THE BOUNDED-OPEN TOPOLOGY AND ITS RELATIVES (*)

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SOMMARIO. - *In questo articolo viene esaminata la topologia aperta-limitata di Buchwalter sull'insieme di tutte le funzioni continue a valori reali sopra uno spazio di Tychonoff in condizioni generali. La topologia menzionata viene inoltre confrontata con numerevoli topologie più o meno note.*

SUMMARY. - *This paper studies Buchwalter's bounded-open topology on the set of all continuous real-valued functions on a Tychonoff space in a general setting and compares this topology with several well-known and lesser known topologies.*

1. Introduction.

The set $C(X)$ of all continuous real-valued functions on a completely regular Hausdorff space X has a number of natural topologies.

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In 1970, in [Bu] Buchwalter introduced another natural topology on $C(X)$, the topology of uniform convergence on bounded subsets of X, in order to study certain interesting features from both analytic and measure-theoretic viewpoints. We call it the bounded-open topology – the justification of calling it so is given in Corollary 3.3. In 1976 the study of this new topology was referred to in a monograph by Schmets in [S]. In 1982, in [A] Arhangel'skiǐ considered, apparently independently, the same topology on $C(X)$ in order to study linear homeomorphisms of some function spaces, though he did not study the topological properties of the resulting function space as such. The primary concern of this work is to study the bounded-open topology in detail from the topological point of view in order to have a better understanding of it in relation to some other well-known and lesser known topologies on $C(X)$. We do it in a general setting by defining several topologies on $C(X)$ and study the entire gamut in a unified way determining thereby, in the process, the exact position the Buchwalter's topology occupies in the hierarchy of the topologies to be discussed in the sequel. This is meticulously carried out in Section 3 with the help of a variety of examples which gradually reveals the exact position of this topology amidst other more familiar ones. It is observed that more often than not this topology coincides with other known ones and complicated examples have to be constructed to establish its distinct identity. In Section 2, we develop a theory involving relevant concepts which facilitates a better understanding and deeper analysis of the examples that really hold the centerstage in this paper. The results of Section 2 are presented in a general setitng so as to subsume the known results involving familiar and well-studied topologies on $C(X)$ and are obtained from the proofs of these known facts through necessary modifications. In Section 4, we investigate some additional properties of Buchwalter's topology on $C(X)$ in a general setting.

Throughout the rest of the paper, we use the following conventions. All spaces are completely regular Hausdorff, that is, Tychonoff. If X and Y are any two spaces with the same underlying set, then we use $X = Y$, $X \leq Y$ and $X < Y$ to indicate, respectively, that X and Y have same topology, that the topology on Y is finer than or equal to the topology on X and that the topology on Y is strictly finer than the topology on X. The symbols $\mathbb{R}$ and $\mathbb{N}$ denote the spaces of real numbers and natural numbers, respectively.

Finally, the constant zero-function in $C(X)$ is denoted by f_0.

2. $\mathcal{G}$-open and uniform topologies on $C(X)$ and their basic properties.

Let $\Gamma \subseteq C(X)$. A set $A \subseteq X$ is said to be Γ-bounded if $f(A)$ is a bounded subset of $\mathbb{R}$ for each $f \in \Gamma$. We say that A is bounded in X if A is Γ-bounded for $\Gamma = C(X)$. A bounded subset is also called relatively pseudocompact (see [BNS]). Note that the concept of a bounded set is different from that of pseudocompactness. A pseudocompact subset is trivially bounded, but a bounded subset may not be pseudocompact. Such examples are given by the pseudocompact spaces Ψ of 5I and Λ of 6P) in [GJ]. The space Ψ contains $\mathbb{N}$ which is dense in Ψ and the space Λ contains $\mathbb{N}$ as a proper closed subset. In fact, if every closed subspace of a Tychonoff space is pseudocompact then X is countably compact. Since none of Ψ and Λ is countably compact, each of them contains a proper non-pseudocompact closed bounded subset.

Throughout the paper we use the following notations to denote the particular families of bounded subsets of X.

$$\begin{aligned}
\mathcal{F}(X) &= \text{the collection of all finite subsets of } X.\\
\mathcal{K}(X) &= \text{the collection of all compact subsets of } X.\\
\mathcal{KC}(X) &= \text{the collection of all countably compact subsets of } X.\\
\mathcal{PS}(X) &= \text{the collection of all pseudocompact subsets of } X.\\
\mathcal{B}(X) &= \text{the collection of all bounded subsets of } X.
\end{aligned}$$

Note that $\mathcal{F}(X) \subseteq \mathcal{K}(X) \subseteq \mathcal{KC}(X) \subseteq \mathcal{PS}(X) \subseteq \mathcal{B}(X)$. We also would like to mention that the closure of a pseudocompact (respectively, bounded) subset of X is also pseudocompact (respectively, bounded). Note that for a closed subset A in a normal Hausdorff space X, the following are equivalent:

(i) A is countably compact,

(ii) A is pseudocompact

(iii) A is bounded.

In this work, we are going to talk about two kinds of topologies on $C(X)$. If α is a collection of some subsets of a space X, then the set-open topology on $C(X)$ generated by α is as follows. The subbasic open sets are of the form

$$[A, V] = \left\{ f \in C(X) : \overline{f(A)} \subseteq V \right\} \tag{S}$$

where $A \in \alpha$ and V is open in $\mathbb{R}$. Note we obtain the point-open (respectively, compact-open) topology on $C(X)$ when $\alpha = \mathcal{F}(X)$ (respectively, $\mathcal{K}(X)$). Also for these two topologies, V can always be taken as a bounded open interval.

To define $\mathcal{G}$-open topology on $C(X)$ for some $\mathcal{G} \subseteq \mathcal{B}(X)$, we take the subbasic open sets of the form (S) where $A \in \mathcal{G}$ and V is open in $\mathbb{R}$. We denote the space $C(X)$ with $\mathcal{G}$-open topology by $C_{\mathcal{G}}(X)$. When $\mathcal{G} = \mathcal{F}(X)$, $\mathcal{K}(X)$, $\mathcal{KC}(X)$, $\mathcal{PS}(X)$ or $\mathcal{B}(X)$, we call the corresponding $\mathcal{G}$-open topologies on $C(X)$ point-open, compact-open, countably compact-open, pseudocompact-open and bounded-open respectively. The corresponding spaces are denoted by $C_p(X)$, $C_k(X)$, $C_{kc}(X)$, $C_{ps}(X)$and $C_b(X)$ respectively.

Let $\overline{\mathcal{G}} = \{\overline{A} : A \in \mathcal{G}\}$. Note that the same $\mathcal{G}$-open topology is obtained if $\mathcal{G}$ is replaced by $\overline{\mathcal{G}}$. This is because for each $f \in C(X)$, $f(\overline{A}) \subseteq \overline{f(A)}$; and hence $\overline{f(\overline{A})} = \overline{f(A)}$. Consequently, $C_{\mathcal{G}}(X) = C_{\overline{\mathcal{G}}}(X)$. In particular. in case of $C_{ps}(X)$ and $C_b(X)$ we can use closed pseudocompact and closed bounded subsets respectively. Also note if $\mathcal{G} \subseteq \mathcal{PS}(X)$, then $f(A) = \overline{f(A)}$.

Now we look at the uniform topology on $C(X)$ generated by $\mathcal{G}$, a subcollection of $\mathcal{B}(X)$. Throughout this paper, whenever we consider the uniform topology generated by $\mathcal{G}$ (to be defined shortly), we assume that $\mathcal{G}$ satisfies the following condition:

$$\begin{gathered}\text{If } A,\ B\ \in \mathcal{G}\text{, then there exists a } C \in \mathcal{G} \\ \text{such that } A \cup B \subseteq C \text{ holds.}\end{gathered} \tag{U}$$

For each $A \in \mathcal{G}$ and $\varepsilon > 0$, let

$$A_\varepsilon = \{(f, g) \in C(X) \times C(X) : |f(x) - g(x)| < \varepsilon \ \text{ for all } \ x \in A\}.$$

It can be easily verified that the collection $\{A_\varepsilon : A \in \mathcal{G},\ \varepsilon > 0\}$ is a base for some uniformity on $C(X)$. We denote the space $C(X)$ with the topology induced by this uniformity by $C_{\mathcal{G},u}(X)$. This topology is called the topology of uniform convergence on $\mathcal{G}$. For each $f \in C(X)$, $A \in \mathcal{G}$ and $\varepsilon > 0$, let

$$< f, A, \varepsilon >= \{g \in C(X) : |f(x) - g(x)| < \varepsilon \quad \text{for all}\, x \in A\}.$$

Then for each $f \in C(X)$, the collection $\{< f, A, \varepsilon >: A \in \mathcal{G}, \varepsilon > 0\}$ forms a neighborhood base at f in $C_{\mathcal{G},u}(X)$. Since the topology comes from a uniformity, $C_{\mathcal{G},u}(X)$ is completely regular. Note that here again $\mathcal{G}$ may be replaced by $\overline{\mathcal{G}}$, that is, $C_{\mathcal{G},u}(X) = C_{\overline{\mathcal{G}},u}(X)$.

The fact that the topology of uniform convergence on $\mathcal{G}$ is completely regular may be obtained in a functional analytic way by showing that $C_{\mathcal{G},u}(X)$ is actually a locally convex space.

For each $A \in \mathcal{G}$, define the seminorm p_A on $C(X)$ by

$$p_A(f) = \sup\{|f(x)| : x \in A\}$$

Also for each $A \in \mathcal{G}$ and $\varepsilon > 0$, let

$$V_{A,\varepsilon} = \{f \in C(X) : p_A(f) < \varepsilon\}.$$

Let $\mathcal{V} = \{V_{A,\varepsilon} : A \in \mathcal{G}, \varepsilon > 0\}$.

It can be easily shown that for each $f \in C(X)$, $f + \mathcal{V} = \{f + V : V \in \mathcal{V}\}$ forms a neighborhood base at f. Since this topology is generated by a collection of seminorms, it is locally convex. Note for each $f \in C(X)$, we have

$$\begin{aligned} f + V_{A,\varepsilon} &\subseteq < f, A, \varepsilon > \\ \text{and } < f, A, \frac{\varepsilon}{2} > &\subseteq f + V_{A,\varepsilon} \end{aligned}$$

for all $A \in \mathcal{G}$. This shows that the topology of uniform convergence on $\mathcal{G}$ is the same as the topology generated by the collection of seminorms $\{p_A : A \in \mathcal{G}\}$. So $C_{\mathcal{G},u}(X)$ is actually a locally convex space.

When $\mathcal{G} = \mathcal{F}(X)$, $\mathcal{K}(X)$, $\mathcal{KC}(X)$, $\mathcal{PS}(X)$ or $\mathcal{B}(X)$, $C_{\mathcal{G},u}(X)$ is denoted by $C_{p,u}(X)$, $C_{k,u}(X)$, $C_{kc,u}(X)$, $C_{ps,u}(X)$, and $C_{b,u}(X)$ respectively. The space $C_{b,u}(X)$ is the one which Buchwalter introduced in [Bu].

Suppose $\mathcal{G}_1$ and $\mathcal{G}_2$ are two families of bounded subsets of X. Then we say that $\mathcal{G}_1$ approximates $\mathcal{G}_2$ (or equivalently, $\mathcal{G}_2$ is approximated by $\mathcal{G}_1$) if for any $A \in \mathcal{G}_2$ and any open set V containing A, there exists a finite number of members of $\mathcal{G}_1$, say $B_1, \ldots, B_n$ such that $A \subseteq B_1 \cup B_2 \cup \ldots \cup B_n \subseteq V$ holds. Let $\mathcal{G} \subseteq \mathcal{B}(X)$. Then $\mathcal{G}$ is called a network on X if for each $x \in X$ and any open set V containing x, there exists an $A \in \mathcal{G}$ such that $x \in A \subseteq V$ holds. Note $\mathcal{G}$ approximates $\mathcal{F}(X)$ if and only if $\mathcal{G}$ is a network. We also say $\mathcal{G}_1 \leq \mathcal{G}_2$ if for each $A \in \mathcal{G}$, there exists $B \in \mathcal{G}_2$ such that $A \subseteq B$ holds.

PROPOSITION 2.1. *Suppose $\mathcal{G} \subseteq \mathcal{B}(X)$ and $\mathcal{G}$ is a network on X. Then $C_{\mathcal{G}}(X)$ is a completely regular Hausdorff space.*

Proof. Since $\mathcal{G}$ is a network on X, $C_p(X) \leq C_{\mathcal{G}}(X)$. But $C_p(X)$ being Hausdorff, $C_{\mathcal{G}}(X)$ is also Hausdorff. The complete regularity of $C_{\mathcal{G}}(X)$ can be proved in a manner similar to the proof of Lemma 5.1 in [MN1]. $\diamondsuit$

COROLLARY 2.2. *$C_p(X)$, $C_k(X)$, $C_{kc}(X)$, $C_{ps}(X)$ and $C_b(X)$ are completely regular Hausdorff spaces.*

PROPOSITION 2.3. *Suppose $\mathcal{G} \subseteq \mathcal{B}(X)$ satisfying the condition* (U). *Then $C_{\mathcal{G},u}(X)$ is Hausdorff if and only if $\cup\{A : A \in \mathcal{G}\}$ is dense in X.*

Proof.

$$C_{\mathcal{G},u}(X) \quad \text{is Hausdorff}$$
$$\iff$$
$$\bigcap\{V_{A,\varepsilon} : A \in \mathcal{G},\, \varepsilon > 0\} = \{f_0\}$$
$$\iff$$
$$\bigcup\{A : A \in \mathcal{G}\} \quad \text{is dense in } X.$$

$\diamondsuit$

COROLLARY 2.4. *$C_{p,u}(X)$, $C_{k,u}(X)$, $C_{kc,u}(X)$, $C_{ps,u}(X)$, and $C_{b,u}(X)$ are completely regular Hausdorff spaces.*

Let $C^*(X)$ denote the set of all bounded functions in $C(X)$. It is well-known that $C^*(X)$ is dense in both $C_p(X)$ and $C_k(X)$. In the following, we obtain similar results for $C_{\mathcal{G}}(X)$ and $C_{\mathcal{G},u}(X)$.

THEOREM 2.5. *For every space X, $C^*(X)$ is dense in $C_{\mathcal{G}}(X)$.*

Proof. If $\cap_{i=1}^n [A_i, V_i]$ is a basic open set in $C_{\mathcal{G}}(X)$ containing f, then because of compactness of $\cup_{i=1}^n \overline{f(A_i)}$, $\cup_{i=1}^n \overline{f(A_i)}$ is contained in some interval (a, b) which, in turn, is contained in $\cup_{i=1}^n V_i$. Then if g is defined by $g(x) = f(x)$ if $a \leq f(x) \leq b$, $g(x) = a$ if $f(x) < a$ and $g(x) = b$ if $b < f(x)$, we have that

$$g \in C^*(X) \bigcap (\bigcap_{i=1}^n [A_i, V_i])$$

as desired. ♢

The proof of Theorem 2.5 can be modified to obtain the following result.

THEOREM 2.6. *For every space X, $C^*(X)$ is dense in $C_{\mathcal{G},u}(X)$.*

REMARK 2.7. Theorems 2.5 and 2.6 hold whenever $\mathcal{G} = \mathcal{F}(X)$, $\mathcal{K}(X)$, $\mathcal{KC}(X)$, $\mathcal{PS}(X)$ or $\mathcal{B}(X)$.

3. Comparison of topologies.

First we will establish relationship between the $\mathcal{G}$-open topology on $C(X)$ and the corresponding topology of uniform convergence. In this section, we consider only those families of bounded subsets of X which satisfy the condition (U).

A subset A of a space X satisfying the condition $A = \overline{\text{int} A}$ is called a closed domain. A family $\mathcal{G}$ of bounded subsets of X is said to be hereditary with respect to closed domains if it satisfies the following condition: whenever $A \in \mathcal{G}$ and B is a closed subdomain of A, then $B \in \mathcal{G}$ also.

THEOREM 3.1. *Suppose $\mathcal{G} \subseteq \mathcal{B}(X)$. Then $C_{\mathcal{G}}(X) \leq C_{\mathcal{G},u}(X)$.*

Proof. First we prove that if $f \in [A, V]$ where $f \in C(X)$, $A \in \mathcal{G}$ and V is open in $\mathbb{R}$, then there exists $\varepsilon > 0$ such that $f \in < f, A, \varepsilon > \subseteq [A, V]$ holds. Since $\overline{f(A)}$ is compact, there exist $\varepsilon > 0$ and a closed subset C of $\mathbb{R}$ such that for all $z \in \overline{f(A)}$, $[z-\varepsilon, z+\varepsilon] \subseteq C \subseteq V$ holds. Now if $g \in < f, A, \varepsilon >$, then $\overline{g(A)} \subseteq C \subseteq V$ which, in turn, implies that $g \in [A, V]$.

Now if $W = \cap_{i=1}^{n}[A_i, V_i]$ is a basic neighborhood of f in $C_{\mathcal{G}}(X)$, then there exists $\varepsilon > 0$ such that $< f, A_1 \cup A_2, \cdots, \cup A_n, \varepsilon > \subseteq W$ holds. $\diamondsuit$

THEOREM 3.2. *Suppose $\mathcal{G}$ is a family of bounded subsets of X hereditary with respect to closed domains. Then $C_{\mathcal{G},u}(X) \leq C_{\mathcal{G}}(X)$.*

Proof. Let $< f, A, \varepsilon >$ be a basic neighborhood of f in $C_{\mathcal{G},u}(X)$ where $A \in \mathcal{G}$ and $\varepsilon > 0$. Since $f(A)$ is bounded, there are open intervals $V_1, \dots, V_n$ of length $\frac{\varepsilon}{2}$ such that $\overline{f(A)} \subseteq \cup_{i=1}^{n} V_i$. For each i, let $A_i = \mathrm{cl}_A(A \cap f^{-1}(V_i))$. So $A_i \in \mathcal{G}$. Suppose for each $1 \leq i \leq n$, $V_i = (a_i, b_i)$. Let $W_i = (a_i - \frac{\varepsilon}{4}, b_i + \frac{\varepsilon}{4})$. Define $W = \cap_{i=1}^{n}[A_i, W_i]$ which is a basic neighborhood of f in $C_{\mathcal{G}}(X)$. It is routine to check that $W \subseteq < f, A, \varepsilon >$. $\diamondsuit$

COROLLARY 3.3. *For any space X, $C_j(X) = C_{j,u}(X)$ where $j = p$, k, kc, ps, b.*

THEOREM 3.4. *Let $\mathcal{G}_1$, $\mathcal{G}_2 \subseteq \mathcal{B}(X)$ and suppose $\mathcal{G}_1$ approximates $\mathcal{G}_2$. Then $C_{\mathcal{G}_2}(X) \leq C_{\mathcal{G}_1}(X)$.*

Proof. Easy and straightforward. $\diamondsuit$

COROLLARY 3.5. *For any space X, $C_p(X) \leq C_k(X) \leq C_{kc}(X) \leq C_{ps}(X) \leq C_b(X)$.*

Now the question is whether the converse of Theorem 3.4 is true or not. We do not know the complete answer yet, but if $\mathcal{G}_1$, $\mathcal{G}_2 \subseteq \mathcal{K}(X)$, then the converse is also true. For the proof, see Theorem 2.1 in [MN3].

THEOREM 3.6. *Let $\mathcal{G}_1$, $\mathcal{G}_2 \subseteq \mathcal{B}(X)$ satisfying the condition* (U). *Then $\mathcal{G}_1 < \overline{\mathcal{G}_2}$ if and only if $C_{\mathcal{G}_1,u}(X) \leq C_{\mathcal{G}_2,u}(X)$.*

Proof. If $\mathcal{G}_1 < \overline{\mathcal{G}_2}$, then it can be easily verified that $C_{\mathcal{G}_1,u}(X) \leq$

$C_{\overline{\mathcal{G}_2},u}(X)$. But $C_{\overline{\mathcal{G}_2},u}(X) = C_{\mathcal{G}_2,u}(X)$.

Conversely, suppose $C_{\mathcal{G}_1,u}(X) \leq C_{\mathcal{G}_2,u}(X)$. Choose any $A \in \mathcal{G}_1$. Consider $< f_0, A, \frac{1}{2} >$. Then there exist a $B \in \mathcal{G}_2$ and $\delta > 0$ such that

$$< f_0, B, \delta > \subseteq < f_0, A, \frac{1}{2} > .$$

Suppose, by way of contradiction, that there is an $x \in A \backslash \overline{B}$. Then there would be some $f \in C(X)$ with $f(x) = 1$ and $f(y) = 0$ for all $y \in \overline{B}$. But this would mean that

$$f \in < f_0, B, \delta > \setminus < f_0, A, \frac{1}{2} > .$$

which is a contradiction. It follows that $A \subseteq \overline{B}$. ♢

Corollary 3.7. *For any space X,*

(a) $C_{p,u}(X) \leq C_{k,u}(X) \leq C_{kc,u}(X) \leq C_{ps,u}(X) \leq C_{b,u}(X) \leq C_u(X)$.

(b) (i) $C_{p,u}(X) = C_{k,u}(X)$ *if and only if every compact subset of X is finite.*

(ii) $C_{k,u}(X) = C_{kc,u}(X)$ *if and only if the closure of every countably compact subset of X is compact.*

(iii) $C_{kc,u}(X) = C_{ps,u}(X)$ *if and only if every closed pseudocompact subset of X is contained in the closure of some countably compact subset.*

(iv) $C_{ps,u}(X) = C_{b,u}(X)$ *if and only if every closed bounded subset of X is contained in a closed pseudocompact subset.*

(v) $C_{b,u}(X) = C_u(X)$ *if and only if X is pseudocompact and in this case* $C_{ps,u}(X) = C_{b,u}(X) = C_u(X)$.

(vi) $C_{k,u}(X) = C_{b,u}(X)$ *if and only if every closed bounded subset of X is compact.*

Now we would like to pay special attention to the part (b)-(vi) of the Corollary 3.7, that is, we would like to investigate the spaces for which every closed bounded subset is compact.

Definition 3.8. *A space X is called a Nachbin-Shirota space (NS-space for brevity, but some authors also call it a μ-space, see* [B]*) if every closed bounded subset of X is compact.*

So $C_{k,u}(X) = C_{b,u}(X)$ if and only if X is an NS-space. An immediate example of an NS-space is a space which is realcompact. But to show this we need the following result.

LEMMA 3.9. *A subset A of a space X is bounded if and only if* $\mathrm{cl}_{\beta X} A \subseteq \nu X$ *where νX is the realcompactification of X and* $\mathrm{cl}_{\beta X} A$ *means the closure of A in βX =the Stone-Cěch compactification of X.*

Proof. See page 28 in [BNS]. ◊

There are NS-spaces which are not realcompact.Such a space is given in the following Example 3.10.

EXAMPLE 3.10. Let $W(\omega_2)$ be the set of all ordinals $< \omega_2$ equipped with the order topology where ω_2 is the smallest ordinal of cardinal $\aleph_2$ and let X be the subspace of $W(\omega_2)$ obtained by deleting all limit ordinals having a countable local base. This space X appears in 9L of [GJ] as well as in the pages 103-106 of [BNS]. X is a P-space, but is not realcompact. Note that every P-space is an NS-space, (see 4K in [GJ]). For this space X, we have,

$$C_{p,u}(X) = C_{k,u}(X) = C_{kc,u}(X) = C_{ps,u}(X) = C_{b,u}(X) < C_u(X).$$

REMARK 3.11. A discrete space is realcompact if and only if its cardinal is nonmeasurable. Also every metrizable space of nonmeasurable cardinal is realcompact (see chapters 12 and 15 in [GJ]).

EXAMPLE 3.12. The space Ψ described in 5I of [GJ]. Since Ψ is first countable, every countably compact subset of Ψ is closed. Also every countably compact subset of Ψ is compact (see [T]) and Ψ is locally compact (see [SS1]). Since Ψ is a non-discrete k-space, there exists a compact subset of Ψ which is not finite (see 2.3-2, page 71 in [BNS]). For $X = \Psi$, we have

$$C_{p,u}(X) < C_{k,u}(X) = C_{kc,u}(X) < C_{ps,u}(X) = C_{b,u}(X) = C_u(X).$$

EXAMPLE 3.13. The space Λ described in 6P of [GJ].

$\Lambda = \beta\mathbb{R}\backslash(\beta\mathbb{N}\backslash\mathbb{N})$. Here $\mathbb{N}$ gives an example of a closed bounded subset which is not pseudocompact. For $X = \Lambda$, we have

$$C_{p,u}(X) < C_{k,u}(X) \leq C_{kc,u}(X) \leq C_{ps,u}(X) = C_{b,u}(X) = C_u(X).$$

Note that $C_{k,u}(X) = C_{kc,u}(X)$ and $C_{kc,u}(X) = C_{ps,u}(X)$ cannot hold simultaneously since X is not compact.

EXAMPLE 3.14. Let $\mathbb{N}^* = \beta\mathbb{N}\backslash\mathbb{N}$ and $X = \beta\mathbb{N}\backslash\{p\}$ where $p \in \mathbb{N}^*$. Then X is countably compact but neither normal nor compact. X, being locally compact, is a k-space. Since X is not discrete, there exists a compact subset of X which is not finite. Therefore

$$C_{p,u}(X) < C_{k,u}(X) < C_{kc,u}(X) = C_{ps,u}(X) = C_{b,u}(X) = C_u(X).$$

EXAMPLE 3.15. Let $X = \Psi \oplus (\beta\mathbb{N}\backslash\{p\}) \oplus \mathbb{R}$ where Ψ is the space of Example 3.12 and $\beta\mathbb{N}\backslash\{p\}$ is the space of Example 3.14. For this space we have

$$C_{p,u}(X) < C_{k,u}(X) < C_{kc,u}(X) < C_{ps,u}(X) = C_{b,u}(X) < C_u(X).$$

The ultimate in this chain of examples is attained if we can get an affirmative answer to the following question. Does there exist a space X such that

$$C_{p,u}(X) < C_{k,u}(X) < C_{kc,u}(X) < C_{ps,u}(X) < < C_{b,u}(X) < C_u(X) \quad \text{holds?}$$

To answer this question we first present the following example.

EXAMPLE 3.16. Let ω_0, ω_1 denote respectively the first infinite and the first uncountable ordinal. Let D be the Dieudonné Plank where the underlying set is $[0, \omega_1] \times [0, \omega_0]\backslash\{(\omega_1, \omega_0)\}$. Let B, C and Y stand for the subsets $[0, \omega_1] \times \{\omega_0\}$, $\{\omega_1\} \times [0, \omega_0]$ and $[0, \omega_1) \times [0, \omega_0)$ of D respectively. All the points of Y are isolated, B and C are closed subsets whose relative topologies are discrete. It is known that D is a Tychonoff space which is neither normal nor countably compact (see [SS]).

First we study the pseudocompact subsets of D. Let A be a non-empty pseudocompact subset of D. If $A \cap Y = \emptyset$, then $A \subset B \cup C$.

The relative topology of $B \cup C$ being discrete, A must be finite. So assume $A \cap Y \neq \emptyset$. Then $A \cap Y = \bigcup_{n<\omega_0} A \cap ([0,\omega_1) \times \{n\})$. Since each horizontal line $[0,\omega_1) \times \{n\}$ of Y consists of isolated points of D alone, A has at most a finite number of points from each such line and hence $A \cap Y$ is countable. Using the property of ordinals we can find $\lambda_0 < \omega_1$ such that $A \cap Y \subset [0,\lambda_0) \times [0,\omega_0)$. Note that

$$A = (A \cap ([0,\lambda_0] \times [0,\omega_0])) \bigcup (A \cap (D \backslash [0,\lambda_0] \times [0,\omega_0]))$$

and $[0,\lambda_0] \times [0,\omega_0]$ is a countable clopen subset of D. Observe that $A \cap (D \backslash [0,\lambda_0] \times [0,\omega_0])$ is a clopen subspace of the pseudocompact set A and is contained in $B \cup C$, so that it is a pseudocompact subset of the discrete closed subspace $B \cup C$ of D. Thus $A \cap (D \backslash [0,\lambda_0] \times [0,\omega_0])$ is finite. Consequently A is countable and can have at most a finite intersection with $B \cup C$. We conclude that every pseudocompact subset of D must be countable and, a fortiori, compact and can have finite number of points in common with C.

Now we show that C is a bounded subset of D. Let f be a continuous function on D and for $(\omega_1, n) \in C$, let $f((\omega_1,n)) = a_n$, $n < \omega_0$. Using the properties of ordinals, the continuity of f and the neighborhood system at (ω_1, n), $n < \omega_0$, we can find an ordinal $\beta < \omega_1$ such that $f = a_n$ on $(\beta,\omega_1] \times \{n\}$ for each $n < \omega_0$. Let $\lambda = \beta + 1 < \omega_1$. Let $U_m(\lambda) = \{(\lambda,\alpha) : m < \alpha \leq \omega_0\}$ be a typical basic neighborhood of (λ,ω_0). For $n > m$, $(\lambda, n) \in U_m(\lambda)$ and $f((\lambda,n)) = a_n$ for all $n < \omega_0$. By continuity of f at (λ,ω_0), $a_n \to f((\lambda,\omega_0))$ as $n \to \infty$ and hence f is bounded on C.

As C is finite and closed and the pseudocompact subsets of D are compact and have at most a finite intersection with C, C provides an example of a bounded subset which is not contained in any closed pseudocompact subset of D. Since D has infinite compact subsets, for this space we have

$$C_{p,u}(D) < C_{k,u}(D) = C_{kc,u}(D) = C_{ps,u}(D) < C_{b,u}(D) < C_u(D).$$

EXAMPLE 3.17. Let $X = \Psi \oplus (\beta\mathbb{N} \backslash \{p\}) \oplus D$ where Ψ, $\beta\mathbb{N}\backslash\{p\}$ and D are as in Examples 3.12, 3.14 and 3.16 respectively. For this space we have

$$C_{p,u}(X) < C_{k,u}(X) < C_{kc,u}(X) < C_{ps,u}(X) < C_{b,u}(X) < C_u(X).$$

REMARK 3.18. In [T], Todd constructed an example (Example 3.2) of a space T which has an infinite bounded subset , but pseudocompact subsets of which are finite. For T we have

$$C_{p,u}(T) = C_{k,u}(T) = C_{kc,u}(T) = C_{ps,u}(T) < C_{b,u}(T) < C_u(T).$$

Let $X = \Psi \oplus (\beta\mathbb{N}\backslash\{p\}) \oplus \mathbb{R} \oplus T$ where Ψ and $\beta\mathbb{N}\backslash\{p\}$ are in Example 3.12 and 3.14. For this space X we have

$$C_{p,u}(X) < C_{k,u}(X) < C_{kc,u}(X) < C_{ps,u}(X) < C_{b,u}(X) < C_u(X).$$

4. Additional properties.

First we consider the separability of $C_{\mathcal{G}}(X)$ and $C_{\mathcal{G},u}(X)$.

THEOREM 4.1. *Suppose $\mathcal{G} \subseteq \mathcal{B}(X)$ and $\mathcal{G}$ is a network satisfying the condition* (U). *Then the following are equivalent.*

(i) $C_{\mathcal{G},u}(X)$ *is separable*

(ii) $C_{\mathcal{G}}(X)$ *is separable*

(iii) $C_{p,u}(X) = C_p(X)$ *is separable*

(iv) $C_{k,u}(X) = C_k(X)$ *is separable*

(v) X *has a coarser separable metrizable topology.*

Proof. First note that by Corollary 4.2.2 in [MN2], (iii), (iv) and (v) are equivalent. Since $C_{\mathcal{G}}(X) \le C_{\mathcal{G},u}(X)$, (i) $\Rightarrow$ (ii). Again since $C_{p,u}(X) \le C_{\mathcal{G}}(X)$, (ii) $\Rightarrow$ (iii).

(v) $\Rightarrow$ (i) If X has a coarser separable metrizable topology, then X is realcompact (see page 278 in [E]) and consequently by Lemma 3.9 the closure of every element of $\mathcal{G}$ is compact. Hence $C_p(X) \le C_{\mathcal{G}}(X) \le C_{\mathcal{G},u}(X) = C_{\overline{\mathcal{G}},u}(X) \le C_k(X)$. Since (v) $\Rightarrow$ (iv), $C_{\mathcal{G},u}(X)$ is separable. $\diamondsuit$

To prove the next theorem we need the well-known result which says that if the topology of a locally convex T_1-space is generated by a countable family of seminorms, then it is metrizable (see page 119 in [TL]).

THEOREM 4.2. *Suppose $\mathcal{G} \subseteq \mathcal{B}(X)$ satisfying the condition* (U). *Then the following are equivalent.*

(i) *There exists a countable subfamily $\mathcal{G}_1 = \{G_1, G_2, \ldots, G_n, \ldots\}$ of $\overline{\mathcal{G}}$ such that for any $G \in \mathcal{G}$ there exists G_n such that $G \subseteq G_n$ holds and $\bigcup\{G_n : n \in \mathbb{N}\}$ is dense in X.*

(ii) *$C_{\mathcal{G},u}(X)$ is metrizable.*

(iii) *$C_{\mathcal{G},u}(X)$ is first countable and Hausdorff.*

Proof. (i) $\Rightarrow$ (ii). Since $\mathcal{G}_1 < \overline{\mathcal{G}}$, $C_{\mathcal{G}_1,u}(X) \leq C_{\overline{\mathcal{G}},u}(X) = C_{\mathcal{G},u}(X)$. For each $G \in \mathcal{G}$, there exists a $G_n \in \overline{\mathcal{G}}$, such that $G \subseteq G_n$. So for any $\varepsilon > 0$, $V_{G_n,\varepsilon} \subseteq V_{G,\varepsilon}$. This shows that $C_{\mathcal{G},u}(X) \leq C_{\mathcal{G}_1,u}(X)$. Hence $C_{\mathcal{G},u}(X) = C_{\mathcal{G}_1,u}(X)$. But $C_{\mathcal{G}_1,u}(X)$ is metrizable.

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i). Since $C_{\mathcal{G},u}(X)$ is first countable, it must have a countable neighborhood base at f_0 of the form

$$V_{G'_n,\varepsilon_n} = \{f \in C(X) : p_{G'_n}(f) < \varepsilon_n\}$$

where $\{\varepsilon_n : n \in \mathbb{N}\}$ is a sequence of positive numbers and the G'_n are elements of $\mathcal{G}$. Thus for each $G \in \mathcal{G}$, there is a natural number n such that $V_{G'_n,\varepsilon_n} \subseteq V_{G,1}$. Now suppose $G \not\subseteq \overline{G'_n}$. So there exists an $x \in G \backslash \overline{G'_n}$. Since X is completely regular, there is an $f \in C(X)$ such that $f(x) = 1$ and $f(y) = 0$ for all $y \in \overline{G'_n}$. Hence $f \in V_{G'_n,\varepsilon_n}$ while $f \notin V_{G,1}$. This contradiction implies that $G \subseteq \overline{G'_n}$. Let $G_n = \overline{G'_n}$ and $\mathcal{G}_1 = \{G_n : n \in \mathbb{N}\}$. Since $C_{\mathcal{G},u}(X)$ is Hausdorff, $\cup\{G : G \in \mathcal{G}\}$ is dense in X. But $\cup\{G : G \in \mathrm{G}\} \subseteq \cup\{G_n : n \in \mathbb{N}\}$. Hence $\bigcup\{G_n : n \in \mathbb{N}\}$ is dense in X. ◇

A topological group E is complete provided that every Cauchy net in E converges to some element in E, where a net (x_α) in E is Cauchy if for each neighborhood U of 0 in E there is an α_0 such that $x_{\alpha_1} - x_{\alpha_2} \in U$ for all α_1, $\alpha_2 \geq \alpha_0$ (for E additive).

The topology on $C_{\mathcal{G},u}(X)$ is generated by the uniformity of uniform convergence on $\mathcal{G}$. When this uniformity is complete, $C_{\mathcal{G},u}(X)$

is said to be uniformly complete. One can check that $C_{\mathcal{G},u}(X)$ is uniformly complete if and only if it is complete as an additive topological group. Also $C_{\mathcal{G},u}(X)$ is completely metrizable if and only if it is complete and metrizable (see pages 34 and 46 in [Be]). Now we examine when $C_{\mathcal{G},u}(X)$ is complete. But in order to do this, first we need the following definitions.

DEFINITION 4.3. A function $f : X \longrightarrow \mathbb{R}$ is said to be *$\mathcal{G}$-continuous* if for every $A \in \mathcal{G}$, there is a continuous functions $g : X \longrightarrow \mathbb{R}$ such that $g|_A = f|_A$.

DEFINITION 4.4. A space X is called a $\mathcal{G}_f$-space if every $\mathcal{G}$-continuous function on X is continuous.

REMARK 4.5. When $\mathcal{G} = \mathcal{B}(X)$, we call a $\mathcal{G}$-continuous function b-continuous and the corresponding $\mathcal{G}_f$-space a b_f-space. Note that when $\mathcal{G} = \mathcal{K}(X)$, the corresponding $\mathcal{G}_f$-space turns out to be well-known k_R-space. When $\mathcal{G} = \mathcal{F}(X)$, every real-valued function on X is $\mathcal{G}$-continuous and consequently in this case, X will be a $\mathcal{G}_f$-space if and only if X is discrete.

THEOREM 4.6. *Suppose $\mathcal{G} \subseteq \mathcal{B}(X)$ satisfying the condition* (U) *and $X = \cup\{A : A \in \mathcal{G}\}$. Then $C_{\mathcal{G},u}(X)$ is complete if and only if X is a $\mathcal{G}_f$-space.*

Proof. Suppose $C_{\mathcal{G},u}(X)$ is complete. Let $f : X \longrightarrow \mathbb{R}$ be a $\mathcal{G}$-continuous function. So for each $B \in \mathcal{G}$, there exists $g_B \in C(X)$ such that $f = g_B$ on B. Consider the net $\{g_B : B \in \mathcal{G}\}$ and a basic neighborhood $< f_0, A, \varepsilon >$ of f_0. This shows that $\{g_B : B \in \mathcal{G}\}$ is a Cauchy net and hence there exists $g \in C(X)$ such that $g_B \to g$. Pick up any $x \in X$ and so there exists $A_0 \in \mathcal{G}$ such that $x \in A_0$. For the neighborhood $< f_0, A_0, \varepsilon >$ of f_0, there exists $B_\varepsilon \in \mathcal{G}$ such that $g_B - g \in < f_0, A_0, \varepsilon >$ for all $B \supseteq B_\varepsilon$. Now there exists $C_\varepsilon \in \mathcal{G}$ such that $A_0 \cup B_\varepsilon \subseteq C_\varepsilon$ and hence $g_{C_\varepsilon} - g \in < f_0, A_0, \varepsilon >$. But this implies that $|f(x) - g(x)| < \varepsilon$. Since ε was arbitrarily chosen, $f(x) = g(x)$. This shows that $f = g$, that is, every $\mathcal{G}$-continuous function is continuous.

Now suppose X is a $\mathcal{G}_f$-space and let $\{f_\alpha\}$ be a Cauchy net in $C_{\mathcal{G},u}(X)$. Since the set of all real-valued functions equipped with the topology of uniform convergence is complete, there exists $f : X \longrightarrow \mathbb{R}$ such that $f_\alpha(x) \to f(x)$ for each $x \in X$. In fact $p_B(f_\alpha - f) \to 0$ for each $B \in \mathcal{G}$. For each $n \geq 1$, let α_n be an index such that $p_B(f_{\alpha_n} - f) < \frac{1}{2^n}$.

Now for $x \in B$,

$$\begin{aligned} |f_{\alpha_{n+1}}(x) - f_{\alpha_n}(x)| &\leq |f_{\alpha_{n+1}}(x) - f(x)| + |f_{\alpha_n}(x) - f(x)| \\ &\leq p_B(f_{\alpha_{n+1}} - f) + p_B(f_{\alpha_n} - f) \\ &< \frac{1}{2^{n+1}} + \frac{1}{2^n} < \frac{1}{2^{n-1}}. \end{aligned}$$

This gives $p_B(f_{\alpha_{n+1}} - f_{\alpha_n}) < \frac{1}{2^{n-1}}$. Let $c_1 = p_B(f_{\alpha_1})$ and $c_{n+1} = p_B(f_{\alpha_{n+1}} - f_{\alpha_n}) < \frac{1}{2^n}$, for $n \geq 1$.

Define for $x \in X$,

$$g_1(x) = \begin{cases} c_1 & \text{if } \ f_{\alpha_1}(x) > c_1 \\ f_{\alpha_1}(x) & \text{if } \ -c_1 \leq f_{\alpha_1}(x) \leq c_1 \\ -c_1 & \text{if } \ f_{\alpha_1}(x) < -c_1 \end{cases}$$

and

$$g_{n+1}(x) = \begin{cases} c_{n+1} & \text{if } \ \Delta > c_{n+1} \\ f_{\alpha_{n+1}}(x) - f_{\alpha_n}(x) & \text{if } \ -c_{n+1} \leq \Delta \leq c_{n+1} \\ -c_{n+1} & \text{if } \ \Delta \leq -c_{n+1} \end{cases}$$

where $\Delta = f_{\alpha_{n+1}}(x) - f_{\alpha_n}(x)$.

Since $\sup\{|g_1(x)| : x \in X\} \leq c_1$ and $\sup\{|g_{n+1}(x)| : x \in X\} \leq c_{n+1} < \frac{1}{2^{n-1}}$, $\sum_{n=1}^{\infty} g_n$ converges uniformly to a real-valued function g on X. But each g_n is continuous and so g is continuous, that is, $g \in C(X)$. Now note for $x \in B$, $g_1(x) = f_{\alpha_1}(x)$ and $g_{n+1}(x) = f_{\alpha_{n+1}}(x) - f_{\alpha_n}(x)$. Hence $f(x) = g(x)$ for $x \in B$. This shows that f is $\mathcal{G}$-continuous. But X is a $\mathcal{G}_f$-space and so f is continuous. But for each $B \in \mathcal{G}$, $p_B(f_\alpha - f) \to 0$ and consequently $C_{\mathcal{G},u}(X)$ is complete. $\diamondsuit$

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