A FEW REMARKS CONCERNING THE STRONG LAW OF LARGE NUMBERS FOR NON-SEPARABLE BANACH SPACE VALUED FUNCTIONS (*)

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1. Introduction

Throughout (Ω, Σ, μ) denotes a complete probability space, $\mathcal{M}(\mu)$ is the set of all μ-measurable real-valued functions (functions that are μ-equivalent are not identified) and X is a Banach space. λ always denotes the Lebesgue measure on the real line $\mathbb{R}$ or on an interval and Σ_μ^+ denotes the family of all elements of Σ which are of positive μ-measure. μ_k is the direct product of k copies of μ. μ^* is the outer measure induced by μ. The set of natural numbers is denoted by $\mathbb{N}$. $\mathcal{L}_\infty(\Omega, \Sigma, \mu)$ is the Banach space of all bounded real-valued measurable functions defined on (Ω, Σ, μ) (functions that are μ-equivalent are not identified) endowed with the supremum norm and $B_\infty(\mu)$ is the closed unit ball in $\mathcal{L}_\infty(\Omega, \Sigma, \mu)$. Similarly the space $\mathcal{L}_\infty(\Omega, \Sigma)$ is defined if no measure on (Ω, Σ) is taken into account. $\mathcal{B}$ is the algebra of Borel subsets of $\mathbb{R}$.

The study of laws of large numbers is an important part of probability. The theory of such laws for strongly measurable Banach space valued functions is well known (cf [PT]). It is the aim of these lectures to present a few facts concerning the strong law of large numbers that have been discovered during last few years by Talagrand [T] and Hoffmann-Jørgensen [HJ]. We consider mainly functions that take their values in a non-separable Banach space. The results show, that inside the classical probability theory, the true

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non-separable Pettis integral can be found.

2. Stable Sets.

We begin our considerations with the following well known result, that will be applied several times.

LEMMA 2.1 *Let* $\Omega_n \subseteq \Omega$ *be such that* $\mu^*(\Omega_n) = 1$ *for all* $n \geq 1$. *Then*

$$(\mu_\infty)^*(\prod_{n=1}^{\infty} \Omega_n) = 1 \ .$$

A similar equality holds for a finite product too.

Proof. Let $\Sigma_n = \{E \cap \Omega_n : E \in \Sigma\}$ and $\nu_n(E) = \mu^*(E)$ for $E \in \Sigma_n$. Then Σ_n is a σ-algebra on Ω_n and ν_n is a probability measure on (Ω_n, Σ_n). Let $(\Omega_\infty, \Sigma_\infty, \mu_\infty)$ be the direct product of the spaces $(\Omega_n, \Sigma_n, \nu_n)$, $n \in \mathbb{N}$. Notice then, that for all $E_1, E_2, \ldots \in \Sigma$ we have

$$\mu_\infty(\prod_{n=1}^{\infty} E_n \cap \Omega_n) = \prod_{n=1}^{\infty} \nu_n(E_n \cap \Omega_n) = \prod_{n=1}^{\infty} \mu(E_n) = \mu_\infty(\prod_{n=1}^{\infty} E_n)$$

and so

$$\mu_\infty(E \cap \Omega_\infty) = \mu_\infty(E)$$

for all $E \in \Sigma_\infty$.

In particular, if $E \supseteq \Omega_\infty$ and $E \in \Sigma_\infty$ then $\mu_\infty(E) = 1$.

This proves the required equality. ◊

LEMMA 2.2. *If* $f : \Omega \to \mathbb{R}$ *is non-measurable then there exist numbers* $\alpha < \beta$ *and* $A \in \Sigma_\mu^+$ *such that*

$$\mu^*(A \cap \{f < \alpha\}) = \mu^*(A \cap \{f > \beta\}) = \mu(A) \ .$$

Proof. Choose $\gamma \in \mathbb{R}$ such that $\{f \leq \gamma\} \notin \Sigma$. Let E be a μ-measurable cover of $\{f \leq \gamma\}$. Notice that $\mu^*(E \cap \{f > \gamma\}) > 0$

(otherwise $\{f \leq \gamma\} = E - E \cap \{f > \gamma\} \in \Sigma$). Hence,

$$\mu^*(E \cap \{f > \gamma\}) = \mu^*[E \cap (\bigcup_{n=1}^{\infty} \{f > \gamma + 1/n\})] > 0 \ .$$

In particular there exists $n \in \mathbb{N}$ such that for $\beta = \gamma + 1/n$ we have

$$\mu^*(E \cap \{f > \beta\}) > 0 \ .$$

Let now F be a measurable cover of the set $E \cap \{f > \beta\}$ and $A = E \cap F$. Since $A \supseteq E \cap \{f > \beta\}$ we have $\mu(A) > 0$. If $\gamma < \alpha < \beta$ then

$$\mu^*(A \cap \{f \geq \alpha\}) \leq \mu^*(A \cap \{f > \gamma\}) \leq \mu^*(E \cap \{f > \gamma\}) = 0$$

and

$$\mu^*(A \cap \{f \leq \beta\}) \leq \mu^*(F \backslash E \cap \{f > \beta\}) = 0 \ .$$

Thus

$$\mu^*(A \cap \{f < \alpha\}) = \mu^*(A \cap \{f > \beta\}) = \mu(A) \ . \qquad \diamondsuit$$

Suppose now that a set $\mathcal{H}$ is pointwise relatively compact as a subset of $\mathbb{R}^\Omega$ but has a non-measurable pointwise cluster point h. Hence there are numbers $\alpha < \beta$, and $A \in \Sigma_\mu^+$ such that the sets

$$U = A \cap \{h < \alpha\} \text{ and } V = A \cap \{h > \beta\}$$

satisfy the equalities $\mu^*(U) = \mu^*(V) = \mu(A)$.

The definition of pointwise convergence shows that for every $k, l \in \mathbb{N}$ and arbitrary $s_1, s_2, \ldots, s_k \in U$, $t_1, t_2, \ldots, t_l \in V$ there exists $f \in \mathcal{H}$ with

$$f(s_i) < \alpha \text{ and } f(t_j) > \beta; \ i \leq k, \ j \leq l \ .$$

So

$$\forall k, l \in \mathbb{N} \quad U^k \times V^l \subseteq \bigcup_{f \in \mathcal{H}} \{f < \alpha\}^k \times \{f > \beta\}^l \ .$$

Hence

$$\forall k, l \in \mathbb{N} \quad U^k \times V^l \subseteq \bigcup_{f \in \mathcal{H}} \{f < \alpha\}^k \times \{f > \beta\}^l \cap A^{k+l} \ .$$

and so

$$\forall k,l \in \mathbb{N} \quad \mu^*_{k+l}\left(\bigcup_{f\in\mathcal{H}} \{f<\alpha\}^k \times \{f>\beta\}^l \cap A^{k+l}\right) = (\mu(A))^{k+l}\ .$$

DEFINITION. Let $\mathcal{H}$ be an arbitrary collection of real valued functions defined on Ω. A set $A \in \Sigma^+_\mu$ for which there exist numbers $\alpha < \beta$ such that

$$\forall k,l \in \mathbb{N}\mu^*_{k+l}\left(\bigcup_{f\in\mathcal{H}} \{f<\alpha\}^k \times \{f>\beta\}^l \cap A^{k+l}\right) < (\mu(A))^{k+l}$$

is called a *critical set* for $\mathcal{H}$. A pointwise bounded set $\mathcal{H}$ is called μ-stable if there exists no critical set for $\mathcal{H}$. In other words $\mathcal{H}$ is μ-stable if for all $A \in \Sigma^+_\mu$ and all $\alpha < \beta$ there exist $k,l \in \mathbb{N}$ such that

$$\mu^*_{k+l}\left(\bigcup_{f\in\mathcal{H}} \{f<\alpha\}^k \times \{f>\beta\}^l \cap A^{k+l}\right) < \mu(A))^{k+l}\ .$$

It can be easily seen that in the above definition of stability one may assume $k = l$.

REMARK 2.3. It is obvious that each subset of a stable set $\mathcal{H}$ is itself stable. In particular single functions being elements of $\mathcal{H}$ are stable. It follows from Lemma 2.2 that they are measurable. Thus a stable set is always a subset of $M(\mu)$. It is also worth to notice that if A is critical then all its subsets of positive measure are critical too.

PROPOSITION 2.4. *If $\mathcal{H}$ is stable then it is pointwise relatively compact in $M(\mu)$. Moreover, its pointwise closure is also stable.*

EXAMPLE *of a pointwise compact collection of measurable functions that is not stable.*

Let $\prec$ be a well ordering of [0,1] and let

$$\mathcal{H} = \{\chi_A \leq \text{ and } \preceq \text{ coincide on } A\}\ .$$

Then $\mathcal{H}$ is pointwise compact in $M(\lambda)$. Moreover, since each uncountable subset of [0,1] contains a decreasing (in the sense of ordinary order) sequence, each element of $\mathcal{H}$ is zero outside a countable set. It can be shown that [0,1] is a critical λ-set.

Perhaps more interesting is the following example:

EXAMPLE *of a sequence of measurable functions that is convergent in measure but is not stable.*

For each $n \in \mathbb{N}$ let π_n be a partition of [0,1] into 2^{2n} closed intervals of equal length. Let

$\mathcal{F} = \{F \subset [0,1] : \exists\ \ n \in \mathbb{N}$ such that F is a union of 2^n elements of $\pi_n\}$.

If $\mathcal{F} = (F_n)_{n=1}^{\infty}$ then for each finite $H \subset [0,1]$ there is a sequence (n_k) such that $\chi_{F_{n_k}} \to \chi_H$ pointwise. Hence $\{\chi_F : F \in \mathcal{F}\}$ is pointwise dense in $\{0,1\}^c$ (c is the cardinality of the continuum) and so it is not λ-stable. On the other hand it is clear that $\chi_{F_n} \to 0$ in λ-measure.

It can be shown that if the continuum is real measurable and $\tilde{\lambda}$ is a universal countably additive extension of λ then all pointwise cluster points of $\mathcal{F}$ are measurable but $\mathcal{F}$ is not $\tilde{\lambda}$-stable.

Fortunately almost everywhere convergence behaves much better.

PROPOSITION 2.5. *If (f_n) is a sequence of μ-measurable functions that is μ-a.e. convergent to a μ-a.e. finite function f, then the family $(f_n : n \in \mathbb{N})$ is μ-stable.*

Proof. Suppose there is $A \in \Sigma_\mu^+$ and $\alpha < \beta$ such that for each $k, l \in \mathbb{N}$

$$\mu_{k+l}\left(\bigcup_{n=1}^{\infty} \{f_n < \alpha\}^k \times \{f_n > \beta\}^l \cap A^{k+l}\right) = \mu(A)^{k+l}$$

and take $0 < \varepsilon < \min\left(\frac{1}{4}(\beta - \alpha),\ \mu(A)\right)$.

According to the Jegoroff theorem we can find $B \in \Sigma_\mu^+$ and $m \in \mathbb{N}$ such that $\mu(B) < \varepsilon$ and $|f_n(\omega) - f(\omega)| < \varepsilon$ for all $\omega \notin B$ and $n > m$.

Let $C = A \backslash B$. Notice that C is critical for (f_n), with the same numbers α and β.

On the other hand, if $\alpha' = \alpha + \varepsilon$ and $\beta' = \beta - \varepsilon$ then we have for all k, l

$$\bigcup_{n=1}^{\infty} \{f_n < \alpha\}^k \times \{f_n > \beta\}^l \cap C^{k+l} \subseteq$$

$$\subseteq C^{k+l} \cap \left[\left(\bigcup_{n=1}^{m} \{f_n < \alpha'\}^k \times \{f_n > \beta'\}^l \right) \cup \left(\{f < \alpha'\}^k \times \{f > \beta'\}^l \right) \right]$$

because $|f_n(\omega) - f(\omega)| < \varepsilon$ for all $\omega \in C$ and $n > m$.

Then notice that for each n at least one of the sets $\{f_n < \alpha'\}$ and $\{f_n > \beta'\}$ is of measure $\leq 1/2$. The same holds for the sets $\{f < \alpha'\}$ and $\{f > \beta'\}$. As a result we get for $k = l$

$$\mu_{2k} \left\{ C^{2k} \cap \left[\left(\bigcup_{n=1}^{m} \{f_n < \alpha'\}^k \times \{f_n > \beta'\}^k \right) \cup \right. \right.$$

$$\left. \left. \cup \left(\{f < \alpha'\}^k \times \{f > \beta'\}^k \right) \right] \right\} <$$

$$< \frac{m+1}{2^k} \mu(C)^{2k} \; .$$

For sufficiently large k we get a contradiction with the critical property of the set C. This completes the proof. ◊

Let $A \in \Sigma_\mu^+$, u, v be two real-valued functions on Ω and $\mathcal{H}$ be a family of real-valued functions. Throughout the paper we will use the following notation:

$$B_{k,l}(\mathcal{H}, A, u, v) =$$

$$= \{(s_1, \ldots, s_k, t_l, \ldots t_l) \in A^{k+l}; \exists h \in \mathcal{H} \;\; \forall i \leq k \;\; h(s_i) < u(s_i),$$

$$\forall j \leq l \; h(t_j) > v(t_j)\} \; .$$

LEMMA 2.6. *Let $\mathcal{H}$ be a uniformly bounded family of measurable real-valued functions. Assume that $\mathcal{H}$ is not stable, and let $A \in \Sigma_\mu^*$ and numbers $\alpha < \beta$ be such that*

$$\mu_{2n}^*[B_{n,n}(\mathcal{H}, A, \alpha, \beta)] = \mu(A)^{2n}$$

for each $n \in \mathbb{N}$. Then there exists a function $g \in B_\infty(\mu)$ such that for each weak neighbourhood V of g in $L_2(\mu)$ the equality

$$\mu_{2n}^*[B_{n,n}(\mathcal{H} \cap V, A, \alpha, \beta)] = \mu(A)^{2n}$$

holds for all n.

Proof. Without loss of generality, we may assume that $\mathcal{H} \subseteq B_\infty(\mu)$. Suppose the theorem does not hold, that is

$$\forall g \in B_\infty(\mu)\ \exists V\ \exists n\ \mu_{2n}^*[B_{n,n}(\mathcal{H} \cap V, A, \alpha, \beta)] < \mu(A)^{2n} .$$

Since $B_\infty(\mu)$ is weakly compact in $L_2(\mu)$, we can find a finite cover $V_1, \ldots, V_k$ of $B_\infty(\mu)$ such that

$$\forall i \leq k\ \ \exists n_i\ \mu_{2n_i}^*\ [B_{n_i,n_i}(\mathcal{H} \cap V_i, A, \alpha, \beta)] < \mu(A)^{2n_i} .$$

Let $n = \max\{n_i : i \leq k\}$.

It follows that we have

$$\forall i \leq k\ \mu_{2n}^*[B_{n,n}(\mathcal{H} \cap V_i, A, \alpha, \beta)] < \mu(A)^{2n} .$$

Let $p \in \mathbb{N}$ be such that $[\mu(A)^{2n}]^p < \frac{1}{k}\mu(A)^{2n}$ and let $m = np$.

Then

$$\forall i \leq k \mu_{2m}^*[B_{m,m}(\mathcal{H} \cap V_i, A, \alpha, \beta)] < \mu(A)^{2m} .$$

Since

$$\mathcal{H} \subseteq \bigcup_{i \leq k} V_i \cap \mathcal{H}$$

we get

$$\mu_{2m}^*[B_{m,m}(\mathcal{H}, A, \alpha, \beta)] < k\mu(A)^{2m} < \mu(A)^{2n}$$

which gives a contradiction with the initial assumption about $\mathcal{H}$. $\diamondsuit$

LEMMA 2.7. *Let $\mathcal{H}$ be a uniformly bounded non-stable family of measurable functions. Let $A \in \Sigma_\mu^*$ and $\alpha < \beta$ be such that for each $n \in \mathbb{N}$*

$$\mu_{2n}^*[B_{n,n}(\mathcal{H}, A, \alpha, \beta)] = \mu(A)^{2n} .$$

Then there exist two measurable functions u, v with $\int v > \int u + (\beta - \alpha)\mu(A)/3$ such that for each n

$$\mu_{2n}^*[B_{n,n}(\mathcal{H}, \Omega, u, v)] = 1 .$$

Proof. As in the proof of the previous lemma, we assume $\mathcal{H} \subseteq B_\infty(\mu)$. Let $a = (\beta - \alpha)\mu(A)/3$. Moreover let

$$u = \begin{cases} h + a & \text{on } \Omega \backslash A \\ \alpha & \text{on } A \end{cases} \qquad v = \begin{cases} h - a & \text{on } \Omega \backslash A \\ \beta & \text{on } A \end{cases} .$$

We have $\int v \geq \int u + a$.

For two subsets I, J of $\{I, \ldots, n\}$ let

$$K_{I,J} =$$

$$= \{(s_1, \ldots, s_n, t_1, \ldots, t_n) \in \Omega^{2n} : s_i \in A \Leftrightarrow i \in I,\ t_j \in A \Leftrightarrow j \in J\}$$

and

$$\tilde{K}_{I,J} = \{(s_1, \ldots, s_n, t_1, \ldots t_n) \in K_{I,J} :$$
$$\exists h \in \mathcal{H} \ \ \forall i \leq n \ \ h(s_i) < u(s_i),\ h(t_i) > v(t_i)\} .$$

We shall prove, that $\mu_{2n}^*(\tilde{K}_{I,J}) = \mu_{2n}(K_{I,J})$.

To do it let us fix I and J. Moreover, take $C \subseteq K_{I,J}$ of positive μ_{2n}-measure. Assuming that *card* $I = k$, *card* $J = l$, let $\delta = \left(\frac{3a}{14}\right)^{2n-k-1}$.

Moreover, let $B_i, \ldots, B_{2n}$ be measurable sets of positive measure, with $B_i \subseteq A$ for $i \in I \cup J$ and $B_i \subseteq \Omega \backslash A$ whenever $i \notin I \cup J$ and such that

$$\mu_{2n}(C \cap \prod_{i \leq 2n} B_i) > (1 - \delta)\mu_{2n}(\prod_{i \leq 2n} B_i) .$$

Since the required equality is obvious if $k + l = 2n$, we assume that $k + l < 2n$, $k > 0$ and $l > 0$.

For $(s_1, \ldots, s_{k+l}) \in A^{k+l}$, let

$$c(s_1, \ldots, s_{k+l}) = \{(t_1, \ldots, t_{2n-k-l}) \in (\Omega \backslash A)^{2n-k-l} : (s_1, \ldots, s_{k+l},\ t_1, \ldots, t_{2n-k-l}) \in C \cap \prod_{i \leq 2n} B_i\} .$$

Then put

$$D = \{(s_1, \ldots, s_{k+l}) \in A^{k+l} : \mu_{2n-k-l}(C(s_1, \ldots, s_{k+l})) > (1-\delta)\mu_{2n-k-l}(\prod_{i > k+l} B_i)\} .$$

Clearly

$$\mu_{k+l}(D) > 0$$

Now let

$$T_{I,J} = \{(s_1, \ldots, s_{k+l}) \in A^{k+l} : \exists h \in \mathcal{H}\ \forall (i \leq k)\ h(s_i) < \alpha,\ \forall (k < i \leq k+l) h(s_i) > \beta\ \forall (k+l < i \leq 2n)\ \ |\int_{B_i} h d\mu - \int_{B_i} g d\mu| < \frac{a}{2}\mu(B_i)\} .$$

In the above formulae g is the function chosen in Lemma 2.6.

In virtue of Lemmata 2.6 and 2.1

$$\mu^*_{k+l}(T_{I,J}) = \mu(A)^{k+l} .$$

Hence

$$D \cap T_{I,J} \neq \emptyset .$$

In particular, there exist $(s_1, \ldots, s_{k+l}) \in D$ and $h \in \mathcal{H}$ such that

$$\forall (i \leq k) \quad h(s_i) < \alpha$$

$$\forall (k < i \leq k+l) \quad h(s_i) > \beta$$

$$\forall (k+l < i \leq 2n) |\int_{B_i} h d\mu - \int_{B_i} g d\mu| < \frac{a}{2}\mu(B_i) .$$

For $k+l < i \leq n+l$, let

$$D_i = B_i \cap \{h < g + a\} .$$

Similarly, let

$$D_i = B_i \cap \{h > g - a\}$$

whenever $n+l<i\le 2n$.

Since $-1\le h,\ g\le 1$ and $a\le 1/3$ we get in the case $k+l<i\le n+l$

$$\int_{B_i} h=\int_{B_i\setminus D_i} h+\int_{D_i} h\ge\int_{B_i\setminus D_i}(g+a)-\mu(D_i)=$$

$$=\int_{B_i} g-\int_{D_i} g+a\mu(B_i\setminus D_i)-\mu(D_i)\ge\int_{B_i} g+a\mu(B_i)-(2+a)\mu(D_i)\ge$$

$$\ge\int_{B_i} g+a\mu(B_i)-\frac{7}{3}\mu(D_i)\ .$$

Since

$$\int_{B_i} h\le\int_{B_i} g+\frac{1}{2}a\mu(B_i)$$

we have

$$a\mu(B_i)-\frac{7}{3}\mu(D_i)\le\frac{1}{2}a\mu(B_i)$$

and so

$$\mu(D_i)\ge\frac{3a}{14}\mu(B_i)\ .$$

In a similar way, using the inequalities $h\le 1$, $g\ge -1$ and $a\le 1/3$, we get the same inequality for $n+l<i\le 2n$. Thus

$$\mu(D_i)\ge\frac{3a}{14}\mu(B_i)$$

for every $i\in\{k+l+1,\ldots,2n\}$, and so

$$\mu_{2n-k-l}\left(\prod_{i=k+l+1}^{2n} D_i\right)>\delta\mu_{2n-k-l}\left(\prod_{i=k+l+1}^{2n} B_i\right)\ .$$

But

$$\mu_{2n-k-l}(C(s_1,\ldots,s_{k+l}))>(1-\delta)\mu_{2n-k-l}\left(\prod_{i=k+l+1}^{2n} B_i\right)$$

and so

$$C(s_1,\ldots,s_{k+l})\cap\prod_{i=k+l+1}^{2n} D_i\ne\emptyset\ .$$

This yields the existence of $(t_1, \ldots, t_{2n-k-l}) \in C(s_1, \ldots, s_{k+l})$ such that

$$f(t_i) < g(t_i) + a$$

for each $i \in \{k+l+1, \ldots, n+l\}$
and

$$f(t_i) > g(t_i) - a$$

for each $i \in \{n+l+1, \ldots, 2n\}$.

But $(s_1, \ldots, s_{k+l}, t_1, \ldots, t_{2n-k-l}) \in C$ and so we get $C \cap \tilde{K}_{I,J} \neq \emptyset$.

This proves the equality $\mu^*_{2n}(\tilde{K}_{I,J}) = \mu_{2n}(K_{I,J})$ for positive k and l satisfying the condition $k + l < 2n$.

Assume now that $k = l = 0$.

Applying Lemma 2.6 we get a function $h \in \mathcal{H}$ satisfying for each $i \leq 2n$ the inequality

$$\left| \int_{B_i} h d\mu - \int_{B_i} g d\mu \right| < \frac{a}{2}\mu(B_i) \ .$$

With the sets D_i defined in the same way as before we obtain the inequality

$$\mu_{2n}(\prod_{i=1}^{2n} D_i) > \delta\mu_{2n}(\prod_{i=1}^{2n} B_i) \ .$$

that yields

$$\prod_{i=1}^{2n} D_i \cap (\Omega \backslash A)^{2n} \neq \emptyset$$

proving again the required equality.

We leave to the reader to prove by the same method the remaining cases with only one of the numbers k, l equal zero.

The summation over all I, J gives $\mu^*_{2n}[B_{n,n}(\mathcal{H}, \Omega, u, v)] = 1$. $\diamondsuit$

3. The law of large numbers for Banach space valued functions.

Definition. We say that a function $f : \Omega \to X$ satisfies the law of large numbers if there exists $a_f \in X$ such that

$$\lim_{n\to\infty} \left\| a_f - \frac{1}{n}\sum_{j=1}^{n} f(\omega_j) \right\| = 0 \quad \text{for } \mu_\infty - \text{a.a.} \quad (\omega_j) \in \Omega^\infty \ .$$

We denote the linear space of all X-valued functions satisfying the law of large numbers on (Ω, Σ, μ) by $LLN(\mu, X)$.

LEMMA 3.1. *If f satisfies the law of large numbers, then* $\int_\Omega^* \|f\| d\mu < \infty$.

Proof. By the assumption

$$\lim_{n\to\infty} \left\| a_f - \frac{1}{n} \sum_{j=1}^{n+1} f(\omega_j) \right\| = 0 \quad \text{for } \mu_\infty - \text{a.a.} \;\; (\omega_n) \in \Omega^\infty \; .$$

But

$$\left\| a_f - \frac{1}{n} \sum_{j=1}^{n+1} f(\omega_j) \right\| = \frac{n}{n+1} \left\| \frac{1}{n} \sum_{j=1}^{n+1} f(\omega_j) - \frac{n+1}{n} a_f \right\|$$

and so

$$\lim_{n\to\infty} \left\| \frac{1}{n} \sum_{j=1}^{n+1} f(\omega_j) - \frac{n+1}{n} a_f \right\| = 0 \text{ for } \mu_\infty - \text{a.a.} \;\; (\omega_n) \in \Omega^\infty \; .$$

Hence

$$\lim_{n\to\infty} \left\| \frac{1}{n} f(\omega_{n+1}) - \frac{1}{n} a_f \right\| = 0 \text{ for } \mu_\infty - \text{a.a.} \;\; (\omega_n) \in \Omega^\infty$$

and further

$$\lim_{n\to\infty} \left\| \frac{1}{n} f(\omega_n) \right\| = 0 \quad \mu_\infty - \text{a.e.}$$

Let

$$\Omega_n = \{\omega \in \Omega : \|f(\omega)\| \geq n\}$$

and let W_n be a measurable cover of Ω_n.

Put

$$g = 1 + \sum_{n=1}^{\infty} \chi_{W_n} \; .$$

g is measurable and $\|f\| \leq g$. It is enough to show that g is μ-integrable. Observe that such a conclusion follows at once from the inequality $\sum_{n=1}^{\infty}\mu(W_n) < \infty$ so we shall prove it. Suppose it does not hold, i.e. $\sum_{n=1}^{\infty}\mu(W_n) = \infty$. Then there is an increasing sequence (k_n) such that

$$\mu_\infty\{\omega \in \Omega^\infty : \exists i \quad k_n < i \leq k_{n+1},\ \omega_i \in W_i\} \geq 1 - 2^{-n} \ .$$

Since $\bigcup_{i \leq n} W_i$ is a measurable cover of $\bigcup_{i \leq n}\Omega_i$ we get

$$\mu^*_\infty\{\omega \in \Omega^\infty : \exists i \quad k_n < i \leq k_{n+1},\ \omega_i \in \Omega_i\} \geq 1 - 2^{-n} \ .$$

Hence, setting

$$W = \{\omega \in \Omega^\infty : \forall n \exists i\ k_n < i \leq k_{n+1},\ \omega_i \in \Omega_i\}$$

we obtain $\mu^*_\infty(W) \geq \prod_{n=1}^{\infty}(1 - 2^{-n}) > 0$.

In particular $\limsup_n \|\frac{1}{n} f(\omega)_n\| \geq 1$ for each $\omega \in W$. This contradiction proves that g is integrable. $\diamondsuit$

It is well known that a real-valued measurable $f \in LLN(\mu, \mathbb{R})$ if and only if $f \in L_1(\mu)$ but in the case of a general real-valued function more can be said.

Lemma 3.2. *$LLN(\mu, \mathbb{R}) \subseteq L_1(\mu)$.*

Proof. Let f be a real-valued function satisfying the law of large numbers. As it has been shown in Lemma 3.1, there is $g \in L_1(\mu)$ such that $\|f(\omega)\| \leq g(\omega)$ for each $\omega \in \Omega$. So to prove the integrability of f it is enough to show that f is measurable.

Let f^* and f_* be μ-upper and μ-lower measurable envelopes of f and suppose that $f^* \neq f_*$ on a set of positive measure. Then take arbitrary measurable functions h_0 and h_1 satisfying the following conditions:

$$|h_0|,\ |h_1| \leq g + 1$$

$$f_*(\omega) = h_0(\omega) = h_1(\omega) = f^*(\omega) \quad \text{if } f_*(\omega) = f^*(\omega)$$

$$f_*(\omega) < h_0(\omega) < h_1(\omega) < f^*(\omega) \quad \text{if } f_*(\omega) < f^*(\omega) \ .$$

Then we have

$$\mu_*\{\omega : h_0(\omega) < f(\omega)\} \leq \mu_*\{\omega : f_*(\omega) < h_0(\omega) \leq f(\omega)\} = 0$$

$$\mu^*\{\omega : f(\omega) < h_1(\omega)\} \leq \mu^*\{\omega : f(\omega) \leq h_1(\omega) < f^*(\omega)\} = 0 \ .$$

Hence, if

$$A = \{\omega : f(\omega) \leq h_0(\omega)\} \text{ and } B = \{\omega : h_1(\omega) \leq f(\omega)\}$$

then

$$\mu^*(A) = \mu^*(B) = 1 \ .$$

Let $(n(k))$ be an increasing sequence of natural numbers with $\lim_k n(k)/n(k+1) = 0$. Then let

$$C_n = A \text{ for } n(2k+1) < n \leq n(2k+2)$$

and

$$C_n = B \text{ for } n(2k) < n \leq n(2k+1) \ .$$

If $C = \prod_{n=1}^{\infty} C_n$, then $\mu^*_\infty(C) = 1$ in view of Lemma 2.1. Since h_0, h_1 and g are integrable they satisfy the law of large numbers. Let

$$\tilde{C} = \{\omega \in C : \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} h_0(\omega_j) = \int h_0 d\mu; \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} h_1(\omega_j) =$$

$$= \int h_1 d\mu; \quad \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} g(\omega_j) = \int g d\mu\} \ .$$

Clearly $\mu^*_\infty(\tilde{C}) = 1$.

Take $\omega \in \tilde{C}$. We'll prove that in spite of the assumption the sequence

$$c_k = \frac{1}{n(k)}\sum_{j=1}^{n_k} f(\omega_j) \quad k = 1, 2, \ldots$$

is not convergent.

Suppose it is convergent to some c. We have

$$c_{2k+1} \geq \frac{n(2k)}{n(2k+1)} c_{2k} + \frac{1}{n(2k+1)} \sum_{n(2k)<i\leq n(2k+1)} h_1(\omega_i) \geq$$

$$\geq \frac{n(2k)}{n(2k+1)} c_{2k} + \frac{1}{n(2k+1)} \sum_{i\leq n(2k+1)} h_1(\omega_i) - \frac{1}{n(2k+1)} \sum_{i\leq n(2k)} [g(\omega_i)+1] \, .$$

This shows that $c \geq \int h_1 d\mu$. Similarly it can be shown that $c \leq \int h_0 d\mu$. This gives a contradiction and so f is measurable. $\diamondsuit$

Let us consider now the case of functions that take their values in a separable subset of a Banach space. We assume for the simplicity that X is separable.

THEOREM 3.3. *Let X be a separable Banach space and f be an X-valued function. Then f satisfies the law of large numbers if and only if f is Bochner integrable. In such a case $a_f = \int_\Omega f d\mu$.*

Proof. Assume that $f \in LLN(\mu, X)$ and observe that our assumption yields $x^* f \in LLN(\mu, \mathbb{R})$ for each functional x^* from X^*. It is a consequence of the two previous lemmata that f is scalarly measurable and pointwise bounded by an integrable function. Hence it is Bochner integrable.

Assume now the Bochner integrability of f. Without loss of generality, we may assume that $\int f = 0$. Moreover, let ε be a positive number and $h : X \to X$ be a simple function, measurable with respect to the norm Borel algebras of sets in X and satisfying the inequality

$$\int_X \|x - h(x)\| d\mu f^{-1}(x) < \varepsilon$$

and

$$\int_X h(x) d\mu f^{-1}(x) = 0 \ .$$

Let $g = h \circ f$. We have $\int g = 0$ and since the range of g is contained in the finite dimensional subspace of X spanned by $h(X)$, we may apply the finite dimensional strong law of large number to get the convergence

$$\frac{1}{n} \sum_{i=1}^{n} g(\omega_i) \to 0 \quad \mu_\infty\text{-a.e.}$$

If $\xi = \|f - g\|$, then again

$$\frac{1}{n} \sum_{i=1}^{n} \|(f - g)(\omega_i)\| \to \int_\Omega \|f - g\| d\mu \leq \varepsilon \quad \mu_\infty - \text{a.e.}$$

Hence

$$\limsup_{n\to\infty} \left\| \frac{1}{n} \sum_{i=1}^{n} f(\omega_i) \right\| \le$$

$$\le \limsup_{n\to\infty} \left\| \frac{1}{n} \sum_{i=1}^{n} g(\omega_i) \right\| + \limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \|(f-g)(\omega_i)\| \le \varepsilon$$

μ_∞-a.e. This proves the theorem. ◊

Consider now a sequence (ξ_i) of independent identically distributed real random variables defined on (Ω, Σ, μ). Let $F(t) = \mu(\xi_i \le t\}$ be their common distribution function, and let F_n be the empirical distribution function based on $\xi_1, \dots, \xi_n$, i.e.

$$F_n(t,\omega) = \frac{1}{n} \sum_{i=1}^{n} \chi_{\{\xi_i \le t\}}(\omega) \ .$$

According to the Glivenko-Cantelli Theorem, we have

$$\sup_t |F_n(t,\omega) - F(t)| \to 0 \quad \mu - \text{a.e.}$$

This result can be reformulated in the following way: Let

$$X_i(\omega, t) = \chi_{\{\xi_i \le t\}}(\omega)$$

and

$$X_i(\omega) = X_i(\omega, \cdot) \in \mathcal{L}_\infty(\mathbb{R}, \mathcal{B}) \ .$$

The Glivenko-Cantelli Theorem says now that

$$\frac{1}{n} \sum_{i=1}^{n} X_i \to F$$

μ-a.e. in the norm of $\mathcal{L}_\infty(\mathbb{R}, \mathcal{B})$.

This means that a strong law of large numbers holds for the sequence (X_n) of $\mathcal{L}_\infty(\mathbb{R}, \mathcal{B})$-valued functions, in spite of the non-measurability of X_n in the sense of Bochner (To see it one can take for ξ_i such random variables that for some $A \in \Sigma_\mu^+$ the sets $\xi_i(A \backslash N)$ are uncountable for each set N of measure zero. The functions X_i are essentially non-separably valued).

Still the G-C theorem can be reformulated in a different way: Define $f : \mathbb{R} \to \mathcal{L}_\infty(\mathbb{R}, \mathcal{B})$ by the formulae

$$f(r) = \chi_{(-\infty, r]} \ .$$

Then

$$\chi_{\{\xi_i \le t\}}(\omega) = \chi_{(-\infty, \xi_i(\omega)]}(t) = f(\xi_i(\omega))(t)$$

and so the transformed form of the Glivenko-Cantelli Theorem looks as follows:

$$\lim_{n\to\infty} \|F - \frac{1}{n}\sum_{i=1}^{n} f(\xi_i(\omega))\|_{\mathcal{L}_\infty(\mathbb{R},\mathcal{B})} = 0 \quad \mu\text{−a.e.}$$

or

$$\lim_{n\to\infty} \|F - \frac{1}{n}\sum_{i=1}^{n} f(t_i)\|_{\mathcal{L}_\infty(\mathbb{R},\mathcal{B})} = 0 \quad \text{for } \nu_\infty\text{−a.e.} \quad (t_i) \in \mathbb{R}^\infty$$

where ν is the distribution of ξ_i on $(\mathbb{R}, \mathcal{B})$. This means that $f \in LLN(\nu, X)$.

We shall prove now that f is Pettis integrable with respect to ν (in fact f is Pettis integrable with respect to an arbitrary finite measure defined on Borel subsets of the real line $\mathbb{R}$).

According to Theorem 8.2 of [M] it is enough to find a bounded sequence of simple functions $f_n : \mathbb{R} \to \mathcal{L}_\infty(\mathbb{R}, \mathcal{B})$ such that for each functional $\eta \in \mathcal{L}_\infty(\mathbb{R}, \mathcal{B})^*$ the sequence $(\langle \eta, f_n \rangle)$ is ν-a.e. convergent to $\langle \eta, f \rangle$. We leave to the reader the case of purely atomic ν and we assume that ν is non-atomic.

Let us notice first that for each $\eta \in \mathcal{L}_\infty(\mathbb{R}, \mathcal{B})^*$ the function $\langle \eta, f \rangle$ is of bounded variation and hence it is Borel measurable.

Denote now for each $n \in \mathbb{N}$ by π_n the partition of the interval $(-n, n]$ consisting of the intervals $((i-1)/2^n, i/2^n]$, $-n2^n + 1 < i \le n2^n$ and let

$$f_n(t) = \begin{cases} 0 & \text{if } t \le -n \\ \chi_{(-\infty, i/2^n]} & \text{if } t \in ((i-1)/2^n,\ i/2^n] \\ 1 & \text{if } n < t \end{cases} \ .$$

Clearly $f_n : \mathbb{R} \to \mathcal{L}_\infty(\mathbb{R}, \mathcal{B})$ and $\|f_n\|_{\mathcal{L}_\infty}(\mathbb{R}, \mathcal{B}) \le 1$.

Each η can be identified with an additive real-valued set function of bounded variation defined on $\mathcal{B}$ (cf. [DS]). Hence

$$\langle \eta, f(t) \rangle = \eta((-\infty, t])$$

for each $t \in \mathbb{R}$, and

$$\langle \eta, f_n(t) \rangle = \eta((-\infty, i/2^n])$$

for each $t \in ((i-1)/2^n,\ i/2^n], \quad -n2+1 < i \leq n2^n$.

If $E_{t,n} \in \pi_n$ is that element which contains t, then we have

$$|\langle \eta, f_n(t) \rangle - \langle \eta, f(t) \rangle| \leq |\eta|(E_{t,n}) \ .$$

It follows from the boundedness of η that

$$\lim_n |\eta|(E_{t,n}) = 0$$

for all but countably many $t \in \mathbb{R}$.

Thus, $\lim_n \langle \eta, f_n \rangle = \langle \eta, f \rangle$ ν-a.e. and f is Pettis integrable with respect to ν.

In fact a more general result holds:

THEOREM 3.4. *If $f \in LLN(\mu, X)$, then f is μ-Pettis integrable.*

Proof. The equality $\lim_{n\to\infty} \|a_f - \frac{1}{n}\sum_{j=1}^n f(\omega_j) = 0$ for μ_∞-a.a. $(\omega_n) \in \Omega^\infty$ implies the relation

$$\lim_{n\to\infty} |x^* a_f - \frac{1}{n}\sum_{j=1}^n x^* f(\omega_j)| = 0 \text{ for } \mu_\infty\text{–a.a. } (\omega_n) \in \Omega^\infty \ .$$

Since moreover $\int_\Omega^* \|f\| d\mu < \infty$, we see that each function $x^* f$ is integrable. It is the consequence of the scalar law of large numbers that

$$\lim_{n\to\infty} |\int x^* f d\mu - \frac{1}{n}\sum_{j=1}^n x^* f(\omega_j)| = 0 \text{ for } \mu_\infty\text{–a.a. } (\omega_n) \in \Omega^\infty \ .$$

This gives the equality

$$\int_\Omega x^* f d\mu = x^* a_f \quad \text{for each} \quad x^* \ .$$

Applying similar consideration to an arbitrary set $E \in \Sigma$ we get the Pettis integrability of f. $\diamondsuit$

The Pettis integrability of f is however a too weak condition to guarantee $f \in LLN(\mu, X)$. f has to behave better. To formulate the main result we need yet some new notions.

DEFINITION. *A function $f : \Omega \to X$ is said to be properly measurable if the set $\{x^* f : \|x^*\| \leq 1\}$ is μ-stable.*

DEFINITION. *f is an X-valued function, then the Glivenko-Cantelli norm of f is given by*

$$\|f\|_{GC} = \lim_n \sup \int^* \sup \left\{ \frac{1}{n} \sum_{j=1}^n |x^* f(\omega_j)| : \|x^*\| \leq 1 \right\} d\mu_\infty(\omega) \ .$$

It is clear that for each x^* from the unit ball of X^*, we have

$$\frac{1}{n} \sum_{j=1}^n |x^*(\omega_j)| \leq \sup \left\{ \frac{1}{n} \sum_{j=1}^n |x^* f(\omega_j)| : \|x^*\| \leq 1 \right\}$$

and so

$$\int^* |x^* f| d\mu \leq \int^* \sup \left\{ \frac{1}{n} \sum_{j=1}^n |x^* f(\omega_j)| : \|x\|^* \leq 1 \right\} d\mu_\infty(\omega) \ .$$

In particular, if f is Pettis integrable then we get $\|f\|_P \leq \|f\|_{GC}$, where

$$\|f\|_P = \sup \left\{ \int |x^* f| d\mu : \|x^*\| \leq 1 \right\}$$

is the ordinary norm in the space of Pettis integrable functions.

For technical reasons we introduce yet for each real-valued function h the following notation:

$$Q_n(\omega)(h) = \frac{1}{n} \sum_{j=1}^n h(\omega_j)$$

for each $\omega = (\omega_j) \in \Omega^\infty$.

THEOREM 3.5. *For a function* $f : \Omega \to X$ *the following conditions are equivalent:*

(i) f *satisfies the law of large numbers;*

(ii) f *is properly measurable and* $\int_\Omega^* \|f\| d\mu < \infty$.

Proof (i) $\Rightarrow$ (ii). We have already proved that if $f \in LLN(\mu, X)$ then $\int_\Omega^* \|f\| d\mu < \infty$ and f is weakly measurable. We shall prove that f is properly measurable.

For the simplicity, we shall denote the set $\{x^* f : \|x^*\| \le 1\}$ by $\mathcal{H}$. We have to prove the stability of $\mathcal{H}$. If $\mathcal{H}$ is not stable, then there exist $A \in \Sigma_\mu^+$ and $\alpha < \beta$ with $\mu_{2n}^*[B_{n,n}(\mathcal{H}, A, \alpha, \beta)] = \mu(A)^{2n}$ for each n. Let $a = (\beta - \alpha)\mu(A)/9$ and $b > \max(|\alpha|, |\beta|)$ be such that $\int g' < a$, where $g' = g\chi_{\{g>b\}}$. For each $h \in \mathcal{H}$ denote by h' its truncation at $-b$ and b. If $\mathcal{H}' = \{h' : h \in \mathcal{H}\}$ then we also have $\mu_{2n}^*[B_{n,n}(\mathcal{H}', A, \alpha, \beta)] = \mu(A)^{2n}$ for all n. Applying Lemma 2.7 we get two bounded measurable functions u and v and v on Ω, with

$$\int v \ge \int u + 3a \quad \text{and} \quad \mu_{k+1}^* C(k,l) = 1 \ ,$$

for each k, l, where

$$C(k,l) = \{(s_1, \ldots, s_k,\ t_1, \ldots, t_l) \in \Omega^{k+l} :$$

$$\exists h \in \mathcal{H}\ \ \forall (i \le k)\ \ h'(s_i) < u(s_i),\ \forall (j \le l)\ \ h'(t_j) > v(t_j)\} \ .$$

We can assume $u \le g + 1$ and $v \ge -g - 1$.

Now let $(n(p))$ be a sequence with $\lim_p n(p)/n(p+1) = 0$ and let

$$C = \{\omega \in \Omega^\infty : \forall p(\omega_{n(2p)+1}, \ldots, \omega_{n(2p+2)})$$

$$\in C(n(2p+1) - n(2p), n(2p+2) - n(2p+1))\} \ .$$

It follows that $\mu_\infty^*(C) = 1$. Let

$$C' = \{\omega \in C : \lim_n Q_n(\omega)(g) = \int g;\ \lim_n Q_n(\omega)(g') = \int g';$$

$$\lim_n Q_n(\omega)(u) = \int u;\ \lim_n; Q_n(\omega)(v) = \int v\} \ .$$

It follows from the scalar law of large numbers that $\mu_\infty^*(C') = 1$.

Fix $\omega \in C'$. For each p let $h_p \in \mathcal{H}$ be such that

$$h_p'(\omega_i) < u(\omega_i) \text{ for } n(2p) < i \leq n(2p+1) \ ,$$

$$h_p'(\omega_i) > v(\omega_i) \text{ for } n(2p+1) < i \leq n(2p+2) \ .$$

We have

$$Q_{n(2p+1)}(\omega)(h_p) \leq \frac{1}{n(2p+1)} \sum_{i \leq n(2p+1)} u(\omega_i) +$$

$$+ \frac{2}{n(2p+1)} \sum_{i \leq n(2p)} (g(\omega_i) + 1) + \frac{1}{n(2p+1)} \sum_{i \leq n(2p+1)} g'(\omega_i)$$

$$Q_{n(2p+2)}(\omega)(h_p) \geq \frac{1}{n(2p+2)} \sum_{i \leq n(2p+2)} v(\omega_i) -$$

$$- \frac{2}{n(2p+2)} \sum_{i \leq n(2p+1)} (g(\omega_i) + 1) - \frac{1}{n(2p+2)} \sum_{i \leq n(2p+2)} g'(\omega_i)$$

and so

$$2 \lim_n \sup \|a_f - \frac{1}{n} \sum_{j=1}^{n} f(\omega_j)| \geq$$

$$\geq \lim_p \sup \Big(|Q_{n(2p+1)}(\omega)(h_p) - a_{h_p} - a_{h_p}| + |Q_{n(2p+2)}(\omega)(h_p) - a_{h_p}| \Big) \geq$$

$$\geq \lim_p \sup |Q_{n(2p+2)}(\omega)(h_p) - Q_{n(2p+1)}(\omega)(h_p)| \geq \int v - \int u - 2a > 0 \ .$$

This contradiction shows that $\mathcal{H}$ is stable. ◊

(ii) ⇒ (i). Assume that f is properly measurable and $\int^* \|f\| d\mu < \infty$. According to [T2][1], for each $k \in \mathbb{N}$ there exists a simple function

1 The proof of this fact is quite long and technically complicated so we decide to omit it hoping that somebody will give a shorter and simpler one.

$f_k : \Omega \to X$ with the property $\|f - f_k\|_{GC} \leq 2^{-k}$. Hence

$$\limsup_{n\to\infty} \|\frac{1}{n}\sum_{i=1}^{n}[f(\omega_i) - f_k(\omega_i)]\| \leq 2^{-k} \ \mu_\infty\text{−a.e.}$$

Since f_k takes only finitely many values, the finite dimensional law of large numbers yields

$$\limsup_{n\to\infty} \|\frac{1}{n}\sum_{i=1}^{n} f_k(\omega_i) - \int f_k\| = 0 \quad \mu_\infty - \text{a.e.}$$

and so

$$\limsup_{n\to\infty} \|\frac{1}{n}\sum_{i=1}^{n} f(\omega_i) - \int f_k\| \leq 2^{-k} \quad \mu_\infty - \text{a.e.}$$

Now it is easy to see that

$$\|\int f_k - \int f_{k+1}\| \leq 2^{-k+1}$$

and so the sequence $(\int f_k)$ is convergent in norm of X to an element a_f satisfying (i). ♢

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