).$$

Let $U$ be an open neighbourhood of $f$ in $\Theta_\tau(Y,Z)$. Similarly we can prove that $g\in Cl(U)$ and $\widehat{F}(Cl(\mathbf{S}))=\widehat{F}(\mathbf{S})\subseteq Cl(U)$. Thus the the map $\widehat{F}$ is $\vartheta-$continuous and the topology $\tau$ is $\mathbf{S}-\vartheta-$splitting.$\diamondsuit$

2.2.1. Corollary.

*If $Z$ is a discrete space, then the discrete topology and, hence, every topology on $\Theta(Y,Z)$ is $\mathbf{S}-\vartheta-$splitting.*

*Proof.* Indeed, suppose that $Z$ is a discrete space, then by the condition $g \prec f$, where $g, f \in \Theta(Y, Z)$, it follows that $g = f$. Hence, $g \overset{\tau}{\prec} f$, for every topology $\tau$ on $\Theta(Y, Z)$. Thus, by Theorem 2.2, every topology on $\Theta(Y, Z)$ is $\mathbf{S}-\vartheta-$splitting. ♢

2.3. Theorem.

*The following propositions are true:*

*(1) If a topology $\tau$ on $\Theta(Y, Z)$ is $\mathbf{S}-\vartheta-$jointly continuous then from the condition $g \overset{\tau}{\prec} f$ it follows that $g \prec f$.*

*(2) If from the condition $g \overset{\tau}{\prec} f$ it follows that $g \prec f$ and $Z$ regular, then the topology $\tau$ on $\Theta(Y, Z)$ is $\mathbf{S}-\vartheta-$jointly continuous.*

*Proof.* (1) Let $\tau$ be an $\mathbf{S}-\vartheta-$jointly continuous topology on $\Theta(Y, Z)$ and let $g \overset{\tau}{\prec} f$, where $g, f \in \Theta(Y, Z)$. We prove that $g \prec f$.

Let $G : \mathbf{S} \to \Theta_\tau(Y, Z)$ be a map for which $G(1) = f$ and $G(0) = g$. We prove that $G$ is $\vartheta-$continuous. Let $U$ be an open neighbourhood subset of $f$ in $\Theta_\tau(Y, Z)$. Since $g \overset{\tau}{\prec} f$ we have that $g \in Cl(U)$. Hence

$$G(Cl(\mathbf{S})) = G(\mathbf{S}) \subseteq Cl(U).$$

Similar if $V \in \mathcal{N}_{\Theta_\tau(Y,Z)}(g)$, then

$$G(Cl(\mathbf{S})) = G(\mathbf{S}) \subseteq Cl(V).$$

Hence, the map $G$ is $\vartheta-$continuous. Since $\tau$ is $\mathbf{S}-\vartheta-$jointly continuous, the map $\widetilde{G} : \mathbf{S} \times Y \to Z$ is also $\vartheta-$continuous.

Let $y \in Y$ and let $W \in \mathcal{N}_Z(g(y))$. We must prove that $f(y) \in Cl(W)$. Indeed, since the map $\widetilde{G}$ is $\vartheta-$continuous at the point $(0, y) \in \mathbf{S} \times Y$ there exists an open neighbourhood $V \times U$ of $(0, y)$ in $\mathbf{S} \times Y$ such that

$$\widetilde{G}(Cl(V \times U)) = \widetilde{G}(\mathbf{S} \times Cl(U)) \subseteq Cl(W).$$

Thus $\widetilde{G}(1, y) = f(y) \in Cl(W)$.

Similar, if $W \in \mathcal{N}_Z(f(y))$, then $g(y) \in Cl(W)$. Hence $g \prec f$.

(2) Let $\tau$ be a topology on $\Theta(Y,Z)$ such that from the condition $g \overset{\tau}{\prec} f$ it follows that $g \prec f$. We prove that $\tau$ is $\mathbf{S}-\vartheta-$jointly continuous.

Let $G : \mathbf{S} \to \Theta_\tau(Y,Z)$ be a $\vartheta-$continuous map and let $G(1) = f$ and $G(0) = g$. We prove that $g \overset{\tau}{\prec} f$.

Indeed, let $U$ be an open neighbourhood of $g$ in $\Theta_\tau(Y,Z)$. Since $G$ is $\vartheta-$continuous, there exists $V \in \mathcal{N}_{\mathbf{S}}(0)$ such that

$$G(Cl(V)) = G(\mathbf{S}) \subseteq Cl(U).$$

Hence $f \in Cl(U)$. Similar, if $U \in \mathcal{N}_{\Theta_\tau(Y,Z)}(f)$, then $g \in Cl(U)$. Thus, $g \overset{\tau}{\prec} f$. By assumption $g \prec f$.

Consider the map $\widetilde{G} : \mathbf{S} \times Y \to Z$. Then we have $\widetilde{G}(0,y) = g(y)$ and $\widetilde{G}(1,y) = f(y)$. We prove that the map $\widetilde{G}$ is $\vartheta-$continuous.

Indeed, let $W$ be an open neighbourhood of $g(y)$ in $Z$. Since $g$ is $\vartheta-$continuous there exists an open neighbourhood $V$ of $y$ in $Y$ such that $g(Cl(V)) \subseteq Cl(W)$. We prove that

$$\widetilde{G}(Cl(\mathbf{S} \times V)) = \widetilde{G}(\mathbf{S} \times Cl(V)) \subseteq Cl(W).$$

Let $(0,y_1) \in Cl(\mathbf{S} \times V)$. Then $\widetilde{G}(0,y_1) = g(y_1) \in Cl(W)$. Now, let $(1,y_1) \in Cl(\mathbf{S} \times V)$. Then $\widetilde{G}(1,y_1) = f(y_1)$. Since $g \prec f$, we have that $f(y_1) \in Cl(W)$. This means that the map $\widetilde{G}$ is $\vartheta-$continuous at the point $(0,y)$. Similarly the map $\widetilde{G}$ is $\vartheta-$continuous at the point $(1,y)$. Thus the map $\widetilde{G}$ is $\vartheta-$continuous and the topology $\tau$ is $\mathbf{S}-\vartheta-$jointly continuous. ◊

2.3.1. Corollary.

*If $Z$ is a regular space, then the discrete topology $\tau$ on the set $\Theta(Y,Z)$ is $\mathbf{S}-\vartheta-$jointly continuous.*

*Proof.* Let $\tau$ be the discrete topology on $\Theta(Y,Z)$. Then from the condition $g \overset{\tau}{\prec} f$ it follows that $g = f$ and hence $g \prec f$. Thus by Theorem 2.3 the topology $\tau$ is $\mathbf{S}-\vartheta-$jointly continuous. ◊

2.4. Theorem.

*A topology $\tau$ on $\Theta(Y, Z)$, where $Z$ is a regular space, is simultaneously $\mathbf{S} - \vartheta-$splitting and $\mathbf{S} - \vartheta-$jointly continuous if and only if the relations "$\overset{\tau}{\prec}$" and "$\prec$" coincide.*

*Proof.* The proof of this theorem follows by Theorems 2.3 and 3.3. ♢

2.5. Remarks.

(1) Relevant results for continuous functions there exist in [G-I-$P_2$].

(2) The Theorems 2.2, 2.3 and 2.4 are also true if we replace the space $\mathbf{S}$ by the space $\mathbf{D}$.

2.6 Problems.

We give some problems concerning topologies on $\Theta(Y, Z)$.
Let $\mathcal{A}$ be an arbitrary family of spaces.

(1) Does there exist a characterization of the $\mathcal{A} - \vartheta-$splitting and $\mathcal{A} - \vartheta-$jointly continuous topologies with the relations "$\prec$" and "$\overset{\tau}{\prec}$"?

(2) Does there exist the finest $\mathcal{A} - \vartheta-$splitting topology on $\Theta(Y, Z)$? It is known that on the set $C(Y, Z)$ of all continuous maps of a space $Y$ into a space $Z$ there exists the finest $\mathcal{A}$-splitting topology. (See [G-I-$P_1$]).

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