STATES ON ORTHOMODULAR POSETS

(Recent Results in Noncommutative Measure Theory) (*)

by PAVEL PTÁK (in Prague)(**)

1. Introduction.

In 1936, G. Birkhoff and J. von Neumann gave a basis of the so-called "logico-algebraic" foundation of quantum mechanics (see [1]). In this approach, a quantum system is supposed to be associated with an orthocomplemented set L ("a quantum logic") such that the elements of L correspond to the propositions on the system. The physical states of the system are then modelled by the (generally noncommutative) probability measures on L ("states"). The development of ideas in the logico-algebraic foundation of quantum theories can be seen in the monographs [2], [3], [4], [5], [6], etc.

In this exposition we will give an account of recent results on states on quantum logics. The motivation for the problems investigated comes from quantum physics, of course. We assume that the quantum logics be *orthomodular posets* (this assumption was found appropriate within quantum theories).

1.1. DEFINITION. An orthomodular poset (abbr. an OMP) is a triple $(L, \leq, ')$, where L is a set that is partially ordered by $\leq$ and that fulfils the following requirements:

(i) L possesses a least and a greatest element, $0, 1$,

(ii) if $a, b \in L$ and $a \leq b$, then $b' \leq a'$,

(iii) the unary operation $' \colon L \to L$ satisfies the following condition: $(a')' = a$ for any $a \in L$,

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(**) Indirizzo dell'Autore: Department of Mathematics, Faculty of Electrical Engineering, Czech Technical University, 166 27 Praha (Repubblica Ceca).

(iv) if $a, b \in L$ and if $a \leq b'$, then the supremum, $a \vee b$, exists in L,

(v) if $a, b \in L$ and $a \leq b$, then $b = a \vee (b \wedge a')$ (the orthomodular law).

Thus, technically speaking, an OMP is a common generalization of the notion of a Boolean algebra and the lattice of projectors in a Hilbert space (the lattice condition is dropped and the distributivity law is relaxed to the orthomodular law). When an OMP is viewed as an event structure of a quantum experiment, the lattice condition on L does not seem to be justified and the distributivity law does seem to be superfluous — its presence could actually bring us outside the quantum physics. From the mathematical standpoint, too, it seems restrictive to admit lattices only (we would e. g. loose the OMP of projections in a C^*-algebra, the OMP of skew projections in a Hilbert space, the OMP of splitting subspaces in a (noncomplete) inner product space, many interesting set-representable OMPs, etc.). In these notes, we shall not require OMPs to be lattices. — Let us denote by L an OMP throughout the notes.

1.2. Definition. A mapping $s: L \to \langle 0, 1 \rangle$ is called a **state** on L if it fulfils the following two conditions:

(i) $s(1) = 1$,

(ii) if $a \leq b'$, then $s(a \vee b) = s(a) + s(b)$.

Thus, by a state we mean a probability measure on L. For the intuition, one may consider Boolean states (=commutative states), states on the lattice $L(H)$ of projectors in a Hilbert space, states on "concrete" (=set-representable) OMPs, some "exotic" states (Greechie's pasting constructions), etc. In the sequel, we will deal with all these categories of states.

Concluding the introduction, let us note that recent lines of research in the noncommutative measure theory comprise several areas: the states on operator algebras, noncommutative probability theory, noncommutative measure theory on quantum logics and orthoalgebras, state space determination of varieties in universal algebra, special measure-theoretic problems of quantum axiomatics, nonstandard mathematical logics, combinatorial methods of constructing state spaces with preassigned properties, "fuzzy" states, etc. The collaboration in these areas of mathematicians, physicists and philosophers has initiated the foundation of an association (International Quantum Structures Association). One of its goals is a coherent study of physically motivated problems in noncommutative probability theory and noncommutative measure theory.

References

[1] Birkhoff G. and von Neumann J., *The logic of quantum mechanics*, Ann. Math. **37** (1936), 823-843.

[2] Mackey G., *The Mathematical Foundations of Quantum Mechanics*, Benjamin, New York, (1963).

[3] Jauch J., *Foundations of Quantum Mechanics*, Addison-Wesley, Reading, Massachussets, (1968).

[4] Varadarajan V., *Geometry of Quantum Theory*, Vols 1 and 2, Van Nostrand, Princeton (1968).

[5] Gudder S., *Stochastic Methods in Quantum Mechanics*, Elsevier/ North-Holland, Amsterdam, (1979).

[6] Pták P. and Pulmannová S., *Orthomodular Structures as Quantum Logics*, Kluwer Academic Publishers, (1991).

2. Enlargements of orthomodular posets (OMPs with given state spaces).

With the motivation coming from theoretical physics, we ask if we can construct OMPs (="quantum logics") with given state spaces and other "attributes" important within the mathematical foundation of quantum mechanics. Let us first introduce basic notions (by L we denote an OMP).

2.1. The state space, $\mathcal{S}(L)$, of L.

By a state on L we mean a mapping $s: L \to \langle 0, 1\rangle$ such that

(i) $s(1) = 1$,

(ii) $s(a \vee b) = s(a) + s(b)$ provided $a \leq b'$.

Let us stand $\mathcal{S}(L)$ for the set of all states on L and let us consider $\mathcal{S}(L)$ with its natural affine and topological structure ($\mathcal{S}(L) \subset \langle 0,1\rangle^L \subset R^L$, where we regard R^L as a topological linear space).

THEOREM [15]. *Let S be a set. Then $S = \mathcal{S}(L)$ if and only if S is a compact convex subspace in a locally convex topological linear space (the symbol = means an affine homeomorphism).*

It should be observed that - in view of the theorem above - $\mathcal{S}(L)$ may be empty or $\mathcal{S}(L)$ may be a singleton (see also [2], [7] and [10] for relevant results). In fact, every OMP can be embedded into such an "exotic" OMP (see [11]).

2.2. The centre, $\mathcal{C}(L)$, of L.

Put $\mathcal{C}(L) = \{c \in L|$ for any $d \in L$, the set $\{c, d\}$ is contained in a Boolean subalgebra of $L\}$. Let us call $\mathcal{C}(L)$ the *centre* of L.

THEOREM [1], [2]. *$\mathcal{C}(L)$ is a Boolean subalgebra of L. Moreover, $\mathcal{C}(L) = L$ if and only if L is a Boolean algebra.*

2.3. The automorphism group, $\mathcal{G}(L)$, of L.

THEOREM [14], [4]. *If G is a group, then there is an OMP, L, such that $G = \mathcal{G}(L)$.*

It may be noted that the OMP of the latter theorem can be required concrete (= set-representable) for any G ([8]).

A natural question arises of whether we can construct OMPs with any given interplay of its state space, its centre and its automorphism group. The following theorem provides an affirmative answer to this question. As it turns out, we can even ensure an arbitrary degree of "noncompatibility" (i. e. we may also prescribe arbitrary subOMPs).

THEOREM. *Suppose that*
K is an OMP with $\mathcal{S}(K) \neq \emptyset$,
S is a compact convex set in a locally convex topological space,
C is a Boolean algebra, and
G is a group.
Then there is an OMP, L, such that
$K \hookrightarrow L$ (i. e. K is a sublogic of L),
$S = \mathcal{S}(L)$,
$C = \mathcal{C}(L)$, and
$G = \mathcal{G}(L)$.

Proof. See [7], [9], [10] and [12]. The final result formulated in the theorem above was proved in [5].

Comments and consequences.

1. Every OMP can be embedded into an OMP with an arbitrary centre.
2. Every OMP which is not stateless can be embedded into an OMP whose state space is arbitrary.
3. In the course of proving the theorem above one needed to develop an advanced "pasting technique" which helped resolve problems in other areas, too (see e. g. [6] and [13]). (To verify the legitimacy of pastings it has sometimes required computer-proving, see e. g. [7] and [17].)

PROBLEMS.

1. Can one prove the σ-complete version of THEOREM? (In particular, can every σ-complete OMP be embedded into a σ-complete OMP whose centre is preassigned?)
2. Can we prove THEOREM for lattice OMPs?

References

[1] Foulis D., *A note on orthomodular lattice*, Portugal. Math. **21**, (1962), 65-72.

[2] Greechie R., *Orthomodular lattices admitting no states*, Journ. Comb. Theory **10A**, (1971), 119-132.

[3] Gudder S., *Stochastic Methods in Quantum Mechanics*, North-Holland, Amsterdam, (1979).

[4] Kalmbach G., *Automorphism groups of orthomodular lattices*, Bull. Aust. Math. Soc. **29**, (1984), 309-313.

[5] Navara M., *On the independence of state space, centre and automorphism group in quantum logics*, Int. Journ. Theor. Phys. 1993 (to appear).

[6] Navara M., *An orthomodular lattice admitting no group-valued measure*, Proc. Amer. Math. Soc. 1993 (to appear).

[7] Navara M. and Rogalewicz V., *The pasting constructions for orthomodular posets*, Math. Nachr. **154**, (1991), 275-284.

[8] Navara M. and Tkadlec J., *Automorphisms of concrete logics*, Com. Math. Univ. Carolinae **32**, (1991), 15-25.

[9] Navara M., Pták P. and Rogalewicz V., *Enlargements of quantum logics*, Pacific Journ. Math. **135**, (1988), 361-369.

[10] Pták P., *Logics with given centers and state spaces*, Proc. Amer. Math. Soc. **88**, (1983), 106-109.

[11] Pták P., *Exotic logics*, Colloquium Math. **54**, (1987), 1-7.

[12] Pták P. and Pulmannová S., *Orthomodular Structures as Quantum Logics*, Kluwer, Dordrecht/Boston/London, (1991).

[13] Pulmannová S. and Rogalewicz V., *Orthomodular lattices with almost orthogonal set of atoms*, Com. Math. Univ. Carolinae **32** (1991), 423-429.

[14] Schrag G., *Every finite group is the automorphism group of some finite orthomodular lattice*, Proc. Amer. Math. Soc. **55** (1976), 243-249.

[15] Schultz F., *A characterization of state spaces of orthomodular lattices*, Journ. Comb. Theory **17A** (1974), 317-328.

[16] Varadarajan V., *Geometry of Quantum Theory*, Vols 1 and 2, Van Nostrand, Princeton (1968).

[17] Weber H., *An orthomodular lattice without any group-valued measure*, Journ. Math. Anal. Appl. 1993 (to appear)

3. Extensions of states (state-universal OMPs).

In this part we prove two results on extensions of states. Recall ([13]) that L is called *unital* if for any $a \in L$, $a \neq 0$, there is a state $s \in S(L)$ such that $s(a) = 1$.

3.1. DEFINITION. Let K be an OMP. Let us call K *state-universal* if the following condition is satisfied: If K is embedded into an OMP L (i.e., if L is an enlargement of K) and if L is unital, then every state on K can be extended over L.

We shall prove that Hilbertian as well as Boolean OMPs are state universal. We shall need one more definition.

3.2. DEFINITION. Let K be an OMP and let $s \in S(K)$. Then s is called *hyperpure* if there is an element $a \in K$ such that $s(a) = 1$ and moreover, the state s is the only state with the latter property.

Let us denote by $\mathcal{S}_{hp}(K)$ (resp. $\mathcal{S}_p(K)$) the set of all hyperpure (resp. pure) states on K. Obviously, every hyperpure state has to be pure (a state is called pure if it cannot be written as a convex combination of two distinct states). Thus, $\mathcal{S}_{hp}(K) \subset \mathcal{S}_p(K)$ for any K. One sees easily that if, for instance, K is a finite Boolean OMP or if $K = L(H)$ for a Hilbert space H with $\dim H < \infty$, then $\mathcal{S}_{hp}(K) = \mathcal{S}_p(K)$. According to [11], there are finite OMPs (even orthomodular lattices) which possess pure states that are not hyperpure.

Our first result indicates the significance of hyperpure states in our context. (Recall that if $S \subset \mathcal{S}(K)$, then the symbol $\overline{conv S}$ denotes the topological closure in $\mathcal{S}(K)$ of the convex hull, $conv S$, of S).

3.3. THEOREM. *Let K be a unital OMP. If $\mathcal{S}_p(K) \subset \overline{conv \mathcal{S}_{hp}}(K)$, then K is state-universal.*

Proof. Suppose that K is a subOMP of L and suppose further that L is unital. By the Krein-Millman theorem, $\overline{conv \mathcal{S}_p}(K) = \mathcal{S}(K)$

and so $\overline{conv}\mathcal{S}_{hp}(K) = \mathcal{S}(K)$. This implies that if every hyperpure state in K admits an extension over L, then so does every state of K. Indeed, if $s = \sum_{i \leq n} \alpha_i s_i$, where every s_i admits an extension over $\mathcal{L}$, then so does s. Moreover, suppose that s_α $(\alpha \in I)$ is a net in $\mathcal{S}(K)$ that converges to a $t \in \mathcal{S}(K)$. Suppose that $\tilde{s}_\alpha$ $(\alpha \in I)$ is a net in $\mathcal{S}(L)$ of extensions of s_α. Since $\mathcal{S}(L)$ is compact, we obtain that a subset of $\tilde{s}_\alpha$ converges to a state $\tilde{t} \in \mathcal{S}(L)$. Since the convergence in $\mathcal{S}(K)$ is pointwise, we infer that $\tilde{t}|K = t$ which we wanted to check. What remains to show is that every hyperpure state of K admits an extension over L. To do this, let $s \in \mathcal{S}_{hp}(K)$. Then there is an element $a \in K$ such that $s(a) = 1$ and, moreover, s is the only state of $\mathcal{S}(K)$ with the latter property. Since L is unital, there is a state $t \in \mathcal{S}(L)$ such that $t(a) = 1$. Put $u = t|K$. Since u is a state on K with $u(a) = 1$, we see that $u = s$. Thus, $s = t|K$ and this completes the proof of Th. 3.3.

3.4. THEOREM. [7]. *Let $K = L(H)$, where H is a complex Hilbert space and dim $H \neq 2$. Then K is state-universal.*

Proof. Let us first prove the following auxiliary result.

3.5 LEMMA. *Let H be a complex Hilbert space and let x be a unit vector of H. Let s_x denote the state on $L(H)$ determined by the formula $s_x(P) = \|P(x)\|^2$ where $P \in L(H)$. Then $s_x \in \mathcal{S}_{hp}(L(H))$.*

Proof. Let P_x denote the orthogonal projection onto the linear span of x in H and let s be a state in $\mathcal{S}(L(H))$ such that $s(P_x) = 1$. By the theorem of Aarnes (see [1]), we can write $s = s_1 + s_2$, where s_1 is a completely additive measure on $L(H)$ and s_2 is a measure on $L(H)$ which vanishes on every finite-dimensional projection. We see therefore that $s_2(P_x^\perp) \leq s(P_x^\perp) = 0$ and, also, $s_2(P_x) = 0$. This means that $s_2(I) = s_2(P_x \cup X^\perp) = s_2(P_x) + s_2(P_x^\perp) = 0$ (I denotes the identity operator). This implies that $s_2(P) = 0$ for any $P \in L(H)$. We infer that $s = s_1$ and therefore s is completely additive. By a generalized version of Gleason's theorem ([4] and [5]), we see that $s = \sum_{n=1}^{\infty} \alpha_n s_n$, where the states s_n's $(n \in N)$ live on mutually orthogonal one-dimensional projections of H. Since $s(P_x) = 1$ and since $\sum_{n=1}^{\infty} \alpha_n = 1$, we conclude that for all but one $n \in N$ we have $\alpha_n = 0$. There is a single $n_0 \in N$ such that $\alpha_{n_0} = 1$ and it follows that $s_{n_0} = s_x$. This completes the proof of Lemma 3.5.

Let us return to the proof of Th. 3.4. We shall make use of the results of [4] again: The states on $L(H)$ are in a "true" one-to-one correspondence with the "functional" states on the von Neumann algebra $\mathcal{B}(H)$ of all bounded linear operators on H. (Here we have used the word true to indicate that the correspondence preserves the natural algebraic and topological properties.) In particular, the pure states on $L(H), \mathcal{S}_p(L(H))$, are in a one-to-one correspondence with pure states on $\mathcal{B}(H), \mathcal{S}_p(\mathcal{B}(H))$, and the hyperpure state on $L(H), \mathcal{S}_{hp}(L(H))$, are in a one-to-one correspondence with the functional hyperpure states on $\mathcal{B}(H)$, $\mathcal{S}_{hp}(\mathcal{B}(H))$. Since $A \in \mathcal{B}(H)$ is nonnegative if and only if $s_x(A) \geq 0$ for all $x \in H$ ($\|x\| = 1$), we have $\mathcal{S}(\mathcal{B}(H)) = \overline{conv}\{s_x | x \in H, \|x\| = 1\} = \overline{conv}\mathcal{S}_{hp}(\mathcal{B}(H))$ (see e. g. [10, Th. 4.3.9, p. 262]). It follows that $\mathcal{S}_p(L(H)) \subset \overline{conv}\mathcal{S}_{hp}(L(H))$ and the proof is finished by applying Th. 3.3. This completes the proof of Th. 3.4.

It should be noted that the latter theorem has been further generalized in [6] for the von Neumann algebra projection OMPs.

Let us now take up another basic case of OMPs — the case of K being a Boolean OMP. Obviously, if K is atomistic, then K is state-universal by Th. 3.3 (Indeed, if K is Boolean, then $\mathcal{S}_p(K)$ consists of two-valued states (see e. g. [13]). If K is atomistic, we immediately obtain that $\mathcal{S}_p(K) = \overline{\mathcal{S}_{hp}(K)}$). However, the following fully general result is here in force.

3.6. Theorem. [12]. *Let K be a Boolean OMP. Then K is state-universal.*

Proof. Suppose that K is Boolean and suppose further that K is a subOMP of a unital OMP L. Put $\mathcal{S} = \{s \in \mathcal{S}(K) | s$ admits an extension over $L\}$. We are going to show that $\mathcal{S} = \mathcal{S}(K)$. We will do that by proving that $\mathcal{S}$ is both closed and dense in $\mathcal{S}(K)$.

Obviously, $\mathcal{S}$ is closed in $\mathcal{S}(K)$. Indeed, if $s_\alpha \in \mathcal{S}(K)$ is a net in $\mathcal{S}$ that converges to s and if $t_\alpha \in \mathcal{S}(L)$ is an extension of s_α, then the compactness of $\mathcal{S}(L)$ ensures that a subnet of t_α converges to a state. If $t \in \mathcal{S}(L)$ is this state, it is evident that t extends s and therefore $\mathcal{S}$ is closed in $\mathcal{S}(K)$.

To show that $\mathcal{S}$ is dense in $\mathcal{S}(K)$, let us assume that $s \in \mathcal{S}(K)$. Let $\mathcal{O}^{\varepsilon}_{a_1,a_2,\dots,a_n}$ be a standard neighbourhood of s in $\mathcal{S}(K)$ (here $\varepsilon > 0$ and $a_1, a_2, \dots, a_n \in K$). We shall prove that there is a state $\tilde{s}$ in $\mathcal{O}^{\varepsilon}_{a_1,a_2,\dots,a_n}$ which belongs to $\mathcal{S}$. As one obtains easily from the definition of a subOMP, the set $\{a_1, a_2, \dots, a_n\}$ generates a finite subOMP in L that is Boolean. Let us denote the latter finite Boolean subOMP of L by B and let $\{b_1, b_2, \dots, b_n\}$ be its atoms. Since L is unital, there are states $t_i \in \mathcal{S}(L)\,(i \leq m)$ such that $t_i(b_i) = 1$. Obviously, $\sum_{i\leq m} s(b_i) = 1$ and therefore $t = \sum_{i\leq m} s(b_i) \cdot t_i$ is a state on L. Moreover, a simple computation yields that $t(a_i) = s(a_i)$ for any $i\,(i \leq m)$. Thus, t restricted to K is a state on K which is "near to s" within $\mathcal{O}^{\varepsilon}_{a_1,a_2,\dots,a_n}$ (whatever ε may be!). By the construction, $t \in \mathcal{S}$. We have thus shown that $\mathcal{S}$ is dense in $\mathcal{S}(K)$ and this completes the proof of Th. 3.6.

Let us note in concluding this section that an analogous problem can be pursued for vector-valued states, too (see [2], [6], [8], [9], etc.).

References

[1] Aarnes J. F., *Quasi-states on $C^\star$-algebras*, Trans. Amer. Math. Soc., Vol. 149, (1970), 601-625.

[2] Avallone A. and Hamhalter J., *Extension theorems (vector measures on quantum logics)*, to appear.

[3] Bunce L., Navara M., Ptak P. and Wright J. D. M., *Quantum logics with Jauch–Piron states*, Quart. Journ. Math. Oxford **36** (1985), 261-271.

[4] Bunce L.J. and Wright J. D. M., *The Mackey-Gleason problem*, Bull. Amer. Math. Soc., Vol. 26 (2), (1992), 288-293.

[5] Gleason A., *Measures on closed subspaces of a Hilbert space.* J. Math. Mech. **6** (1957), 428–442.

[6] Hamhalter J., *The Gleason property and extensions of states*, to appear in J. London Math. Soc (1993).

[7] Hamhalter J., Navara M. andPták, P., *States on orthoalgebras*, to appear.

[8] Hamhalter J. and Pták P., *Hilbert-space-valued measures on Boolean algebras (extensions)*, Acta Math. Univ. Comen. LX, **2** (1991), 1-6.

[9] Hamhalter J. and Pták P., *Hilbert-space-valued states on quantum logics*, Applications Math. **37** (1992), 51-61.

[10] Kadison R. V. and Ringrose J. R., *Fundamentals of the theory of operator algebras*, Vol. I, Academic Press, Inc., (1986).

[11] Navara M. and Pták P., *Two-valued measures on σ-classes.* Čas. Pěst. Mat. **108** (1983), 225-229.

[12] Pták P., *Extensions of states on logics*, Bull. Acad. Polon. Sci. Ser. Math. 33, (1985), 493-494.

[13] Pták P. and Pulmannová S., *Orthomodular Structures as Quantum Logics.* Kluwer Academic Publishers, Dordrecht/Boston/London, 1991.

4. Jauch–Piron property in noncommutative measure theory.

4.0.1. Definition. A mapping $s: L \to \langle 0, 1\rangle$ is called a **state** on L if it fulfils the following two conditions:

(i) $s(1) = 1$,

(ii) if $a \leq b'$, then $s(a \vee b) = s(a) + s(b)$.

Further, a state $s: L \to \langle 0, 1\rangle$ is called Jauch–Piron (abbr. a J.–P. state) if the following condition holds true: If $s(a) = s(b) = 1$ for $a, b \in L$, then there is an element $c \in L$ such that $c \leq a, c \leq b$ and $s(c) = 1$. If every state on L is Jauch–Piron, we call L **a Jauch–Piron OMP** (abbr. a J.–P. OMP).

Unlike the "ordinary" commutative (= Boolean) case, an OMP may not possess any state at all (see [10]), or — provided it does — none (or most) of them may not be Jauch–Piron (see e. g. [21]). Note that the Jauch–Piron condition may be thought of probabilisticly — one requires that the pairs of "almost sure" events in a given state admit a subordinated "almost sure" event. Technically, the presence of the Jauch–Piron property may often move us nearer to the "classical" (=Boolean) mathematical areas.

As regards examples of J.–P. OMPs, such are Boolean algebras (we put $c = a \wedge b$) and the lattices $L(C_n)$ of all projections in an n-dimensional Hilbert spaces ($n \geq 3$). The former statement is obvious and the latter derives as a direct consequence of the famous Gleason's theorem (see [9]). Typically, an OMP possesses both the J.–P. and the non-J.–P. states.

4.0.2. EXAMPLE. Put $\Omega = \langle 0, 1\rangle^2$ and take for L the collection of all subsets of Ω whose Lebesgue measure is rational. Thus, $L = \{A \subset \Omega | \, \mu(A)$ is a rational + number$\}$. Then L is a (non-Boolean) OMP (we understand L endowed with the inclusion partial ordering and with the set-theoretic orthocomplementation operation ($A' = \Omega - A$). We claim that L is not a J.–P. OMP. Indeed, take a measurable subset of Ω, some B, with $\mu(B) > 0$ and define a state $s\colon L \to \langle 0, 1\rangle$ by putting $s(A) = \frac{s(A \cap B)}{s(B)}$ ($A \in L$). Then s is Jauch–Piron if and only if $\mu(B)$ is a rational number.

Let us now examine miscellaneous aspects of Jauch–Pironness.

4.1. "Discrete" Jauch–Piron OMPs.

Let us start with finite OMPs. Even in this class we have the "Greechie phenomenon" to be aware of - there are finite OMPs without any state at all (see [10]). Obviously, a stateless OMP is also Jauch–Piron by our definition but we are naturally more interested in OMPs whose states spaces are reasonably rich. Recall that an OMP L is said to be **unital** (see [12]) if the following condition is fulfilled: If $a \in L$ and if $a \neq 0$, then there is a state s on L such that $s(a) = 1$. We now have the following result.

4.1.1. THEOREM. [29]. *Let L be a finite unital Jauch–Piron OMP. Then L is Boolean.*

4.1.2. THEOREM. [28]. *Let L be a unital Jauch–Piron OMP. Let*

L contains only finitely many maximal Boolean subalgebras. Then L is Boolean.

4.1.3. THEOREM. [22]. *There is a unital countable Jauch–Piron OMP that is not Boolean. Moreover, the latter OMP can be required a subOMP of the projection OMP $L(C_3)$.*

To complete the schema here, it seems desirable to know if one can construct Greechie OMPs fulfilling the properties of Th. 4.1.3. (Let us call an OMP Greechie if it is atomic and every two maximal Boolean subalgebras in it meet in at most one atom.) This question seems to be open.

In the conclusion of this paragraph, let us note the following strengthening of the J.–P. condition (see [7], [18] and [27]). Let us say that a state s on L is strongly Jauch–Piron if for any couple $a, b \in L$ there is an element $c \in L$ such that $c \geq a$, $c \geq b$ and $s(c) \leq s(a) + s(b)$. Obviously, if s is strongly Jauch–Piron, then it is Jauch–Piron. The "vice versa" statement does not hold: Every lattice OMP that is unital with respect to strongly Jauch–Piron states has to be Boolean (see [27] and [30]).

4.2. The Jauch–Piron property in concrete OMPs.

An OMP is called **concrete** if it can be represented by a collection of subsets of a set. In other words, L is concrete if $L \subset \exp S$, where $\exp S$ is the collection of all subsets of a set S, and if the following conditions are satisfied:

(i) $\emptyset \in L$,

(ii) if $A \in L$ ($A \subset S$), then $S - A \in L$,

(iii) if $A, B \in L$ ($A, B \subset S$) and if $A \cap B = \emptyset$, then $A \cup B \in L$.

Thus, the concrete OMPs are in a sense "nearly Boolean". (It should be noted that such (or very similar) structures appeared already in the classics of the descriptive theory of sets and mathematical analysis many years ago (see e.g. [17]).) The conceptual value of concrete OMPs for quantum axiomatics seems first to be pointed out by S. Gudder (see e. g. [11] and [12]).

When does a concrete Jauch–Piron OMP have to be Boolean? The next result (one of the first results in this line) says that it is so quite often and that it is always "nearly" so.

4.2.1. THEOREM. [20]. *Let L be a concrete Jauch–Piron OMP. Then the following statements hold true:*

(i) If L is a lattice, then L is Boolean.

(ii) If L lives on an at most countable set (i.e., if $L = (S, L_S)$, where S is at most countable), then L is Boolean,

(iii) If $L \subset \exp S$ for a set S and if $A, B \in L$ $(A, B \subset S)$, then there is a finite collection $\{C_1, C_2, \dots, C_n\} \subset \exp S$ such that $C_i \in L$ for any $i (i \leq n)$ and $A \cap B = \bigcup_{i \leq n} C_i$.

Let us consider the condition (iii) for a moment to acquire better insight of the kind of the problems that are pursued here. The proof of the condition (iii) goes approximately as follows: If $A \cap B \neq \emptyset$, then there is a state, s, on L such that $s(A) = s(B) = 1$. Thus, the set $\mathcal{S}_{A,B} = \{t \text{ is a state on } L | t(A) = t(B) = 1\}$ is non-void. Moreover, $\mathcal{S}_{A,B}$ is compact in the pointwise topology. For any $C \in L$ with $C \subset A \cap B$, put $\mathcal{S}_C = \{t \in \mathcal{S}_{A,B} | s(C) > 0\}$. Since L is Jauch–Piron, we have $\mathcal{S}_{A,B} = \bigcup \mathcal{S}_C$, where C varies over all sets $C \in L$ such that $C \subset A \cap B$. Since every $\mathcal{S}_C$ is open in $\mathcal{S}_{A,B}$, we let the compactness of $\mathcal{S}_{A,B}$ work for us to get a finite family C_i $(i \leq n)$ with $\mathcal{S}_{A,B} = \bigcup_{i \leq n} \mathcal{S}_{C_i}$. Obviously, $\bigcup_{i \leq n} C_i = A \cap B$.

Let us return to the question which was asked prior the latter theorem. This question appeared to be fairly nontrivial. However,

4.2.2. THEOREM. [19]. *There is a concrete Jauch–Piron OMP that is* **not** *Boolean.*

In fact, the technique utilized in the latter result guarantees a proper class of Jauch–Piron OMPs that are not Boolean. The following question then announces itself immediately: Can every concrete OMP be embedded (in a compatibility preserving manner) into a concrete Jauch–Piron OMP? The answer to this question seems to be unknown for the time being.

4.2.3. Remark. In [5], the authors succeeded in solving the σ-complete version of the question posed above. They solved it in the affirmative under the set-theoretic assumption $\neg$RM of the nonexistence of real-measurable cardinals. (Can this set theoretic assumption be omitted?).

4.3. Jauch–Pironness in OMPs of projections.

The study of the J.–P. condition in the OMPs of projections in von Neumann algebras started with the paper [1] and was further deepened in [6] and [13]. Let us only state here two results which are directly related to the contents of this survey.

4.3.1. Theorem. [13]. *Let $\mathcal{A}$ be a von Neumann algebra. Then the OMP $\mathcal{P}(\mathcal{A})$ of all projections of $\mathcal{A}$ is Jauch–Piron if and only if $\mathcal{A}$ is a direct sum of a commutative von Neumann algebra and a finite dimensional von Neumann algebra.*

In the projection OMPs, an interesting line of investigations presents also the "individual" Jauch–Piron condition. For instance, the following elegant result is in force (the result has a direct interpretation in quantum foundations — see e.g. [4] and [8]):

4.3.2. Theorem. [6]. *Let $\mathcal{A}$ be a von Neumann algebra which does not contain a central Abelian part. Let s be a pure state on $\mathcal{P}(\mathcal{A})$. Then s is Jauch–Piron if and only if s is σ-additive.*

4.4. A link of Jauch–Piron property with topological representations of OMPs.

In an analogy with Boolean algebras, a natural project is to look for set or topological representations of OMPs. Obviously, the standard Stone representation technique cannot be adopted here — the set of two-valued states on an OMP may be too poor (see e.g. [10]). Several attempts have been made to obtain at least some weaker representations of OMPs (or orthomodular lattices) — see e.g. [3], [15], [24], [31], [32], [33] etc. An interesting topological representation was found in [31] and, to certain surprise, the Jauch–Piron condition is again involved (see [32] for a precise definition of all notions; see also [31] for relevant comments and open problems).

4.4.1. THEOREM. [31]. *Let L be an OMP. Then there is a 0-dimensional closure space, $(S, -)$, such that L can be order-orthoembedded in the orthomodular lattice of clopen subsets of S. Moreover, S can be taken a topological space if and only if L possesses a unital set of weakly additive Jauch–Piron states.*

4.5. Can the Jauch–Piron condition help in extending states?

We call an OMP K **state-universal** if the following implication holds true: Whenever K is embedded in L, where L is a unital OMP, then every state on K can be extended over L. We do not know whether every Jauch–Piron OMP is state universal (though in the "most natural" cases to be tested it is so — see [14] and [25]). It should be observed that, on the other hand, there are state universal OMPs which are not Jauch–Piron (e.g. $L(H)$ for dim $H = \infty$).

REFERENCES

[1] AMANN A., *Jauch–Piron states in W^*-algebraic quantum mechanics*, J. Math. Phys. **28** (1987), 2384–2389.
[2] BELTRANETTI E. and CASSINELLI G., *The logic of quantum mechanics*, Addison-Wesley, Reading, Massachusetts.
[3] BINDER J. and PTÁK P., *A representation of orthomodular lattices*, Acta Univ. Carolinae - Math. Phys. **31** (1), (1990), 21–26.
[4] BUGAJSKI S., BUSCH P., CASSINELLI G., LAHTI P. and QUADT P., *Sigma-convex structures and classical embeddings of quantum mechanical state spaces*, to appear.

[5] BUNCE L., NAVARA M,, PTÁK P. and WRIGHT J. D. M., *Quantum logics with Jauch–Piron states*, Quart. Journ. Math. Oxford **36** (1985), 261–271.

[6] BUNCE L. and HAMHALTER J., *Jauch–Piron states on von Neumann algebras*, to appear.

[7] DE LUCIA P. and PTÁK P., *Quantum probability spaces that are nearly classical*, Bull. Polish Acad. Sciences - Math., Vol. **40**, No. 2, (1992), 163–173.

[8] EMCH G., *Algebraic Method in Statistical Mechanics and Quantum Field Theory*, Wiley Interscience, London, (1972).

[9] GLEASON A.M., *Measures on the closed subspaces of a Hilbert space*, J. Math. Mech. **6** (1957), 885–893.

[10] GREECHIE R., *Orthomodular lattices admitting no states*, J. Comb. Theory **10A** (1971), 119–132.

[11] GUDDER S., *Quantum probability spaces*, Proc. Amer. Math. Soc. **21**, (1969), 296–302.

[12] GUDDER S., *Stochastic Methods of Quantum Mechanics*, North-Holland, Amsterdam, (1979).

[13] HAMHALTER J., *Pure Jauch–Piron states on von Neumann algebras*, Ann. Inst. Henri Poincaré, Vol. **58**, (1993), No. 1, 1–15.

[14] HAMHALTER J., *The Gleason property and extensions of states*, to appear in J. London Math. Soc. (1993).

[15] ITTURIOZ L., *A representation theory for orthomodular lattices by means of closure spaces*, Acta, Math. Hungar. **47**, (1986), 145–151.

[16] JAUCH J.M., *Foundations of Quantum Mechanics*, Addison-Wesley, Reading, Massachusetts, 1968.

[17] KURATOWSKI K., *Topology II*, London, New York, Academic Press 1966.

[18] MAJERNÍK V. and PULMANNOVÁ S., *Bell inequalities on quantum logics*, Journ. Math. Phys. **33** (6), (1992), 2173–2178.

[19] MÜLLER V., *Jauch–Piron states on concrete quantum logics*, to appear in Int. Journ. Theor. Phys. 1993.

[20] NAVARA M. and PTÁK P., *Almost Boolean orthomodular posets*, J. Pure Applied Algebra **60**, (1989), 105–111.

[21] NAVARA M. and ROGALEWICZV V., *The pasting constructions for orthomodular posets*, Math. Nachr. **154**, (1991), 157–168.

[22] OVTCHINIKOFF P., *On countable models of orthomodular lattices and Jauch–Piron property*, Proc. of the 10th Conf. on Probability "Probastat91", Bratislava (Czechoslovakia), 1991.

[23] PIRON C., *Foundations of Quantum Physics*, Benjamin, Reading, Massachusetts, (1976).

[24] PTÁK P., *Weak dispersion-free states and the hidden variables hypoth-*

esis, J. Math. Phys. **24** (1983), 839–840.
[25] Pták P., *Extensions of states on logics*, Bull. Polish Acad. Sciences - Math. (1985), 493–497.
[26] Pták P. and Pulmannová S., *Orthomodular Structures as Quantum Logics*, Kluwer, Dordrecht/Boston/London (1991).
[27] Pták P. and Pulmannová S., *A measure theoretic characterization of Boolean algebras among orthomodular lattices*, Comment. Math. Univ. Carolinae **35**,1 (1994), 205–208.
[28] Rogalewicz V., *Jauch–Piron logics with finiteness conditions*, Int. J. Theor. Phys. **30** (1991), 437–445.
[29] Rüttimann G., *Jauch–Piron states*, J. Math. Phys. **18** (1977), 189–193.
[30] Salvati S., *Una caratterizzazione delle algebre di Boole tramite i p-ideali*, Pubblicazione del Dipartimento di matematica e applicazioni "R. Caccioppoli", Universita' degli studi di Napoli "Federico II", No. 35 (1992).
[31] Tkadlec J., *Partially additive states on orthomodular posets*, Colloquium Math. LXII, (1991), 7–14.
[32] Tkadlec J., *Partially additive measures and set representations of orthoposets*, Journ. Pure Applied Algebra **86** (1993), 79–94.
[33] Zierler N. and Schlessinger M., *Boolean embeddings of orthomodular sets of quantum logics*, Duke Math. J. **32** (1965), 251–262.

5. Inner product spaces (algebraic and measure-theoretic conditions for completeness).

Let S be a real inner product space and let $\langle .,.\rangle$ stand for scalar product of S. Let us denote by $E(S)$ (resp. $F(S)$) the set of all (closed) subspaces of S which fulfil the following condition: $A \in E(S) \Leftrightarrow A \oplus A^{\perp} = S$ (resp. $A \in F(S) \Leftrightarrow (A^{\perp})^{\perp} = A$). The symbol $A^{\perp}$ denotes the set $\{b \in S | \langle a, b\rangle = 0 \text{ for any } a \in A\}$, and $\oplus$ denotes the direct sum. Let us now view $E(S)$ (the *splitting subspaces set*) and $F(S)$ (the *exact subspaces set*) with the ordering given by inclusion and with the orthocomplementation $A \to A^{\perp}$.

Proposition [12].

(i) $E(S)$ is an OMP (thus, $E(S)$ is always orthomodular).

(ii) $F(S)$ is a complete orthocomplemented poset ($F(S)$ does not have to be orthomodular).

Due to the distinguished position of Hilbert space OMPs in "noncommutative" investigations, a natural question occurs of how one can characterize the (topological) completeness of S in terms of $E(S)$ (resp. $F(S)$).

THEOREM [1]. *S is complete (= Hilbert) if and only if $F(S)$ is orthomodular.*

It would be nice if we had the following duality (we however have not been able to establish it for the time being).

CONJECTURE [10]. S is complete if and only if $E(S)$ is a lattice.

It may be noted that in [14] the authors showed (with the help of the method of [9]) that $E(c_{00}(N))$ is *not* a lattice ($c_{00}(N)$ is the subspace of $l^2(N)$ consisting of sequences that are 0 almost everywhere). Observe on passing that this result answers a question posed in [12].

If $E(S)$ satisfies a mild σ-completeness condition, then this is sometimes sufficient to bring about the completeness of S. The following result based on the Amemia-Araki procedure [1] came into existence as a generalization of [3] and [4]. (Recall that an OMP L is said to have the *atomic subsequential completeness property* (the ASCP) if for every sequence $\{a_i | i \in N\}$ of mutually orthogonal atoms in L there is an infinite set $M, M \subset N$ such that the supremum $\bigvee_{i \in M} a_i$ exists in L. See also [2] and [11] for more information on ASCP.)

THEOREM [10]. *If $E(S)$ has the ASCP, then S is complete.*

Let us now return to the lattice $F(S)$. The following result initiated a series of measure-theoretic characterizations of completeness of S (see e. g. [5], [7] and the survey in [5]).

THEOREM [8]. *If $F(S)$ possesses a σ-additive state, then S is complete. (Thus, S is complete if and only if $\mathcal{S}(F(S)) \neq \emptyset$.)*

PROBLEM [13]. Does S have to be complete if $F(S)$ possesses a finitely additive state?

References

[1] Amemia J. and Araki H., *A remark on Piron's paper*, Publ. Res. Inst. Math. Sci. Sect. A 1966/67, 12, 423–427.
[2] d'Andrea A.B. and de Lucia P., *The Brooks-Jewett theorem on an orthomodular lattice*, Journ. Math. Anal. Appl. **15** (1991), 507–522.
[3] Cattaneo G. and Marino G., *Completeness of inner product spaces with respect to splitting subspaces*, Letters Math. Phys. **11** (1986), 15–20.
[4] Dvurečenskij A., *Completeness of inner product spaces and quantum logic of splitting subspaces*, Letters on Math. Phys. **15** (1988), 231–235.
[5] Dvurečenskij A., *Gleason's Theorem and Its Applications*, Kluwer (Ister), 1993.
[6] Dvurečenskij A. and Mišík L. Jr., *Gleason's theorem and completeness of inner product spaces*, Int. Journ. Theor. Phys. **24** (4) (1988), 417–426.
[7] Dvurečenskij A. and Pulmannová S., *A signed measure completeness criterion*, Letters on Math. Phys. **17** (1989), 253–261.
[8] Hamhalter J. and Pták P., *A completeness criterion for inner product spaces*, Bull. London Math. Soc. **19** (1987), 259–263.
[9] Keller J. and Gross H., *On the definition of Hilbert space*, Manuscripta Math. **23** (1977), 67–90.
[10] de Lucia P. and Pták P., *A note on inner product spaces*, Acta Polytechnica — Mathematics, Proc. Czech Technical University of Prague, Vol. **34**, No. 2, (1994), 35–37.
[11] de Lucia P. and Treynor T., *Noncommutative group valued measures on an orthomodular poset*, to appear
[12] Piziak R., *Orthomodular Posets from Sesquilinear Forms*, Journ. Australian Math. Soc. **15** (1973), 265–269.
[13] Pták P., FAT — CAT (in the state space of quantum logics), Proceedings of "Winter School on Measure Theory", Liptovský Ján 1988, (Czechoslovakia).
[14] Pták P. and Weber H., *Notes on inner product spaces*, University of Potenza (Italy), March 1992.

6. Observables as OMP-valued measures.

Let M be a separable Banach space and let L be a σ-complete

OMP. An *observable* is a σ-additive mapping from the σ-algebra $\mathcal{B}(M)$ of Borel sets on M into L. This notion is a quantum version of random variable. Since observables are not, in general, "pointwise" mappings, the question that arises first is how (when) we can endow sets of observables with the structure of a Banach space. We show that certain sets of compatible observables can be indeed made a Banach space. (We also discuss other topical questions on observables.)

6.1. DEFINITION. A mapping $x: \mathcal{B}(M) \to L$ is called an observable (on M and L) if

(i) $x(M) = 1$,

(ii) $x(M - A) = x(A)'$ for any $A \in \mathcal{B}(M)$,

(iii) if $\{A_n | n \in N\}$ is a countable set of Borel sets then $x(\bigcup_{n \in N} A_n) = \bigvee_{n \in N} x(A_n)$.

6.2. DEFINITION. Let $\mathcal{P} = \{x_\alpha | \alpha \in I\}$ be a set of observables. Then we call $\mathcal{P}$ *compatible* if the set $\bigcup_{\alpha \in I} R(x_\alpha)$, where $R(x_\alpha)$ is the range of x_α, admits a Boolean subalgebra of L, some B, such that $\bigcup_{\alpha \in I} R(x_\alpha) \subset B$.

6.3. THEOREM. ("on simultaneous testability"). *There are Borel measurable mappings $f_n: M \to M (n \in N)$ such that the following statement holds true: If $x_n: \mathcal{B}(M) \to L$ is a sequence of compatible observables, then there is an observable, $z: \mathcal{B}(M) \to L$, such that $x_n = z \cdot f_n^{-1}$ for any $n \in N$.*

6.4. DEFINITION. The *spectrum* of an observable $x: \mathcal{B}(M) \to L$ is the least closed subset F of M such that $x(F) = 1$. If the spectrum of x is bounded then x is called bounded (in this case we set $\|x\| = sup\{|m| | m \in F, F$ is the spectrum of $x\}$.

6.5. THEOREM. *Let P be a set of bounded compatible observables. Then P can be extended to a set Q (of observables) such that Q may be converted to a Banach space (with the norm $x \to \|x\|$ defined above).*

Proof. See [9]; for relevant results, see also [1], [2], [4], [5], [10] and [12].

PROBLEMS.

1. **Finitely additive observables**. Give a coherent account of finitely additive observables.
2. (**Uniqueness problem**) Let us restrict ourselves to the *real observables* ($M = R$) and to the *rich OMPs* (an OMP is called rich if for any couple $a, b \in L$ with $a \not\leq b$ we can find a (σ-additive) state $s \in \mathcal{S}(L)$ such that $s(a) = 1$ and $s(b) \neq 1$). If $x : \mathcal{B}(R) \to L$ is an observable and $s \in \mathcal{S}(L)$, s σ-additive, then the composition $s_x = s \cdot x$ is a probability measure on $\mathcal{B}(R)$. Moreover, if x is bounded then the integral $s(x) = \int_R t s_x(dt)$ is finite (t denotes the identity function). The uniqueness problem now reads as follows: Is it true that $x = y$ if and only if $s(x) = s(y)$ for any $s \in \mathcal{S}(L)$? The problem was posed and given partial solutions in [3]. Another contributions were given in [13], [14] and [15]. Recently M.Navara ([8]) showed that for general OMPs the answer to the uniqueness problem is in the negative. Since the uniqueness property is quite important for quantum theories it seems desirable to find the extent of the class of OMPs for which the uniqueness problem answers positively.
3. **Integration on concrete OMPs.** Let $L = (\Omega, \Delta)$ be a concrete σ-complete OMP (thus, $\Delta \subset \exp \Omega$ and Δ forms an OMP when we regard Δ with the partial ordering given by the set-theoretic inclusion and the orthocomplementation given by the set-theoretic complementation). Let $x, y : \mathcal{B}(R) \to L$ be two observables. In this case, x, y are carried by point mappings, i.e. there exist measurable functions $f, g : \Omega \to R$ such that $x = f^{-1}$ and $y = g^{-1}$. Suppose that $f + g : \Omega \to R$ is again a measurable function. When does the equality $\int (f+g)\,\mathrm{d}s = \int f\,\mathrm{d}s + \int g\,\mathrm{d}s$ hold true for any $s \in \mathcal{S}(L)$? The paper [6] brings a large class of functions f, g for which the latter equality holds (an important breakthrough was [17], see also [9]). A characterization does not seem to be known. A similar "monotony" problem (does $f \leq g$ imply $\int f\,\mathrm{d}s \leq \int g\,\mathrm{d}s$?) has been fully solved (see [7] and [16]).

References

[1] Dvurečenskij A. and Pulmannová S., *On the sum of observables in a logic*, Math. Slovaca **30** (1980), 393–399.

[2] Gudder S., *Spectral methods for a generalized probability theory*, Trans. Amer. Math. Soc. **119** (1965), 428–442.

[3] Gudder S., *Uniqueness and existence properties of bounded observables*, Pacific Journ. Math. **19** (1966), 81–93.

[4] Gudder S., *Stochastic Methods in Quantum Mechanics*, North Holland, 1979.

[5] Gudder S. and Mullihin H., *Measure theoretic convergence of observables and operators*, Journ. Math. Phys. **14** (1973), 234–242.

[6] Navara M., *When is the integration on quantum probability space additive?* Real Analysis Exchange **14** (1988), 228–234.

[7] Navara M., *The integral on σ-classes is monotonic*, Reports Math. Phys. **20** (1984), 417–421.

[8] Navara M., *The uniqueness problem for observables*, to appear

[9] Navara M. and Pták P., *Two-valued measures on σ-classes*, Čas. pěst. mat. **108** (1983), 225–229.

[10] Pták P., *Spaces of observables*, Czechoslovak Math. Journ. **34** (1984), 252–261.

[11] Pták P., *An observation on observables*, Acta Polytechnica, Czech Technical University **10**, (1987), 81–86.

[12] Pták P., *Realcompactness and the notion of observables*, Journ. London Math. Soc. **23** (1981), 534–536.

[13] Pták P. and Pulmannová S., *Orthomodular Structures as Quantum Logics*, Kluwer, Dordrecht, 1991.

[14] Pták P. and Rogalewicz V., *Regularly full logics and the uniqueness problem for observables*, Ann. Inst. Henri Poincaré A38, (1983), 69–74.

[15] Rogalewicz V., *On the uniqueness problem for observables for quite full logics*, Ann. Inst. Henri Poincaré A41, (1984), 445–451.

[16] Šipoš J., *The integral on quantum probability spaces is monotonic*, Reports Math. Phys. **21** (1985), 65–68.

[17] Zerbe J. and Gudder S., *Additivity of integrals on generalized measure spaces*, Journ. Comb. Theory **39A** (1985), 42–51.