ON DANIELL INTEGRALS AND COMPACT SUPPORTS (*)

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SOMMARIO. - *Un integrale di Daniell definito su tutto $C(X,\mathbb{R})$ equivale ad una misura di Radon a supporto compatto.*

SUMMARY. - *A Daniell integral defined on all of $C(X,\mathbb{R})$ is a Radon measure with compact support.*

Let X be a completely regular Suslin space (e.g. a Polish space), and $C(X,\mathbb{R})$ be the vector lattice of all real continuous functions on X.

THEOREM. *Let μ be a bounded linear form on $C(X,\mathbb{R})$. Then the following conditions are equivalent:*

(a) *μ^+ and μ^- are Daniell integrals.*

(b) *the finite, signed Radon measure $m = m^+ - m^-$, corresponding to $\mu = \mu^+ - \mu^-$, has compact support.*

(c) *μ^+ and μ^- are continuous relative to the topology of compact convergence.*

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Proof. $(b) \Longrightarrow (c)$. Let the net f_α converge to f uniformly on every compact, in particular on *supp (m)*; since $m^+(X)$ and $m^-(X)$ are finite, the standard argument holds.

$(c) \Longrightarrow (a)$. Let f_α be a net of continuous functions, having a first element f_1, decreasing and converging to zero, therefore converging uniformly on every compact set; then

$$\lim_\alpha \mu^+(f_\alpha) = 0 \qquad \text{and} \qquad \lim_\alpha \mu^-(f_\alpha) = 0.$$

$(a) \Longrightarrow (b)$. It is sufficient to deal with μ^+ and the corresponding measure m^+, defined at least on the Baire σ–algebra $\mathcal{A}(X)$; $\mu^+(1) < +\infty$. Since X is a Suslin space and the continuous functions separate the points, $\mathcal{A}(X) = \mathcal{B}(X)$ and μ^+ is a Radon measure (see [4] p.41 and [3] chap. II, Theorem 10). If $supp(m^+)$ were not compact, by the facts that m^+ is Radon and X is completely regular there would be a sequence of functions $f_n \in C(X, \mathbb{R})$, positive, with disjoint compact supports, such that $\sum_n f_n \in C(X, \mathbb{R})$, for all n $\int f_n \, dm^+ = 1$ and therefore $\mu^+(\sum_n f_n) = +\infty$, against the assumptions (as in [2], vol. III, p. 177, where X was locally compact). ♢

REMARK. The equivalence between "having compact support" and "being continuous in the topology of uniform convergence ... on compact sets" is well-known for Schwartz distributions on $\mathbb{R}^n$. Is it true also for distributions on $\mathbb{R}^\omega$ (i.e. in countably many variables)? See [1], Problem 2.

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REFERENCES

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