

# LOGISTIC DYNAMICS WITH DISTRIBUTED LAGS (\*)

by FABIO GRASSI  
and SERGIO INVERNIZZI (in Trieste)(\*\*)

SOMMARIO. - *Presentiamo alcuni risultati riguardanti il sistema dinamico discreto  $x_t = F(y_t)$  dove  $y$  è una variabile ritardata su  $x$  con ritardo distribuito, ed  $F(s) = \mu s(1 - s)$  è la mappa logistica. Dimostriamo che un'appropriata distribuzione del ritardo produce una notevole semplificazione della complessità dinamica in confronto a tutti i risultati basilari concernenti il caso del differimento fisso  $y_t = x_{t-1}$ .*

SUMMARY. - *We present some results concerning the dynamics of a discrete-time dynamical system  $x_t = F(y_t)$  where  $y$  is a variable lagged on  $x$  by means of a distributed lag, and  $F(s) = \mu s(1 - s)$  is the logistic map. We show that a suitable distribution of the delay produces a significant simplification of the dynamical complexity when compared to all basic results concerning the choice of a single fixed delay  $y_t = x_{t-1}$ .*

---

Presented at the course *Dinamica non lineare, teoria e applicazioni*, organized by Centro per lo Studio della teoria dei Sistemi, CNR, and Politecnico di Milano at the Centro Studi CISL, Florence (Italy), February 6-10, 1995.

---

Partially supported by the Italian *Ministero della Università e della Ricerca Scientifica e Tecnologica*, funds 40%, area 1.13 (Economical Sciences).

---

(\*) Pervenuto in Redazione il 16 Settembre 1995.

(\*\*) Indirizzo degli Autori: Department of Mathematics, University of Trieste, I-34100 Trieste (Italy). E-mail: [inverniz@utsax7.univ.trieste.it](mailto:inverniz@utsax7.univ.trieste.it)

## 1. Introduction.

Consider the logistic one-dimensional map  $F : \mathbb{R} \rightarrow \mathbb{R}$ ,  $F(s) = \mu s(1 - s)$ . The semi-cascade (discrete-time dynamical system)

$$x_t = F(x_{t-1}) \quad (1)$$

$t$  integer,  $t \geq 1$ , is a well-known deterministic model for the discrete time evolution of complex phenomena arising in different applied fields, among which: ecology and population dynamics (R. May [10]), economics (J.-M. Grandmont [7]), and urban modeling (J. R. Beaumont *et al.* [4]). The complex and possibly chaotic behavior exhibited by (1) can be conceived as a product of two features. The former is the mixed feedback assumption (corresponding to the “expanding and folding” character of  $F$ , “tuned” by the parameter  $\mu$ ). The latter is the fixed delay assumption, i.e. the implicit understatement that the state of the system at time  $t$  depends only on its state at the previous time  $t - 1$ . Pointing out these two different aspects, we rewrite (1) by introducing the hidden variable  $y_t = x_{t-1}$ , lagged on  $x$  by a delay of one period, thus:

$$\begin{cases} x_t = F(y_t) \\ y_t = x_{t-1} \end{cases} \quad (2)$$

Accordingly, the study of the dynamics of (1) or (2) can be addressed to the analysis of the rôle either of the nonlinearity (at the different values of  $\mu$ ) or of the implicitly assumed synchronization of all lagged responses in a single fixed delay. The former approach is the most common. On the contrary, the latter has been rather neglected in the literature, even if it should be considered undoubtedly important for applications. Actually, already in 1957, R. G. D. Allen wrote in his classical book on mathematical economics that *‘It is already clear that an essential feature of an economic model in dynamic form is the inclusion of time lags in the relations of the model, in the influence of one variable to another. The existence of time lags is generally recognized; what is not so clearly appreciated is that the form assumed for the lag is very important, particularly when the variables are aggregate’* [2, page 23].

In this paper we propose a study of a model in which the fixed delay in (2) is substituted by a *distributed lag* informally written

thus:

$$y_t = \lambda_1 x_{t-1} + \lambda_2 x_{t-2} + \lambda_3 x_{t-3} + \dots + \lambda_t x_0 \quad (3)$$

where  $\lambda_1 + \lambda_2 + \lambda_3 + \dots + \lambda_t = 1$  [2, page 21]. In particular we shall consider lags distributed according to a geometric progression:

$$y_t = \frac{1 - \rho}{1 - \rho^t} (x_{t-1} + \rho x_{t-2} + \rho^2 x_{t-3} + \dots + \rho^{t-1} x_0) \quad (4)$$

with  $0 < \rho < 1$ , and, more generally, lags whose distributions are “dominated” by geometric ones, in a sense specified by the subsequent Lemma 2.1. The sequence  $w_t = \sum_{i=1}^t \rho^{i-1} x_{t-i}$  is obtained from  $x_t$  through an infinite impulse response (IIR) filter. An important property of IIR filters [3] is that they can increase the Lyapunov dimension of the pertinent attractor. We recall that the Lyapunov dimension of an attractor is the number  $D_L = k + \sum_{i=1}^k \lambda_i / |\lambda_{k+1}|$ , where the Lyapunov exponents of the attractor are  $\lambda_1 \geq \lambda_2 \geq \dots$ , and where  $k$  is determined by  $\sum_{i=1}^k \lambda_i \geq 0 > \sum_{i=1}^{k+1} \lambda_i$ . Therefore,  $y_t$  can be more complex than  $x_t$  when these series are linearly coupled externally to the closed-loop system  $x_t = F(x_{t-1})$ . This is not the case when the coupling of  $x_t$  and  $y_t$  is interior to the nonlinearity:  $x_t = F(y_t)$ . Actually, we shall prove that an interior IIR filtering process (with suitable lag distributions) works against the “expanding and folding” property of the nonlinearity, leading to a considerable simplification of the dynamics. For example, a map which is fully chaotic for the fixed delay (like the logistic at  $\mu = 4$ ) can be forced to a dynamics with a globally attracting fixed point by a suitable simple IIR. This would be a partial answer to the following question raised by R. May: *‘In general, there has been a tendency for the natural populations to exhibit dynamical behavior that is relatively tame: for the difference equations with one critical point ... most of the biological populations appear to exhibit stable point behavior’* [11, page 559], and also by the biologist L. Edelstein-Keshet: *‘Comparison between observations and model predictions indicate that many dynamical behavior patterns, which are theoretically possible, are not observed in nature. We are thereby led to inquire which effects in natural systems have influences on populations that might otherwise behave chaotically’* [5, page 77].

## 2. Basic preliminaries.

### 2.1. Regular, Nörlund, and geometric matrices.

Let us recall for the sake of completeness some basic facts on countably infinite matrices  $(a_{nk})$ ,  $(n, k) \in \mathbb{N} \times \mathbb{N}$ .

DEFINITION 2.1. A matrix  $(a_{nk})$  is *regular* if the following three conditions hold: (i) there is a number  $M > 0$  such that, for every  $n \in \mathbb{N}$ ,  $\sum_{k=0}^{\infty} |a_{nk}| < M$ ; (ii) for every  $k \in \mathbb{N}$ ,  $a_{nk} \rightarrow 0$  as  $n \rightarrow \infty$ ; (iii)  $\sum_{k=0}^{\infty} a_{nk} \rightarrow 1$  as  $n \rightarrow \infty$ .

The classical Toeplitz theorem [8, page 43] states that an arbitrary infinite matrix  $(a_{nk})$  “preserves convergence” (that is: for every real sequence  $(\xi_n)_{n \geq 0}$  convergent to a limit  $l \in \mathbb{R}$ , and for every  $n \in \mathbb{N}$ , the series  $\sum_{k=0}^{\infty} a_{nk} \xi_k$  has a finite sum  $\zeta_n$ , and the sequence  $(\zeta_n)_{n \geq 0}$  converges to  $l$ ) if and only if  $(a_{nk})$  is regular.

DEFINITION 2.2. A matrix  $(a_{nk})$  is a *weighted mean matrix* if, for each row-index  $n \in \mathbb{N}$  (i)  $a_{nk} \geq 0$  and  $a_{nk} = 0$  for each column-index  $k > n$ , and (ii)  $\sum_{k=0}^n a_{nk} = 1$ .

DEFINITION 2.3. Let a real sequence  $(d_n)_{n \geq 0}$  with  $d_n > 0$  be given, and let  $D_n \stackrel{\text{def}}{=} \sum_{k=0}^n d_k$ . Then the *Nörlund matrix* generated by  $(d_n)_{n \geq 0}$  is the countably infinite matrix  $P \stackrel{\text{def}}{=} (p_{nk})$  defined by

$$p_{nk} = \begin{cases} \frac{d_{n-k}}{D_n} & \text{if } k \leq n, \\ 0 & \text{if } k > n. \end{cases}$$

for any  $(n, k) \in \mathbb{N} \times \mathbb{N}$ .

Observe that any Nörlund matrix is by definition a weighted mean matrix. Moreover, it is easy to show that a Nörlund matrix  $P = (d_{n-k}/D_n)_{nk}$  generated by a sequence  $(d_n)_{n \geq 0}$  is regular if and only if  $d_n/D_n \rightarrow 0$  as  $n \rightarrow \infty$ . Actually, (i) and (iii) in Definition 2.1 are trivially satisfied; (ii) becomes: for every  $k$ ,  $d_{n-k}/D_n \rightarrow 0$  as

$n \rightarrow \infty$ . This implies  $d_n/D_n \rightarrow 0$  (for  $k = 0$ ); on the other hand, if  $d_n/D_n \rightarrow 0$ , then for every  $k$

$$0 \leq \frac{d_{n-k}}{D_n} = \frac{d_{n-k}}{D_{n-k}} \cdot \frac{D_{n-k}}{D_n} \leq \frac{d_{n-k}}{D_{n-k}} \rightarrow 0.$$

In particular, any bounded sequence generates a regular Nörlund matrix. A regular Nörlund matrix can be generated by an unbounded sequence as well: for example, if  $d_n = n + 1$  we get  $D_n = \frac{1}{2}(n+1)(n+2)$ , and so the resulting Nörlund matrix is regular.

Let us define a special subset of the class of Nörlund matrices:

DEFINITION 2.4. The *geometric matrix* (with initial value  $v > 0$  and ratio  $\rho > 0$ ) is the Nörlund matrix generated by the geometric progression  $d_n \stackrel{\text{def}}{=} v\rho^n$ ,  $n \geq 0$ .

Since

$$R_n \stackrel{\text{def}}{=} \sum_{k=0}^n v\rho^k = \begin{cases} \frac{v(1-\rho^{n+1})}{1-\rho} & \text{for } \rho \neq 1 \\ v(n+1) & \text{for } \rho = 1 \end{cases}$$

a geometric matrix is regular if and only if  $0 < \rho \leq 1$ . The importance of geometric matrices among regular Nörlund matrices lies in the following comparison lemma, which is an easy consequence of the Hardy Inclusion Theorem for regular summation methods (Theorem 23 in [8, page 69]).

LEMMA 2.1. Let  $P \stackrel{\text{def}}{=} (p_{nk})$  be a regular Nörlund matrix generated by the sequence  $(d_n)_{n \geq 0}$ . Suppose that there exist numbers  $m \in \mathbb{N}$  and  $\rho$ ,  $0 < \rho \leq 1$ , such that  $d_{n+1} \geq \rho d_n$  for all  $n \geq m$ . If  $(x_n)_{n \geq 0}$  is a sequence of real numbers such that  $\sum_{k=0}^n (\rho^{n-k}/R_n)x_k$  converges to a limit  $p^* \in \mathbb{R}$ , then the sums  $\sum_{k=0}^n p_{nk}x_k$  converge to  $p^*$  too, as  $n \rightarrow \infty$ .

## 2.2. Mann iterations.

Let  $J \subset \mathbb{R}$  be a compact interval, let  $f : J \rightarrow J$  be a continuous function, and let  $A \stackrel{\text{def}}{=} (a_{nk})$  be a given real weighted mean matrix.

Obviously, if  $x_0, x_1, \dots, x_n$  are in  $J$ , then  $z_n \stackrel{\text{def}}{=} \sum_{k=0}^n a_{nk}x_k$  is in  $J$ , too. A *Mann iteration* [9] on  $J$  is the iterative process  $(f, A)$

$$\begin{cases} x_{n+1} = f(z_n), \\ x_0 \in J. \end{cases} \quad (5)$$

The sequence  $(x_n)_{n \geq 0}$  defined inductively by (5) is called the *orbit* of  $(f, A)$  starting at  $x_0$ . An *equilibrium* of  $(f, A)$  is a point  $x^* \in J$  which is the limit of an orbit of  $(f, A)$  starting at some point in  $J$ . Observe that any fixed point  $\bar{x} = f(\bar{x})$  of  $f$  is an equilibrium of  $(f, A)$ , but the converse is not necessarily true. However, if the matrix  $A$  is regular, the orbit  $(x_n)_{n \geq 0}$  converges if and only if  $(z_n)_{n \geq 0}$  converges. Moreover, when these sequences converge, they converge to the same limit, which is necessarily a fixed point of  $f$ .

### 2.3. Geometric matrices with $\rho = 1$ .

If  $A$  is the geometric matrix with ratio  $\rho = 1$ , the Mann iteration coincides with the Cesàro iteration scheme. It is known that in this case the dynamics of (5) is totally trivial. Actually [6], for every continuous map  $f : J \rightarrow J$ , and for every initial point  $x_0 \in J$  the sequence in (5) converges to a fixed point of  $f$ . Now it is easy to see that also a regular *non-decreasing* Nörlund matrix does not allow any dynamics different from global convergence towards an equilibrium. In fact we have:

**COROLLARY 2.1.** *Let  $Q = (q_{nk}) = (d_{n-k}/D_n)$  be a regular Nörlund matrix such that  $d_{n+1} \geq d_n$ . Then for every continuous map  $f : J \rightarrow J$ , and for every initial point  $x_0 \in J$  the sequence  $(x_n)_{n \geq 0}$  defined by the iteration*

$$x_{n+1} = f\left(\sum_{k=0}^n q_{nk}x_k\right)$$

*converges to a fixed point of  $f$ .*

*Proof.* Just use the comparison lemma 2.1. ◇

The results of this section allow us to consider in the sequel only geometric matrices with ratio  $\rho < 1$ .

**3. Segmented form of iterations with geometric matrices.**

Consider now the system (5) in the equivalent form:

$$\begin{cases} z_n = \sum_{k=0}^n p_{nk}x_k & ; x_0 = z_0 \in [0, 1] \\ x_{n+1} = f(z_n) \end{cases} \quad (6)$$

We say that the sequence  $(z_n)_{n \geq 0}$  defined in (6) admits a *segmented form* if there exists a *divergent series*  $\sum_{n=0}^{\infty} t_n$  of real numbers  $t_n \in [0, 1]$  such that, for every  $n \geq 0$ ,

$$z_{n+1} = t_n f(z_n) + (1 - t_n)z_n \quad (7)$$

The following result is contained in [1, Prop. 1.2]:

**PROPOSITION 3.1.** *If  $P = (p_{nk})$  is the geometric matrix with ratio  $0 < \rho < 1$ , and initial value  $v$ , letting*

$$t_n \stackrel{\text{def}}{=} \frac{1}{R_{n+1}} = \begin{cases} \frac{1 - r}{v(1 - r^{n+2})} & \text{for } r \neq 1 \\ \frac{1}{v(n + 2)} & \text{for } r = 1 \end{cases}$$

*then the sequence  $(z_n)_{n \geq 0}$  inductively defined in (6) admits the segmented form (7).*

The segmented form of a sequence is often a useful tool for proving its convergence. For example, using this technique one can prove that:

**THEOREM 3.1.** *Let  $f : [0, 1] \rightarrow [0, 1]$  be a globally lipschitzian map with Lipschitz constant  $L$ , and let  $P$  be a geometric memory with ratio  $(L - 1)/(L + 1) < \rho < 1$ . Then for every initial point  $x_0 \in [0, 1]$  the sequence  $(x_n)_{n \geq 0}$  defined by the iteration (6) converges to a fixed point of  $f$ .*

*Proof.* See [1, Prop. 1.2]. ◇

Remark that for any given globally lipschitzian map  $f : [0, 1] \rightarrow [0, 1]$  it is always possible to choose a ratio  $\rho < 1$  forcing the dynamics of (7) to the global convergence to an equilibrium. For example, in the case of the *tent map*

$$T_2(x) = \begin{cases} 2x & 0 \leq x < 1/2 \\ 2(1-x) & 1/2 \leq x \leq 1 \end{cases}$$

which is chaotic on  $[0, 1]$  (for the fixed 1-period delay), we get global convergence to the positive equilibrium as soon as the lag is geometrically distributed with ratio  $\rho > 1/3$ .

#### 4. Main results.

##### 4.1. Reduction to a two-dimensional map.

We shall assume from now on that  $P$  is a geometric memory with ratio  $0 < \rho < 1$  (the case  $\rho = 0$  reduces to the ordinary iteration, the case  $\rho = 1$  has been discussed in the preceding section), and initial value  $v = 1$ , for the sake of simplicity. We shall consider the particular aspect assumed by the segmented form (7) when  $F(s) = \mu s(1-s)$ .

Since

$$R_{n+1} = 1 + \rho R_n,$$

the study of (6) can be reduced to the study of

$$\begin{cases} z_{n+1} = \frac{1}{1 + \rho R_n} F(z_n) + \left(1 - \frac{1}{1 + \rho R_n}\right) z_n \\ R_{n+1} = 1 + \rho R_n \\ z_0 \in [0, 1], R_0 = 1. \end{cases} \quad (8)$$



that is, after simplifications,

$$\begin{cases} z_{n+1} = -\left(\frac{\mu}{1 + \rho R_n}\right) z_n^2 + \left(\frac{\mu + \rho R_n}{1 + \rho R_n}\right) z_n \\ R_{n+1} = 1 + \rho R_n \\ z_0 \in [0, 1], R_0 = 1. \end{cases} \quad (9)$$

Let

$$G_n(s) \stackrel{\text{def}}{=} -\left(\frac{\mu}{1 + \rho R_n}\right) s^2 + \left(\frac{\mu + \rho R_n}{1 + \rho R_n}\right) s, \quad s \in [0, 1]. \quad (10)$$

Since  $t_n \rightarrow t \stackrel{\text{def}}{=} 1 - \rho$  as  $n \rightarrow \infty$ , the sequence  $(G_n)$  converges to

$$G(s) = -\mu(1 - \rho)s^2 + [\mu(1 - \rho) + \rho]s \quad (11)$$

in the  $C^p$ -topology on any compact interval, for any  $p \geq 1$ . The fixed points of  $G$  are 0 and  $p^* = (\mu - 1)/\mu$ ; they do not depend on  $\rho$ .

#### 4.2. Conjectures.

Observe that (11) is topologically conjugate through the change of coordinates

$$s = \frac{\mu(1 - \rho) + \rho}{\mu(1 - \rho)} s'$$

to the logistic map

$$s' \mapsto \mu' s'(1 - s')$$

with parameter

$$\mu' \stackrel{\text{def}}{=} \mu(1 - \rho) + \rho.$$

Since, intuitively, the maps  $G_n$  are closer and closer to  $G$  as  $n \rightarrow \infty$ , and it is natural to conjecture that the asymptotic behavior of the sequence  $(z_n)_{n \geq 1}$  defined in (9) is exactly the same as the asymptotic behavior of the orbits of the one-dimensional map  $s \mapsto G(s)$ . This would imply, for example, that the (increasing) sequence  $(\mu_n)_{n \geq 0}$  of parameter values at which the logistic map undergoes the bifurcation of the stable orbit of period  $2^n$ , starting from  $\mu_0 = 1$  (where the positive equilibrium bifurcates from the origin) and converging to

$\mu_\infty = 3.571456\dots$  (where the chaotic region begins) is more and more expanded as  $r$  increases from 0 to 1. The same would happen for the point  $\mu'_c = 4$  of the “final bifurcation” (where divergent orbits appear). Actually, for the logistic map  $s' \mapsto \mu' s'(1 - s')$ , we have approximately:

$$\begin{aligned}\mu'_0 &= 1, \mu'_1 = 3.000, \mu'_2 = 3.449, \mu'_3 = 3.544, \\ \mu'_4 &= 3.564, \dots, \mu'_\infty = 3.571, \\ \mu'_c &= 4.0,\end{aligned}$$

while for the map  $G$  in (11) we have, for  $\rho = 0.6$ ,

$$\begin{aligned}\mu_0 &= 1, \mu_1 = 6.000, \mu_2 = 7.122, \mu_3 = 7.360, \\ \mu_4 &= 7.410, \dots, \mu_\infty = 7.427, \\ \mu_c &= 8.5,\end{aligned}$$

and, for  $\rho = 0.9$ ,

$$\begin{aligned}\mu_0 &= 1, \mu_1 = 21.00, \mu_2 = 25.49, \mu_3 = 26.44, \\ \mu_4 &= 26.64, \dots, \mu_\infty = 26.71, \\ \mu_c &= 31.\end{aligned}$$

In this sense we could say that the geometric distribution of the delay would produce a significant simplification of the dynamical complexity when compared to the basic results concerning the choice of a single fixed delay.

#### 4.3. Some abstract results.

We formalize and prove the preceding suggestions by setting a simple general result on non-autonomous asymptotically autonomous iterations which should be interesting into itself.

Let  $\mathcal{M}$  be a metric space, and let  $g_n : \mathcal{M} \rightarrow \mathcal{M}$  be a sequence convergent to a continuous map  $g : \mathcal{M} \rightarrow \mathcal{M}$ , uniformly on compact subsets of  $\mathcal{M}$ . For  $x \in \mathcal{M}$ , let  $\mathcal{O}_x$  be the  $(g_n)$ -orbit of  $x$  (i.e. the set  $\{x_n \mid n \in \mathbb{N}\}$  of all points defined by the non-autonomous iteration  $x_{n+1} = g_n(x_n)$ ,  $x_0 = x$ ), and let  $\omega_x$  be the pertinent  $\omega$ -limit (i.e.

the set of limits of the possibly convergent subsequences of  $\mathcal{O}_x = (x_n)_{n \geq 0}$ .

LEMMA 4.1. *For any given  $x \in \mathcal{M}$ , if  $\mathcal{O}_x$  is relatively compact, then  $\omega_x$  is  $g$ -invariant.*

*Proof.* Let  $y \in \omega_x$ : there is a sequence  $(x_{n_k})_{k \geq 0}$  convergent to  $y$ . Then  $x_{n_k+1} = g_{n_k}(x_{n_k}) \rightarrow G(y)$  as  $k \rightarrow \infty$ . Hence  $g(\omega_x) \subseteq \omega_x$ . Since  $\mathcal{O}_x$  is relatively compact, the sequence  $(x_{n_k-1})_{k \geq 0}$  has a subsequence  $(x_{n_{k_l}-1})_{l \geq 0}$  convergent to some  $z \in \omega_x$ . As before,  $x_{n_{k_l}} \rightarrow g(z)$  for  $l \rightarrow \infty$ ; by uniqueness,  $g(z) = y$ , and so  $g(\omega_x) = \omega_x$ .  $\diamond$

We say that a closed set  $A \subseteq \mathcal{M}$  attracts the compact subsets of  $\mathcal{M}$  with respect to  $g$  when for any compact subset  $K \subseteq \mathcal{M}$  and for any  $\epsilon > 0$  there exists an integer  $m = m(\epsilon, K)$  such that for  $n > m$  we have  $g^n(K) \subseteq B_\epsilon(A)$ , being  $B_\epsilon(A)$  the  $\epsilon$ -neighborhood of  $A$ . The following example is obvious:

EXAMPLE 4.1. Let  $\mathcal{M} = J$  be an interval of  $\mathbb{R}$ , let  $g : J \rightarrow J$  be a continuous map, and let  $p$  be an attracting fixed point of  $g$ , with basin of attraction the whole interval  $J$ . If  $g$  is piecewise-monotone, then  $\{p\}$  attracts the compact subsets of  $(0, 1)$  (with respect to  $g$ ).

We say that  $g$  is piecewise-monotone if there are an integer  $N$  and points  $a_1, a_2, \dots, a_N$  such that the restriction of  $g$  to each of the subinterval  $[a_i, a_{i+1}]$  ( $i = 1, 2, \dots, N-1$ ) is monotone. To prove the lemma we have just to observe that for every  $[a, b] \subseteq J$  the image  $g^n([a, b])$  is union of  $N$  subintervals with endpoints some suitable of the  $g^n(a_i)$ , ( $i = 1, 2, \dots, N-1$ ); thus for a given  $\epsilon > 0$  we have only to choose an integer  $m$  such that for  $n > m$  all the endpoints  $g^n(a_i)$  ( $i = 1, 2, \dots, N-1$ ) are in  $[p - \epsilon, p + \epsilon]$ .

COROLLARY 4.1. *Let  $A$  be a closed subset of  $\mathcal{M}$  attracting the compact subsets with respect to  $g$ . Then  $A$  contains all the  $\omega$ -limit sets of the orbits of  $(g_n)$ .*

*Proof.* Any  $\omega_x$  is compact. By assumption, for arbitrary  $\epsilon > 0$  there is an integer  $n_{\epsilon, \omega_x}$  such that for  $n > n_{\epsilon, \omega_x}$  we get  $\omega_x = g^n(\omega_x) \subseteq$

$B_\epsilon(A)$ . Being  $A$  closed, we have  $\omega_x \subseteq A$ .  $\diamond$

#### 4.4. Global convergence to fixed points.

Now we can state the main result of the paper.

**THEOREM 4.1.** *Let  $(d_n)_{n \geq 0}$  be a real sequence,  $d_n > 0$ , and let  $D_n = \sum_{k=0}^n d_k$ . Suppose that*

- (a)  $\lim_{n \rightarrow \infty} d_n/D_n = 0$ ;
- (b) *there are an integer  $n_0$  and a number  $\rho$ ,  $0 < \rho < 1$ , such that  $d_{n+1}/d_n \geq \rho$  for all  $n \geq n_0$ .*

Let  $F : [0, 1] \rightarrow [0, 1]$  be the logistic map  $F(x) = \mu x(1-x)$ ,  $1 < \mu \leq 4$ . Consider the iteration

$$\begin{cases} x_{n+1} = F\left(\frac{1}{D_n} \sum_{k=0}^n d_{n-k} x_k\right), \\ x_0 \in [0, 1]. \end{cases} \quad (12)$$

If

$$\frac{\mu - 3}{\mu - 1} < \rho < 1,$$

then for every initial condition  $0 < x_0 < 1$  the sequence  $(x_n)_{n \geq 0}$  is convergent to the positive fixed point  $p^* = (\mu - 1)/\mu$  of  $F$ .

*Proof.* First, the comparison lemma 2.1 allows us to assume that  $d_0 = 1$  and  $d_{n+1} = \rho d_n$  for all  $n \geq 0$ , that is that  $(d_n)$  is the geometric progression with initial value 1 and ratio  $\rho$ . Then we consider the iteration (12) in this particular case, and we reduce it to the non-autonomous iteration (11) as in section 4.1. Example 4.1 allows us to apply the abstract results of section 4.3 to the case  $g_n = G_n$ ,  $g = G$ , (see resp. (11) and (12)): since the positive equilibrium  $p^*$  of  $G$  attracts the compact subset of  $(0, 1)$ , then all the  $\omega$ -limit sets of  $G_n$  corresponding to orbits starting from the open interval  $(0, 1)$  reduce to the single point  $p^*$ . Hence  $z_n$  (and thus  $x_n$ ) converges to the same limit.  $\diamond$

4.5. Global convergence to periodic points.

The 2-periodic orbits are treated in the same way, by remarking that the composite map  $G_{n+1} \circ G_n$  converges to the second iterate  $G^2$  of  $G$ , and the extension to  $q$ -periodic orbits is similar.

It is also interesting to remark that at any level  $\rho$ ,  $0 < \rho < 1$ , the ratio

$$\frac{\mu_n - \mu_{n-1}}{\mu_{n+1} - \mu_n}$$

does not depend on  $\rho$ , being equal to  $(\mu'_n - \mu'_{n-1})/(\mu'_{n+1} - \mu'_n)$ , and so the period-doubling 'route to chaos' of the map  $G$  in (11) follows the well-known Feigenbaum pattern, with Feigenbaum constant  $\delta = 4.6692\dots$  (See Fig. 1 here and [1, Fig. 1]).

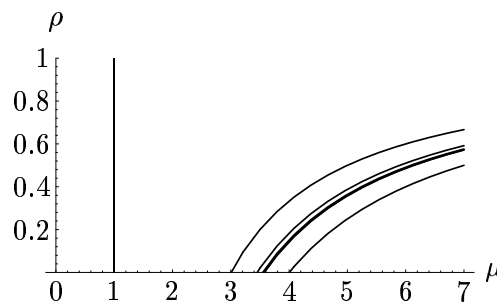


Figure 1: Curves  $\mu(1 - \rho) + \rho = \mu'_k$  ( $k = 0, 1, 2, 3, 4, \infty, c$ ) in the  $(\mu, \rho)$ -plane.

REFERENCES

- [1] AICARDI F. and INVERNIZZI S., *Memory Effects in Discrete Dynamical Systems*, Int. J. Bifurcation and Chaos, **2**(4) (1992), 815-830.
- [2] ALLEN R. G. D., *Mathematical Economics*, MacMillan and Co. ltd, London, 1957.
- [3] BADIO R. and POLITI A., *On the fractal dimension of filtered chaotic signals*, in: Dimensions and Entropies in Chaotic Systems, ed. G. Mayer-Kress, Springer-Verlag, Berlin, 1986.

- [4] BEAUMONT J. R., CLARKE M. and WILSON A. G., *The dynamics of urban spatial structure: some exploratory results using difference equations and bifurcation theory*, Environment and Planning A **13** (1981), 1473-1483.
- [5] EDELSTEIN-KESHET L., *Mathematical Models in Biology*, The Random House / Birkhäuser Mathematics Series, New York, 1988.
- [6] FRANKS R. L. and MARDEC R. P., *A theorem on mean-value iteration*, Proc. Amer. Math. Soc. **30** (1971), 324-326.
- [7] GRANDMONT J.-M., *On Endogenous Competitive Business Cycles*, Econometrica, **53** (1985), 995-1045.
- [8] HARDY G. H., *Divergent Series*, Oxford at the Clarendon Press, Oxford, 1949.
- [9] MANN W. R., *Mean value methods in iteration*, Proc. Amer. Math. Soc., **4** (1953), 506-510.
- [10] MAY R. M., *Simple Mathematical Models With Very Complicated Dynamics*, Nature, **261** (1976), 459-467.
- [11] MAY R. M., *Nonlinear problem in ecology and resource management*, in: Comportement chaotique des systèmes déterministes / Chaotic behavior of deterministic systems, ed. G. Ioos, R. H. G. Helleman, and R. Stora, North-Holland Publ. Co., Amsterdam, 1988.