ON DISCRETE INEQUALITIES INVOLVING ARITHMETIC, GEOMETRIC, AND HARMONIC MEANS (*)

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SOMMARIO. - *Si dimostra: se* $A(n), G(n)$, *e* $H(n)$ *denotano la media aritmetica, geometrica ed armonica dei primi* n *interi positivi , allora si ha che per* $n \geq 2$:

$$\begin{aligned} \frac{H(n)}{H(n-1)} - \frac{H(n+1)}{H(n)} &< \frac{G(n)}{G(n-1)} - \frac{G(n+1)}{G(n)} \\ &< \frac{A(n)}{A(n-1)} - \frac{A(n+1)}{A(n)}. \end{aligned}$$

SUMMARY. - *We prove: if* $A(n), G(n)$, *and* $H(n)$ *denote the arithmetic, geometric, and harmonic means of the first* n *positive integers, then we have for* $n \geq 2$:

$$\begin{aligned} \frac{H(n)}{H(n-1)} - \frac{H(n+1)}{H(n)} &< \frac{G(n)}{G(n-1)} - \frac{G(n+1)}{G(n)} \\ &< \frac{A(n)}{A(n-1)} - \frac{A(n+1)}{A(n)}. \end{aligned}$$

1. Introduction.

The arithmetic mean - geometric mean inequality, which is "probably the most important inequality, and certainly a keystone of the theory of inequalities" [3, p. 3], has attracted the attention of several

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mathematicians, and many articles have been published providing new proofs and various refinements, extensions, and variants of this classical result; see the monographs [3, 4, 6, 8].

In this paper we present some inequalities involving the arithmetic, geometric, and harmonic means of the first n positive integers, that is,

$$\begin{aligned} A(n) &= \frac{1}{n}\sum_{i=1}^{n} i = \frac{n+1}{2}, \\ G(n) &= \left(\prod_{i=1}^{n} i\right)^{1/n} = (n!)^{1/n}, \\ H(n) &= \frac{n}{\sum_{i=1}^{n} \frac{1}{i}}. \end{aligned}$$

This work has been motivated by an interesting paper of H. Minc and L. Sathre [7]. When studying a problem on an upper bound for permanents of $(0,1)$-matrices, they found several inequalities involving $G(n)$ "which are of interest in themselves" [7, p. 41]. Their main result is the following inequality.

If $n \geq 2$, then

$$1 < n\frac{G(n+1)}{G(n)} - (n-1)\frac{G(n)}{G(n-1)}. \tag{1.1}$$

In [1] and [2] the author proved the following refinements of (1.1).

If $n \geq 3$, then

$$\begin{aligned} 1 &< 1 + \frac{G(n)}{G(n-1)} - \frac{G(n+1)}{G(n)} < 1 + \frac{1}{n} - \frac{1}{n+1} \\ &< n\frac{G(n+1)}{G(n)} - (n-1)\frac{G(n)}{G(n-1)}. \end{aligned} \tag{1.2}$$

Since $\frac{A(n)}{A(n-1)} - \frac{A(n+1)}{A(n)} = \frac{1}{n} - \frac{1}{n+1}$ we obtain from the second inequality of (1.2) a companion of the arithmetic mean - geometric mean inequality.

If $n \geq 2$, then

$$\frac{G(n)}{G(n-1)} - \frac{G(n+1)}{G(n)} < \frac{A(n)}{A(n-1)} - \frac{A(n+1)}{A(n)}\,. \tag{1.3}$$

In view of the geometric mean - harmonic mean inequality it is natural to ask whether there exists a related result involving geometric and harmonic means. Using the asymptotic expressions

$$\log\Gamma(n) = \left(n - \frac{1}{2}\right)\log(n) - n + \frac{1}{2}\log(2\pi) + \mathrm{O}\left(\frac{1}{n}\right)$$

and

$$\psi\,(n) = \log(n) + \mathrm{O}\left(\frac{1}{n}\right)$$

(see [5, pp. 823-824]), we obtain

$$n\,(n-1)\left[\frac{G(n)}{G(n-1)} - \frac{G(n+1)}{G(n)}\right] = 1 - \frac{\log(n+1)}{n+1} + \mathrm{O}\left(\frac{1}{n}\right)$$

and

$$n\,(n-1)\left[\frac{H(n)}{H(n-1)} - \frac{H(n+1)}{H(n)}\right] = 1 - \frac{1}{\log(n+1)} + \mathrm{O}\left(\frac{1}{(\log(n))^2}\right)$$

which imply

$$\frac{H(n)}{H(n-1)} - \frac{H(n+1)}{H(n)} < \frac{G(n)}{G(n-1)} - \frac{G(n+1)}{G(n)} \tag{1.4}$$

for all sufficiently large n. It is the aim of this note to show that inequality (1.4) holds for all $n \geq 2$. To prove this theorem we need some inequalities involving Γ, $\psi = \Gamma'/\Gamma$, and ψ' which we present in the next section.

2. Some lemmas.

LEMMA 1. *For all $x > 1$ we have*

$$\log(x) - \frac{1}{x} < \psi(x) < \log(x) - \frac{1}{2x}\,.$$

LEMMA 2. *For all $x > 1$ we have*

$$\frac{1}{x} < \psi'(x)\,.$$

LEMMA 3. *For all $x > 1$ we have*

$$\log\Gamma(x) < \frac{1}{x} + \left(x - \frac{1}{2}\right)\,\log(x) - x + \frac{1}{2}\,\log(2\pi)\,.$$

Proofs for these inequalities can be found in [7].

LEMMA 4. *For all $x \geq 10$ we have*

$$\frac{1}{\log(x)} < \frac{\psi(x) + \mathrm{C} + (x-2)\psi'(x)}{[\,\psi(x) + C]^2}$$

where C=0.5772... *denotes Euler's constant.*

Proof. Let $x \geq 10$; from Lemma 1 and Lemma 2 we obtain

$$\psi(x) + \mathrm{C} + (x-2)\,\psi'(x) > \log(x) - \frac{1}{x} + \mathrm{C} + \frac{x-2}{x} \tag{2.1}$$

and

$$(\psi(x) + \mathrm{C})^2 < \left(\log(x) + \mathrm{C} - \frac{1}{2x}\right)^2\,. \tag{2.2}$$

And, from (2.1), (2.2), and the logarithmic mean - arithmetic mean inequality

$$\frac{x-y}{\log(x) - \log(y)} \leq \frac{x+y}{2} \qquad (x > 0,\ y > 0)$$

(see [8, p. 273]), we conclude

$$\begin{aligned}
[\psi(x) + \mathrm{C}]^2 \quad &- \quad \log(x)\;\,[\psi(x) + \mathrm{C} + (x-2)\,\psi'(x)\,] \\
&< \quad \log(x)\,\left(\frac{2}{x} - 1 + \mathrm{C}\right) + \left(\mathrm{C} - \frac{1}{2x}\right)^2 \\
&\leq \quad \frac{2(x-1)}{x+1}\,\left(\frac{2}{x} - 1 + \mathrm{C}\right) + \left(\mathrm{C} - \frac{1}{2x}\right)^2 \\
&= \quad \mathrm{q}\,(x)\,,\ \text{say}\,.
\end{aligned}$$

Since q is strictly decreasing we get for $x \geq 10$ that

$$\mathrm{q}\,(x) \leq \mathrm{q}\,(10) = -0.086\ldots\,.$$

This proves Lemma 4. ◊

3. The main result.

Now we are in a position to establish the following counterpart of inequality (1.3).

THEOREM. *If* $n \geq 2$, *then*

$$\frac{H(n)}{H(n-1)} - \frac{H(n+1)}{H(n)} < \frac{G(n)}{G(n-1)} - \frac{G(n+1)}{G(n)}\,. \tag{3.1}$$

Proof. Since

$$\Gamma(n+1) = n! \quad \text{and} \quad \psi\,(n+1) = -\mathrm{C} + \sum_{i=1}^{n} \frac{1}{i}$$

(see [9, p. 247]), we obtain

$$\begin{aligned} \frac{G(n)}{G(n-1)} - \frac{H(n)}{H(n-1)} &= \frac{(\Gamma(n+1))^{\frac{1}{n}}}{(\Gamma(n))^{\frac{1}{(n-1)}}} - 1 - \frac{1}{n-1} + \\ &\quad + \frac{1}{(n-1)\ [\psi\,(n+1) + \mathrm{C}]} \\ &= \mathrm{d}_n\,,\ \text{say}\,. \end{aligned}$$

We have to prove that

$$\mathrm{d}_{n+1} < \mathrm{d}_n \qquad \text{for } n \geq 2\,. \tag{3.2}$$

To establish (3.2) we define for positive real x:

$$f\,(x) = g\,(x) - 1 - \frac{1}{x} + \frac{1}{x\ [\psi\,(x+2) + \mathrm{C}]}$$

where

$$g(x) = \frac{(\Gamma(x+2))^{\frac{1}{(x+1)}}}{(\Gamma(x+1))^{\frac{1}{x}}}.$$

In the first part of the proof we show that f is strictly decreasing on $[10,\infty)$.

Let $x \geq 10$; differentiation yields

$$\begin{aligned} x^2 f'(x) &= g(x)\left\{\frac{x^2}{(x+1)^2} - \frac{x}{x+1}\psi(x+1) \ + \right. \\ &+ \frac{2x+1}{(x+1)^2}\log\Gamma(x+1) + \\ &- \left.\frac{x^2}{(x+1)^2}\log(x+1)\right\} + \\ &+ 1 - \frac{\psi(x+2) + \mathrm{C} + x\,\psi'(x+2)}{[\psi(x+2)+\mathrm{C}]^2}. \end{aligned} \tag{3.3}$$

Since the expression in curled brackets is negative (see [7]) and since $g(x) > 1$, we conclude from (3.3) that

$$\begin{aligned} x^2 f'(x) &< \frac{x^2}{(x+1)^2} - \frac{x}{x+1}\psi(x+1) + \\ &+ \frac{2x+1}{(x+1)^2}\log\Gamma(x+1) + \\ &- \frac{x^2}{(x+1)^2}\log(x+1) + 1 + \\ &- \frac{\psi(x+2) + \mathrm{C} + x\psi'(x+2)}{[\psi(x+2)+\mathrm{C}]^2}. \end{aligned} \tag{3.4}$$

Using Lemma 1, Lemma 3, and Lemma 4 we obtain from (3.4) that

$$x^2 f'(x) \ < \ \frac{1}{x+1}\log(x+1) + \mathrm{a}(x) - \frac{1}{\log(x+2)} \tag{3.5}$$

where

$$\begin{aligned} \mathrm{a}(x) &= 1 + \frac{x^2}{(x+1)^2} + \frac{x}{(x+1)^2} + \frac{2x+1}{(x+1)^3} + \\ &- \frac{2x+1}{x+1} + \frac{2x+1}{(x+1)^2}\log\sqrt{2\pi}. \end{aligned}$$

Elementary calculations reveal that

$$\mathrm{a}\,(x) \le \frac{2}{x+1}\,, \tag{3.6}$$

so that (3.5) and (3.6) imply

$$x^2 f'\,(x) < \frac{2+\log(x+1)}{x+1} - \frac{1}{\log(x+2)}\,. \tag{3.7}$$

Let

$$\mathrm{p}\,(x) = \log(x+1)\,\log(x+2) + 2\log(x+2) - x - 1\,.$$

From the geometric mean - logarithmic mean inequality

$$\sqrt{xy} \le \frac{x-y}{\log(x)-\log(y)} \qquad (x>0,\ y>0)$$

(see [8, p. 272]), we conclude

$$\begin{aligned}\mathrm{p}'\,(x) &= \frac{1}{x+1}\log(x+2) + \frac{1}{x+2}\log(x+1) + \frac{2}{x+2} - 1\\ &\le \frac{1}{\sqrt{x+2}} + \frac{x}{(x+2)\sqrt{x+1}} + \frac{2}{x+2} - 1 < 0\,.\end{aligned}$$

Thus, we have for $x \ge 10$ that

$$\mathrm{p}\,(x) \le \mathrm{p}\,(10) = -0.0716\ldots < 0\,.$$

This leads to

$$\frac{2+\log(x+1)}{x+1} < \frac{1}{\log(x+2)}\,, \tag{3.8}$$

so that (3.7) and (3.8) imply

$$f'\,(x) < 0 \quad \text{for } x \ge 10\,.$$

Since $f\,(n) = \mathrm{d}_{n+1}$ we obtain

$$\mathrm{d}_{n+1} < \mathrm{d}_n \quad \text{for } n \ge 11\,. \tag{3.9}$$

The approximate values of d_n for $n = 2, 3, \ldots, 11$, are given in the following table.

n	d (n)
2	0.08088
3	0.05761
4	0.04472
5	0.03651
6	0.03080
7	0.02661
8	0.02341
9	0.02087
10	0.01882
11	0.01713

From (3.9) and the table we conclude that (d_n) is strictly decreasing for all $n \geq 2$. This proves the Theorem. $\diamondsuit$

REMARK. An easy calculation reveals that the sequence

$$n \mapsto H(n)\,/H(n-1) \qquad (n = 2, 3, \ldots)$$

is strictly decreasing, so that inequality (3.1) provides a refinement of the left-hand inequality of (1.2).

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