STOCHASTIC PROCESSES IN BANACH SPACES. QUASIMARTINGALES AND STOCHASTIC INTEGRALS (*)

by JAMES K. BROOKS (in Gainesville)(**)

1. Introduction.

In this paper we shall examine stochastic processes taking their values in a Banach space. The structure of quasimartingales is studied from the point of view of regularity and the existence of a Doob-Meyer decomposition (section 4). The framework for the stochastic analysis is given in section 2; properties of quasimartingales and their associated Doléans function appear in section 3.

In order to develop the stochastic integral and its properties, a bilinear vector integration theory is sketched in section 5, along with a Lebesque space in which convergence theorems are given and a Dunford-Pettis criterion for weak compactness. This foundation is then used in section 6 to define $\int H dX$, the stochastic integral of a predictable process H taking values in a Banach space F relative to a summable cadlag process X with values in $L(F, G)$, the space of bounded operators from F into the Banach space G.

References are supplied for the reader interested in the interesting interplay between functional analysis and stochastic processes.

2. Definitions for the stochastic structure.

Throughout this paper we shall assume the following stochastic setting.

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(**) Indirizzo dell'Autore: Department of Mathematics, University of Florida, Gainesville, Florida 32611-8000 (U.S.A.).

$(\Omega, \mathcal{F}, P)$ is a probability space. The family $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ is a filtration satisfying the "usual conditions" ($[D-M]$). These conditions are the following. For each $t \in \mathbb{R}_+ = [0,\infty)$, $\mathcal{F}_t$ denotes a sub σ-algebra of subsets of $\mathcal{F}$; for $t_1 \leq t_2$, we have $\mathcal{F}_{t_1} \subset \mathcal{F}_{t_2}$. All the P-null sets belong to $\mathcal{F}_0$. Finally, we assume that the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ is right continuous, that is $\mathcal{F}_t = \bigcap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon} := \mathcal{F}_{t_+}$. Without loss of generality, we may assume that $\mathcal{F} = \vee_{t\geq 0} \mathcal{F}_t$.

The probabilistic view of a filtration is that $\mathcal{F}_t$ contains all the events up to (and including) time t.

E is a Banach space with dual space E'. Let $|\cdot|$ denote the norm of any given Banach space.

$X := (X_t)_{t\in\mathbb{R}_+}$ will always denote an E-valued adapted process, with $X_t \in L^1_E(P)$, for each $t \geq 0$. The term adapted means that X_t is $\mathcal{F}_t$-measurable. Thus the process X does "not look into the future." Each $X_t : \Omega \to E$ is a random vector which describes certain events which occur up to time t.

We shall always consider X to be extended to $t = \infty$ by defining $X_\infty \equiv 0$. If X has a limit at ∞, denote this limit by $X_{\infty-}$.

There are two main σ-algebras in stochastic analysis which consist of subsets of $\mathbb{R}_+ \times \Omega$. The first one is $\mathcal{O}$, the optional σ-algebra, which is generated by all real valued processes $(Y_t)_{t\geq 0}$ that are adapted and right continuous. The second σ-algebra $\mathcal{P}$ is the predictable σ-algebra, which is generated by all the real valued adapted processes (Z_t) that are left continuous on $(0,\infty)$. Note that $\mathcal{P} \subset \mathcal{O}$. The term predictable is a good one since the property $Z_t = Z_{t-}$, for $t > 0$, hints of a certain ability to see slightly into the future. We shall give some alternate definitions of these two fundamental σ-algebras which will shed some light on their structure.

The notion of a stopping time, which is due to Doob, is the cornerstone of the general theory of stochastic processes. Intuitively, it is a random time which does not look into the future.

A random variable $T : \Omega \to [0,\infty]$ is a stopping time (relative to the filtration at hand) if the event $\{T \leq t\} \in \mathcal{F}_t$, for each $t \in \mathbb{R}_+$.

The calculus of stopping times is a necessary tool for any stochastic analysis. In this paper, we shall just use the known properties without citing the appropriate theorems. Given a stopping time T, the σ-algebra $\mathcal{F}_T \subset \mathcal{F}$ is the collection of all events A such that $A \cap \{T \leq t\} \in \mathcal{F}_t$. This justifies the statement that the random time T does not look in the future. Obviously, if T is a constant, then

$\mathcal{F}_T$ coincides with the obvious corresponding member of the filtration. Roughly speaking, one can extend to stopping times T and σ-algebras $\mathcal{F}_T$ all that is known for constant times t and σ-algebras $\mathcal{F}_t$.

If T is a stopping time, let $X_T : \Omega \to E$ denote the random vector defined by

$$X_T(\omega) = X_{T(\omega)}(\omega), \text{ for } \omega \in \Omega.$$

The process X^T (the process stopped at T) is defined to be the process $(X_{t\wedge T})_{t\in\mathbb{R}^+}$.

Suppose that U and V are two nonnegative extended real valued functions defined on Ω such that $U \leq V$. We denote by $[U, V]$ the subset of $\mathbb{R}_+ \times \Omega$ defined by $[U, V] = \{(t, \omega) \in \mathbb{R}_+ \times \Omega, U(\omega) \leq t \leq V(\omega)\}$

Likewise we define the <u>stochastic intervals</u> $[U, V)$, (U, V) and $(U, V]$. In particular, the graph of U is $[U] := [U, U]$.

Observe that when U and V are stopping times, then $[U, V] \in \mathcal{O}$. In fact, $\mathcal{O}$ is generated by the class of stochastic intervals $[S, \infty]$, where S is a stopping time. $\mathcal{P}$ is generated by the stochastic intervals of the form $(S, T]$, where S and T are stopping times, and $[O_A], A \in \mathcal{F}_0$, where $O_A(\omega)$ is 0 or depending on whether or not $\omega \in A$. Note that $[O_A] = \{0\} \times A$.

We say that a stopping time T is <u>predictable</u> if there exists a strictly increasing sequence (T_n) of stopping times which converges to T and $T_n < T$ on $\{T > 0\}$. We say that the T_n <u>predict</u> T. Let us mention that T is a predictable stopping time if and only if $[T, \infty) \in \mathcal{P}$.

Now we shall examine the building blocks of $\mathcal{P}$. For every α, where $0 < \alpha \leq \infty$, we denote by $\mathcal{A}(0, \alpha]$ the ring generated by rectangles of the form $(s, t] \times A$, where $0 \leq s < t \leq \alpha$ and $A \in \mathcal{F}_s$. Note that if $t < \infty$, then $(s, t] \times A$ is a predictable rectangle. We set $\mathcal{A}(0, \alpha) = \bigcup_{\beta<\alpha} \mathcal{A}(0, \beta]$.

A nice result is that the ring $\mathcal{A}(0, \infty)$ generates $\mathcal{P} \cap ((0, \infty) \times \Omega)$. Note that $\mathcal{A}(0, \infty)$ consists of all finite disjoint unions of bounded predictable rectangles $(s, t] \times F$, with $t < \infty$ (we always assume that $F \in \mathcal{F}_s$, when we write $(s, t] \times F$.

This result indicates the character of the events belonging to $\mathcal{P} \cap ((0, \infty) \times \Omega)$. In this section of the paper we avoid subsets of the form $\{0\} \times \Omega$, because these sets play no role in the analysis concerning regularity – which we shall first address. Of course, if we add the predictable rectangles $[O_A]$, then we can generate all of $\mathcal{P}$. We shall do this when we turn to stochastic integration.

3. The Doléans function. Quasimartingales.

We define the Doléans function μ_X of the process X on $\mathcal{A}(0,\infty)$ as follows:

For each predictable rectangle $(s,t] \times F$, set

$$\mu_X((s,t] \times F) = E(1_F(X_t - X_s)).$$

In particular, $\mu_X((s,\infty] \times F) = -E(1_F X_s)$.

Note that μ_X is finitely additive on the semi-ring of predictable rectangles, and thus it can be extended to an E-valued, finitely additive measure on the algebra $\mathcal{A}(0,\infty]$. This function is invaluable in the study of quasimartingales, as we shall see.

We define the following regularity condition.

Condition (R). $\lim_{t\downarrow s} E(1_F X_t) = E(1_F X_s)$, for each $F \in \mathcal{F}_s$.

Note that the above condition is obviously equivalent to the condition
$\lim_{t\downarrow s} \mu_X((s,t] \times F) = 0$.

Now we shall list some properties of μ_X.

1. X is a martingale if and only if $\mu_X = 0$ on $\mathcal{A}(0,\infty)$.
2. X is a submartingale if and only if $\mu_X \geq 0$ on $\mathcal{A}(0,\infty)$. X is a negative submartingale if and only if $\mu_X \geq 0$ on $\mathcal{A}(0,\infty]$.
3. If Y is another process, then $\mu_X = \mu_Y$ on $\mathcal{A}(0,\infty)$ if and only if $X - Y$ is a martingale. We have $\mu_X = \mu_Y$ on $\mathcal{A}(0,\infty]$ if and only if X is a modification of Y, i.e. $X_t = Y_t$ a.s. for each t.
4. If Y is a negative submartingale and if $\mid \mu_X(A) \mid \leq \mu_Y(A)$ for each $A \in \mathcal{A}(0,\infty]$, then $\mid X_t \mid \leq -Y_t$ a.s. for each t.
5. For any stochastic interval $(S,T]$ with S and T simple stopping times, we have $\mu_X(S,T] = E(X_T - X_S)$.

For any α, with $0 < \alpha \leq \infty$, we define the mean variation of X on $(0, \alpha]$ to be the number

$$\mathrm{Var}_X(0,\alpha] = \sup\, \Sigma_i \parallel E((X_{t_{i+1}} - X_{t_i}) \mid \mathcal{F}_{t_i}) \parallel_1 \leq \infty,$$

where the supremum is taken over all finite partitions $0 \leq t, \leq \ldots \leq t_n = \alpha$. The mean variation $\mathrm{Var}_X(0,\alpha)$ of X on $(0,\alpha)$ is defined similarly by taking $t_n < \alpha$.

It is important to be able to compute $\mathrm{Var}_X(0,\alpha)$ by taking the supremum over different sets, for example, random partitions. We shall present various representations of $\mathrm{Var}_X(0,\alpha]$, along with the relationship between $\mathrm{Var}_X(0,\alpha]$ and $\mid \mu_X \mid$, the variation of μ_X. But first, some definitions.

We say X is a quasimartingale on $(0,\alpha]$, or on $(0,\alpha)$ if $\mathrm{Var}_X(0,\alpha]$ $< \infty$ or $\mathrm{Var}_X(0,\alpha) < \infty$ respectively. We shall be interested in three types of quasimartingales, namely those on $(0,\infty]$, on $(0,\infty)$, and those on every bounded interval $(0,\alpha]$. We will list some important facts concerning quasimartingales along with some convergence properties. In particular, X is a quasimartingale on $(0,\alpha]$ or $(0,\alpha)$, with $\alpha \leq \infty$ if and only if μ_X has bounded variation on $\mathcal{A}(0,\alpha]$ or $\mathcal{A}(0,\alpha)$ respectively. In fact, $\mid \mu_X \mid ((0,\alpha] \times \Omega)$ is equal to $\mathrm{Var}_X(0,\alpha]$.

We say X is of class (D) if the family $\{X_T : T$ simple stopping time $\}$ is uniformly integrable. Say that X is of class (LD) if for each fixed $\alpha < \infty$, the above family, with $T \leq \alpha$, is uniformly integrable.

Note that the definition of X being a quasimartingale reminds us of the definition of a function being of bounded variation. Let's push this analogy further: think of martingales as constant functions and submartingales as increasing functions. Then it seems reasonable to expect that we can decompose a real valued quasimartingale as a difference of two supermartingales – and this is exactly Rao's beautiful theorem [R]. A measure theoretic way of looking at it is to regard μ_X^+ and μ_X^- as being Doléans functions of supermartingales.

To appreciate the importance of quasimartingales, by the important Bichteler-Dellacherie theorem, the most general process X such that the stochastic integral $\int H dX$ can be defined, is (under a change of measure!) a quasimartingale. As we shall see, even characterizing summable E-valued processes X for stochastic integration in Banach spaces uses the notion of a quasimartingale.

Let's look at more properties involving these fundamental concepts.

6. Assume that X is a quasimartingale on $(0,\alpha]$ for each $\alpha < \infty$, and let Z be a total subset of E'. The following conditions are equivalent for any $s < \infty$.

   (a) $\lim_{t\downarrow s} \mid \mu_X \mid ((s,t] \times \Omega) = 0$;

   (b) $\lim_{t\downarrow s} \mu_X((s,t] \times F) = 0$, for every $F \in \mathcal{F}_s$;

   (c) For any $\alpha \leq \infty$, we have

   $$\mathrm{Var}_X(0,\alpha] = \sup \Sigma \parallel E(1_{F_i}(X_{t_i} - X_{s_i}) \mid \mathcal{F}_{s_i}) \parallel_1,$$

   where the supremum is taken over all finite families $((s_i,t_i] \times F_i)$ of disjoint predictable rectangles contained in $(0,\alpha] \times \Omega$.

7. For any $\alpha \leq \infty$, we have

   $$\mid \mu_X \mid ((0,\alpha] \times \Omega) = \mathrm{Var}_X(0,\alpha]$$

   and

   $$\mid \mu_X \mid ((0,\alpha) \times \Omega) = \mathrm{Var}_X(0,\alpha).$$

8. For any $\alpha < \infty$, we have

   $$\begin{aligned} \mathrm{Var}_X(0,\alpha] &= \sup \Sigma_i \mid E(X_{S_{i+1}} - X_{S_i}) \mid \\ &= \sup \Sigma_i \parallel E(X_{S_{i+1}} - X_{S_i} \mid \mathcal{F}_{S_i}) \parallel_1, \end{aligned}$$

   the supremum being taken over all finite increasing sequences $S_1 \leq \ldots \leq S_n$ of simple stopping times S_i, with $S_n \leq \alpha$.

9. X is a quasimartingale on $(0,\alpha)$ or on $(0,\alpha]$ if and only if μ_X has bounded variation on $\mathcal{A}(0,\alpha)$ or $\mathcal{A}(0,\alpha]$ respectively.

10. Any martingale is a quasimartingale on $(0,\infty)$; it is a quasimartingale on $(0,\infty)$ if and only if $\sup_t \parallel X_t \parallel_1 < \infty$, and this last expression is $\mathrm{Var}_X(0,\alpha]$.

11. Any negative submartingale and any positive supermartingale is a quasimartingale on $(0,\infty]$.

12. A process X with integrable variation, that is $E(\mid X \mid_\infty) < \infty$, is a quasimartingale on $(0,\infty]$.

13. X is a quasimartingale on $(0,\infty]$ if and only if X is a quasimartingale on $(0,\infty)$ and $\sup_t \parallel X_t \parallel_1 < \infty$.

14. X is a quasimartingale on $(0, \alpha]$ if and only if the stopped process X^α is a quasimartingale on $(0, \infty]$.

15. If X is a quasimartingale on $(0, \alpha]$, then $\mid X \mid$ is a quasimartingale on $(0, \alpha]$.

16. X is a quasimartingale on $(0, \infty]$ if and only if the following condition is satisfied:

    for every $t < \infty$, there is a positive function $f_t \in L^1(P)$ such that for every $F \in \mathcal{F}_t$ we have

$$\mid \mu_X \mid ((t, \infty] \times F) \leq E(1_F f_t).$$

The next result is due to Orey [O].

17. Assume that X is a quasimartingale on each bounded interval. X is right continuous in the mean if and only if it is right continuous in probability. If X has a right continuous modification, then it is right continuous in the mean.

Here is another theorem proved by Orey [O].

18. Assume that X is a right continuous quasimartingale on $(0, \infty]$. Then

    (a) X_T is integrable for every stopping time T;

    (b) If (T_n) is a decreasing sequence of stopping times converging to T, then $X_{T_n} \to X_T$ in the mean.

This concludes our list of fundamental properties concerning quasimartingales. Now we turn to regularity properties.

4. Regularity of quasimartingales. The Doob-Meyer decomposition.

As we have seen, quasimartingales are fundamental entities in stochastic theory. Of the upmost concern is when a quasimartingale X has a cadlag modification Y, that is, for each $t, X_t = Y_t$ a.s., and Y is cadlag – right continuous, with left limits existing. Why is being cadlag so important? Look at it in the following light. Given a stopping time T, most calculations involve X_T, but unless X is cadlag, X_T is not measurable. Without being able to work with the random vector X_T, little progress can be made.

Another important aspect in working with a quasimartingale X, is when does X have a nice form which will permit us easy access to the information X contains? If we can express X in the following manner:

$$X = M + A,$$

where M is a local martingale and A is a predictable cadlag process with paths of finite variation, then certainly everyone would agree that X indeed has a very nice form. If X has such a representation, we say X has a Doob-Meyer decomposition (which is unique up to an evanescent set if M is cadlag).

The main result in [B-D.1] is that the regularity of X and the existence of a Doob-Meyer decomposition for X are equivalent. Moreover, weak regularity is equivalent to strong regularity, and both are equivalent to the regularity condition (R), which we stated in section 3. This condition is new even for the scalar case. In [D-M, vol. B], only a sufficient condition is given for the existence of a cadlag modification of a supermartingale.

The main theorem in [B-D.1] gives fourteen equivalent statements concerning the regularity and the existence of a Doob-Meyer decomposition for a quasimartingale X. The proof is long and sheds new light on the structure of quasimartingales. Before stating this theorem, we shall give a quick presentation of the main results and corollaries of the theorem. Remember $X : \mathbb{R}_+ \times \Omega \to E$ is a quasimartingale.

(A) Assume that E has the Radon-Nikodym property (RNP). Then X is cadlag if and only if X is weakly cadlag, that is $< x', X >$ is cadlag for each $x' \in E'$. Furthermore, X is cadlag if and only if X has a Doob-Meyer decomposition of the form $X = M + A$, where M is a cadlag local martingale and A is a predictable process with finite variation and $A_0 = 0$. This decomposition is unique up to an evanescent set.

(B) Assume E has the RNP and X is a quasimartingale. Then X has a cadlag modification if and only if X satisfies the regularity condition (R), or even the following weak regularity condition (WR):

$$(WR) \quad \lim_{t \downarrow s} < E(1_F X_t), z > = < E(1_F X_s), z >,$$

for each $s < \infty, F \in \mathcal{F}_s$, and z in a norming subset $\mathcal{Z} \subset E'$.

(C) Without any assumption on the Banach space E, we have the following results concerning the existence of a cadlag modification of X in terms of its right limit X_+ along the rationals:

(C_1) If X is a local martingales, then X_+ exists a.s., X_+ is a local martingale and a cadlag modification of X.

(We use the fact that a cadlag modification for E-valued martingales exists; this was established in [B-D.2].

(C_2) If X is a quasimartingale with separable range and if X_+ exists a.s., then X_+ is a right continuous modification of X if and only if the regularity condition (R) holds. If X is real valued, then X_+ and X_- exist a.s.

(C_3) If X has integrable variation, then X_+ (which exists everywhere) is a cadlag modification of X if and only if condition (R) holds.

(D) Assume again that E has the RNP. Then X is a quasimartingale of class (D) and satisfies condition (R) if and only if X has a Doob-Meyer decomposition $X = M + A$ with M a (not necessarily right continuous) martingale of class (D) and A a predictable, cadlag process with integrable variation and $A_0 = 0$.

Conditions (D) and (R) together are also equivalent to the associated measure μ_X being σ-additive and with bounded variation.

THE REGULARITY THEOREM. *Assume that E has the RNP and that X is a quasimartingale on each bounded interval and has a separable range. Let $Z \subset E'$ be any set which is norming for the range of X.*

The following assertions are equivalent:

(1) X has a cadlag modification.

(2) X has a right continuous modification.

(3) X is right continuous in the mean, that is, for the strong topology of L^1_E.

(4) X is right continuous for the weak topology of L^1_E.

(5) X is right continuous in probability.

(6) X satisfies condition (R).

(7) $X = M + A$, where M is a (not necessarily right continuous) local martingale and A is a predictable, cadlag process with finite variation and $A_0 = 0$.

$(1') < X, z >$ has a cadlag modification for each $z \in Z$.

$(2') < X, z >$ has a right continuous modification for each $z \in Z$.

$(3') < X, z >$ is right continuous in the mean, that is, in the strong topology of L^1, for each $z \in Z$.

$(4') < X, z >$ is right continuous in the weak topology of L^1, for each $z \in Z$.

$(5') < X, z >$ is right continuous in probability for each $z \in Z$.

$(6')$ X satisfies condition (WR).

$(7') < X, z >= M(z) + A(z)$, for each $z \in Z$, where $M(z)$ is a (not necessarily right continuous) local martingale and $A(z)$ is a predictable, cadlag process of finite variation with $A_0(z) = 0$.

The decomposition in (7) or $(7')$ is unique up to an evanescent set.

Remarks. The above theorem and its corollaries are taken from [B-D.1]. The reader is referred to this paper for the proof of the theorem, along with a number of corollaries and theorems involving quasimartingales, and the proofs of some of the results in section 3, properties 1-18, some of which are established in proving the above theorem. Other pertinent references include [D-M], [F], [F_O], [K], [M.1], [M-P] (the last two references deal with Banach-valued martingales), [O], [R], [P], [Y]. See [B-M] for decompositions of weak quasimartingales.

5. Integration in Banach spaces.

In order to represent the stochastic integral as a genuine integral, we must develop an integration theory for infinite dimensional measures – even if we restrict ourselves to the scalar setting (here the "stochastic measure" is L^P-valued. To develop the stochastic integral in Banach spaces, we establish a bilinear integration theory. This has been given in [B-D.3], [B-D.4], and [B-D.5]. In this section we shall present only the barest of sketches of this theory due to space limitations, and refer the reader to the above references for all undefined terms used in the sequel.

Throughout, E, F, G will be Banach spaces. The unit ball of a Banach space G is denoted by G_1. The space of bounded linear operators from F into G is denoted by $L(F,G)$. We write $E \subset L(F,G)$ to mean that E is continuously embedded in $L(F,G)$. Examples of such embeddings are $E = L(\mathbb{R}, E); E \subset L(E', \mathbb{R}) = E''; E \subset L(F, E\widehat{\otimes}_\pi F)$, if E is a Hilbert space over the reals, $E = L(E, \mathbb{R})$; if E and F are Hilbert spaces, $E \subset L(F, E \otimes_{HS} F)$, where HS denotes the Hilbert-Schmidt norm. We write $c_0 \not\subset G$ to mean that G does not contain a subspace which is isomorphic to c_0. A subspace $Z \subset E'$ is norming for E if $\mid x \mid = \sup\{\mid < x, z > \mid : z \in Z_1\}$, for $x \in E$.

We shall define an integral $\int f dm$, where m is $E \subset L(F,G)$-valued, and f is F-valued. Then in the next section, this integral will be used to define the stochastic integral $\int H dX$ where H is F-valued, and predictable, and X is $L(F,G)$ valued.

$\mathcal{R}$ will denote a ring of subsets of a set S, and Σ is the σ-algebra generated by $\mathcal{R}$. We shall assume that S is a countable union of sets from $\mathcal{R}$.

Let $m : \mathcal{R} \to E \subset L(F,G)$ be a finitely additive measure. We say that m is <u>strongly additive</u> if for any sequence of disjoint sets A_n from $\mathcal{R}$, we have $m(A_n) \to 0$.

We shall need the following extension theorem which is stated in terms of norming subspaces of E, which will be used to establish a summability theorem for stochastic processes [B-D.4].

1. Theorem. *Let $m : \mathcal{R} \to E$ be a finitely additive measure. Suppose that $Z \subset E'$ is norming for E. Then conditions (a), (b) and (c) are equivalent.*

(a) m is strongly additive on $\mathcal{R}$ and for each $z \in Z$, zm is σ-additive on $\mathcal{R}$.

(b) m is strongly additive and σ-additive on $\mathcal{R}$.

(c) m can be extended uniquely to a σ-additive measure $m : \Sigma \to E$.

(d) Assume $c_0 \not\subset E$. If m is bounded on $\mathcal{R}$ and if zm is σ-additive on $\mathcal{R}$ for each $z \in Z$, then m can be extended uniquely to a σ-additive E-valued measure on Σ.

(e) Assume $c_0 \not\subset E$. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, $1 \leq p < \infty$ and let $m : \mathcal{R} \to L^P_E(\mu)$ be finitely additive and bounded. For each $z \in Z$ define the measure $zm : \mathcal{R} \to L^P(\mu)$ by $(zm)(A) =< m(A), z >$ for $A \in \mathcal{R}$. If zm is σ-additive on $\mathcal{R}$ for each $z \in Z$, then m can be uniquely extended to a σ-additive measure $m_1 : \Sigma \to L^P_E(\mu)$.

The semivariation of $m : \mathcal{R} \to E \subset L(F, G)$.

For each set $A \in \mathcal{R}$, define the semivariation $\tilde{m}_{F,G}(A)$ of m on A relative to the pair (F, G) by

$$\tilde{m}_{F,G}(A) = \sup \ \mid \Sigma_{i \in I} m(A_i) x_i \mid,$$

where the supremum is taken over all finite families $(A_i)_{i \in I}$ of disjoint sets from $\mathcal{R}$ with union A, and all finite families $(x_i)_{i \in I}$ of elements from F_1. We then obtain a set function $\tilde{m}_{F,G} : \mathcal{R} \to [0, +\infty]$. We remark that

$$\tilde{m}_{F,G}(A) = \sup \ \mid \int s dm \mid,$$

where the supremum is taken over all F-valued, simple $\mathcal{R}$-measure functions s, such that $\mid s \mid \leq 1_A$; the integral $\int s dm$ is defined in the usual manner.

We say m has <u>finite</u> (respectively <u>bounded</u>) <u>semivariation</u> relative to (F, G) if $\tilde{m}_{F,G}(A) < \infty$ for every $A \in \mathcal{R}$ (respectively $\sup \{\tilde{m}_{F,G}(A) : A \in \mathcal{R}\} < \infty$. If $E = L(\mathbb{R}, E)$, we sometimes write $\tilde{m}$. In this case, m has bounded semivariation on $\mathcal{R}$ if and only if m is bounded on $\mathcal{R}$.

2. Proposition. Assume Z is a closed subspace of G' which is norming for G. If $\mid m_z \mid$ is bounded on $\mathcal{R}$ for each $z \in Z$, then $\tilde{m}_{F,G}$ is bounded on $\mathcal{R}$.

3. Proposition. Let $m : \mathcal{R} \to L(F,G)$ be finitely additive with bounded semivariation relative to (F,G). If $c_0 \not\subset G$, then $\{\mid m_z \mid : z \in Z_1\}$ is uniformly strongly additive on $\mathcal{R}$. If m is σ-additive on $\mathcal{R}$, then this set is uniformly σ-additive on $\mathcal{R}$.

The last two important propositions are due to Dobrakov [Do.1], [Do.2], in a more restrictive setting. Since the set in Proposition 3 occurs often, let us set

$$m_{F,G} = \{\mid m_z \mid : z \in Z_1\}$$

where $m_z : \mathcal{R} \to F'$ is defined by $< x, m_z(A) > = < m(A)x, z >$ where $x \in F, z \in Z$. Of course, $\mid m_z \mid$ is the total variation set function of m_z.

From now on in this section we shall assume that $m : \Sigma \to E \subset L(F,G)$ is finitely additive and has bounded semivariation $\tilde{m}_{F,G}$ and $Z \subset G'$ is norming for G. Also we shall assume that $m_{F,G}$ consists of σ-additive measures. We shall develop an integration theory with respect to m functions $f : S \to F$.

We say that a set $Q \subset S$ is m-negligible if there exists a set $A \in \Sigma$ with $Q \subset A$ such that $m(B) = 0$ for every $B \subset A, B \in \Sigma$. Thus a set $A \in \Sigma$ is m-negligible if and only if $\tilde{m}_{F,G}(A) = 0$. We say that a function $f : S \to F$ is m-negligible (or $f = 0, m - a.e$) if it vanishes outside an m-negligible set. A subset $Q \subset S$ is said to be $m_{F,G}$-negligible if for each $z \in Z$, Q is contained in an $\mid m_z \mid$-negligible set. Note that Q need not belong to Σ.

A function $f : S \to F$ is said to be $m_{F,G}$-measurable if it is m_z-measurable for every $z \in Z$. We say that $f : S \to F$ if m-measurable if it is the m- a.e. limit of a sequence of F-valued, Σ-measurable simple functions. If f is m-measurable, then it is $m_{F,G}$-measurable. The converse is true if $m_{F,G}$ is uniformly σ-additive as the next proposition shows (this is the case if $c_0 \not\subset G$ of if G is weakly complete).

4. Proposition. (a) Let λ be a control measure of m. Then $m_{F,G}$ is uniformly σ-additive if and only if $\tilde{m}_{F,G} << \lambda$.

(b) Suppose $m_{F,G}$ is uniformly σ-additive. Then a function $f : S \to F$ is m-measurable if and only if f is $m_{F,G}$-measurable.

Next we extend the definition of $\tilde{m}_{F,G}$ to functions. For each $f : S \to F$ (or $\overline{\mathbb{R}}$) which is $m_{F,G}$-measurable, we define

$$\tilde{m}_{F,G}(f) = \sup\{\mid \int sdm \mid\},$$

where the supremum is extended over all F-valued, Σ-measurable simple functions s such that $\mid s \mid \leq \mid f \mid$ on S.

One can show that

$$\tilde{m}_{F,G}(f) = \sup\{\int \mid f \mid d \mid m_z \mid : z \in Z_1\}.$$

For simplicity, write $N = \tilde{m}_{F,G}$. We shall now list some properties.

(i) N is sub additive and positively homogeneous on the space of $m_{F,G}$-measurable functions.

(ii) $N(f) = N(\mid f \mid)$

(iii) $N(f) \leq N(g)$ if $\mid f \mid \leq \mid g \mid$.

(iv) $N(f) = \sup\ \{N(1_A f) : A \in \Sigma\} = \sup_n\{N(f1_{(\mid f \mid \leq n)})\}$.

(v) $N(\sup f_n) = \sup N(f_n)$ for every increasing sequence (f_n) of positive $m_{F,G}$-measurable functions.

(vi) $N(\Sigma f_n) \leq \Sigma N(f_n)$ for every sequence of positive $m_{F,G}$-measurable functions.

(vii) $N(\liminf f_n) \leq \liminf N(f_n)$ for every sequence of positive $m_{F,G}$-measurable functions.

(viii) If $N(f) < \infty$, then f is finite $m_{F,G}$-a.e.

(ix) If $f : S \to F$ is $m_{F,G}$-measurable and $c > 0$, then

$$N(\{\mid f \mid > C\}) \leq \frac{1}{c} N(f).$$

If $\tilde{m}_{F,G}(\{\mid f_n - f \mid > \epsilon\}) \to 0$ as $n \to \infty$ for each $\epsilon > 0$, we say that $f_n \to f$ in $m_{F,G}$-measure.

(x) If $N(f_n - f) \to 0$, then $f_n \to f$ in $m_{F,G}$-measure then there exists a subsequence (f_{n_r}) converging $m_{F,G}$-a.e. to f.

We mention that the Egorov theorem is not valid in general, but it can be proved in the case $m_{F,G}$ is uniformly σ-additive.

We denote by $\mathcal{F}_F(m_{F,G})$ the space of all $m_{F,G}$-measureable functions $f : S \to F$ such that $\tilde{m}_{F,G}(f) < \infty$. The mapping $f \to \tilde{m}_{F,G}(f)$ is a seminorm on this vector space, which is complete relative to this seminorm (use property (vi)). We call $\mathcal{F}_F(m_{F,G})$ the space of F-valued $m_{F,G}$-integrable functions, however the reader can choose any closed subspace of $\mathcal{F}_F(m_{F,G})$ as the space of integrable functions, according to the refinements of the specific problem at hand. This will be illustrated in stochastic integration theory.

Note that $\mathcal{F}_F(m_{F,G}) \subset L^1_F(\mid m_z \mid)$ continuously for each $z \in Z$. We can extend Proposition 2 and show that if Z is closed in G', then $\tilde{m}_{F,G}(f) < \infty$ if and only if $\int \mid f \mid d \mid m_z \mid < \infty$ for each $z \in Z$. In this case we have

$$\mathcal{F}_F(m_{F,G}) = \bigcap_{z \in Z} L^1_F(\mid m_z \mid).$$

The set $\mathcal{B}_F$ of all bounded F-valued $m_{F,G}$-measurable functions is contained in the set $\mathcal{F}_F(m_{F,G})$. In particular, the sets $\mathcal{S}_F(\mathcal{R})$ and $\mathcal{S}_F(\Sigma)$ of the F-valued, $\mathcal{R}$-measurable, respectively Σ-measurable simple functions are contained in $\mathcal{F}_F(m_{F,G})$, however, unlike the classical case, these sets are <u>not</u> necessarily dense in $\mathcal{B}_F$ for the seminorm $m_{F,G}$, unless $m_{F,G}$ is uniformly σ-additive. Set $\mathcal{B} = \mathcal{B}_F$.

For any subspace $\mathcal{C} \subset \mathcal{F}_F(m_{F,G})$, we denote by $\mathcal{F}_F(\mathcal{C}, m_{F,G})$ the closure of $\mathcal{C}$ in $\mathcal{F}_F(m_{F,G})$, which is also complete. We have $\mathcal{F}_F(\mathcal{S}_\mathcal{R}, m_{\mathcal{R},E}) = \mathcal{F}_F(\mathcal{B}_\mathcal{R}, m_{\mathcal{R},E})$.

Let us record some properties of $\mathcal{F}_F(m_{F,G})$.

5. THEOREM. (a) *If* $f \in \mathcal{F}_F(\mathcal{B}, m_{F,G})$, *then* $\tilde{m}_{F,G}(f 1_A) \to 0$ *as* $\tilde{m}_{F,G}(A) \to 0$. *The converse is also true if* $m_{F,G}$ *is uniformly* σ-*additive.*

(b) *If* $f \in \mathcal{F}_F(m_{F,G})$ *and if* $\tilde{m}_{F,G}(f 1_{A_n}) \to 0$ *for any sequence of* $m_{F,G}$-*measureable sets* $A_n \downarrow \phi$, *then* $f \in \mathcal{F}_F(\mathcal{B}, m_{F,G})$.

(c) *If* $f : S \to F$ *is* $m_{F,G}$-*measureable and if* $\mid f \mid \leq g \in \mathcal{F}_\mathcal{R}(\mathcal{B}, m_{F,G})$, *then* $f \in \mathcal{F}_F(\mathcal{B}, m_{F,G})$.

(d) *$f \in \mathcal{F}_F(\mathcal{B}, m_{F,G})$ if and only if f is $m_{F,G}$-measureable and $| f | \in \mathcal{F}_{\mathbb{R}}(\mathcal{B}, m_{F,G})$.*

(e) *Suppose (f_n) is a sequence of functions from $\mathcal{F}_F(m_{F,G})$ which converges uniformly on S to f. Then $f \in \mathcal{F}_F(m_{F,G})$ and $f_n \to f$ in $\mathcal{F}_F(m_{F,G})$.*

(f) *If m has finite variation on Σ, then $m_{F,G}$ is uniformly σ-additive and $\mathcal{F}_F(\mathcal{B}, m_{F,G}) = \mathcal{F}_F(m_{F,G})$.*

6. Theorem (Vitali). *Let (f_n) be a sequence from $\mathcal{F}_F(m_{F,G})$ and let $f : S \to F$ be $m_{F,G}$-measurable. If condition (1) below and either of conditions (2a) or (2b) are satisfied, then $f \in \mathcal{F}_F(m_{F,G})$ and $f_n \to f$ in $\mathcal{F}_F(m_{F,G})$.*

(1) $\tilde{m}_{F,G}(f_n 1_A) \to 0$ as $\tilde{m}_{F,G}(1_A) \to 0$, uniformly in m.

(2a)$f_n \to f$ in $m_{F,G}$-measure.

(2b)$f_n \to f$ pointwise and $m_{F,G}$ is uniformly σ-additive.

Converseley, if $f_n \to f$ in $\mathcal{F}_F(\mathcal{B}, m_{F,G})$, then (1) and (2a) hold.

7. Theorem (Lebesgue). *Let (f_n) be a sequence from $\mathcal{F}_F(\mathcal{B}, m_{F,G})$ and let $f : S \to F$ be an $m_{F,G}$-measurable function and suppose $g \in \mathcal{F}_{\mathcal{R}}(\mathcal{B}, m_{F,G})$. If*

(1) *$| f_n | \leq g$, $m_{F,G}$-a.e. for each n, and any one of the conditions (2a) or (2b) holds:*

(2a)*$f_n \to f$ in $m_{F,G}$-measure.*

(2b)*$f_n \to f$ pointwise and $m_{F,G}$ is uniformly σ-additive, then $f \in \mathcal{F}_F(\mathcal{B}, m_{F,G})$ and $f_n \to f$ in $\mathcal{F}_F(m_{F,G})$.*

The integral

If $f \in \mathcal{F}_{F,G}(m) := \mathcal{F}_F(m_{F,G})$, then $f \in L^1_F(| m_z |)$, hence the real number $\int f dm_z$ is defined. The mapping $z \to \int f dm_z$ is linear and continuous from $Z' \to \mathbb{R}$. Suppose $Z = G'$. Then $\int f dm \in G''$, where $\int f dm$ denotes the above map.

Thus

$$< z, \int f dm >= \int f dm_z, \text{ for } z \in G'.$$

Also, $\mid \int f dm \mid \leq \tilde{m}_{F,G}(f)$, that is, the integral is continuous on $\mathcal{F}_{F,G}(m)$. Thus if (f_n) and f satisfy the Vitali or Lebesque theorems, then $\int f_n dm \to \int f dm$ in G''.

We are particularly interested in the case when $\int f dm \in G$. One can prove that if $c_o \not\subset G$, then $\int f dm \in G$, for all $f \in \mathcal{F}_{F,G}(m)$. In general, if $\mathcal{C}$ is a subset of $\mathcal{F}_{F,G}(m)$ such that $\int f dm \in G$ whenever $f \in \mathcal{C}$, then $\int f dm \in G$, whenever $f \in \mathcal{F}_{F,G}(\mathcal{C}, m)$. Note that this is true when $\mathcal{C} = \mathcal{S}_E(\Sigma)$. It is reasonable to call certain subsets of $\mathcal{F}_{F,G}(m)$ integrable functions when they satisfy additional conditions imposed by the specific problem to which the integration theory is applied. In the case of stochastic integration, we impose the condition that the stochastic integral be cadlag.

Weak Completeness and weak compactness in $\mathcal{F}_{F,G}(\mathcal{B}, m)$.

One of the main goals in [B-D.3] was to obtain sufficient conditions for weak completeness and weak compactness (Dunford-Pettis type of theorems) in $\mathcal{F}_{F,G}(\mathcal{B}, m)$. To establish these results, a characterization of elements in $(\mathcal{F}_{F,G}(\mathcal{B}, m))'$ was given, using techniques of Köthe spaces. This theory can be applied to stochastic integration theory to yield new convergence theorems – even in the scalar setting.

A crucial property in establish weak compactness theorems is the Beppo Levi property.

Let $m : \Sigma \to E \subset L(F, G)$ be a σ-additive measure. We say that $m_{F,G}$ has the Beppo Levi property if every sequence (f_n) of positive Σ-measurable simple functions with $\sup_n \tilde{m}_{F,G}(f_n) < \infty$ is a Cauchy sequence in $\mathcal{F}_{\mathbb{R}}(\mathcal{B}, m_{F,G})$. Note that in this case $\sup_n f_n \in \mathcal{F}_{\mathbb{R}}(\mathcal{B}, m_{F,G})$.

8. Theorem. *Let $m : \Sigma \to E \subset L(F, G)$ be σ-additive. Suppose that m has finite semivariation and $m_{F,G}$ is uniformly countably additive (for example, if $c_0 \not\subset G$). If $\int f dm \in G$ for every Σ-measurable function $f \in \mathcal{F}_{F,G}(m)$ then $m_{F,G}$ has the Beppo Levi property.*

In the sequel, $m : E \subset L(F, G)$ is σ-additive and has finite semivariation $\tilde{m}_{F,G}$.

9. THEOREM. *Assume that F is reflexive and $c_o \not\subset G$. Then $\mathcal{F}_{F,G}(\mathcal{B}, m)$ is weakly sequentially complete.*

10. ΠTHEOREM. *Assume $m_{F,G}$ is uniformly σ-additive (e.g. if $c_o \not\subset G$), and suppose F is reflexive, let K be a subset of $\mathcal{F}_{F,G}(\mathcal{B}, m)$ such that*

(1) *K is bounded.*
(2) *$lim_n \tilde{m}_{F,G}(f 1_{A_n}) = 0$ uniformly for $f \in K$, whenever $A_n \in \Sigma$ and $A_n \downarrow \phi$.*

Then K is conditionally weakly compact in $\mathcal{F}_{F,G}(\mathcal{B}, m)$. If moreover $c_o \not\subset G$, then K is relatively compact.

11. THEOREM. *Let $K \subset \mathcal{F}_{\mathcal{R},E}(\mathcal{B}, m)$ be a set satisfying the following conditions:*

(1) K is bounded.
(2) $\int_{A_n} f dm \to 0$ uniformly for $f \in K$

whenever $A_n \in E$ and $A_n \downarrow \phi$.

Then K is conditionally weakly compact. If, moreover, E does not contain a copy of c_o, then K is relatively weakly compact.

12. THEOREM. *Assume $c_o \not\subset E$. Let (f_n) be a sequence of elements from $\mathcal{F}_{\mathcal{R},E}(\mathcal{B}, m)$. If $\int_A f_n dm \to \int_A f_0 dm$ for every $A \in \Sigma$, then $f_n \to f_0$ weakly in $\mathcal{F}_{\mathcal{R},E}(\mathcal{B}, m)$.*

6. Stochastic Integration in Banach spaces.

Now we are in a position to develop a stochastic integration theory in Banach spaces. We shall define stochastic integrals $\int H dX$, where H is F-valued and predictable, and X is a cadlag, $L(F, G)$-valued process.

Throughout the remainder of this paper, we shall assume the stochastic structure presented in section 2. For $1 \leq p < \infty$, we fix a cadlag process $X : \mathbb{R}_+ \times \Omega \to E \subset L(F, G)$ such that $X_t \in L^P_E(P) \equiv L^P_E$, for each t.

Summable Processes

The key concept in the theory of stochastic integration in Banach spaces is that of a summable process. First we define the stochastic measure I_X as follows:

$$I_X : \mathcal{R} \to L^P_E$$

is defined first for predictable rectangles by $I_X[O_A] = I_A X_0$, and $I_X(s,t] \times A = 1_A(X_t - X_s)$, and then extend it in an additive fashion to $\mathcal{R}$. Since $E \subset L(F,G)$, we can consider $L^P_E \subset L(F, L^P_G)$, and thus the semivariation of I_X can be computed relative to the pair (F, L^P_G). We denote the semivariation of I_X relative to (F, L^P_G) by $\tilde{I}_{F,G}$-rather than by $\tilde{I}_{F,L^P_G}$; here $I := I_X$.

Thus we have

$$\tilde{I}_{F,G}(A) = \sup \ \| I_X(A_i)x_i \|_{L^P_G}, A \in \mathcal{R}$$

where the supremum is extended over all families of vectors $x_i \in F_1$ and disjoint sets A_i from $\mathcal{R}$ contained in A. If I_X can be extended to $\mathcal{P}$, we define $\tilde{I}_{F,G}$ on $\mathcal{P}$ in an analogous fashion. I_X is said to have bounded semivariation relative to (F, L^P_G) if $\tilde{I}_{F,G}$ is bounded on $\mathcal{R}$.

We say X is p-summable relative to (F,G) if I_X has a σ-additive L^P_G-valued extension, still denoted by I_X to the predictable σ-algebra $\mathcal{P}$ such that I_X has bounded semivariation on $\mathcal{P}$ relative to (F, L^P_G). If $p = 1$, we say, simply, that X is summable relative to (F,G). If X is p-summable relative to $(\mathbb{R}, E)$, regarding E as $L(\mathbb{R}, E)$, we say that X is p-summable.

Remarks. Observe that X is p-summable if and only if I_X has a σ-additive extension to $\mathcal{P}$, since in this case I_X is bounded in L^P_E on $\mathcal{P}$, which implies bounded semivariation relative to $(\mathbb{R}, L^P_E)$. If $1 \leq p' < p < \infty$, and if X is p-summable relative to (F,G), then it is

p'-summable relative to (F, G). In particular, p-summable relative to (F, G) implies summable relative to (F, G), hence most results proved for summable processes remain valid for p-summable processes. If X is p-summable relative to (F, G), then X is summable relative to $(\mathbb{R}, E)$.

We mention that if X is a process with integrable variation, then X is p-summable relative to any pair (F, G) such that $E \subset L(F, G)$. If E and G are Hilbert spaces, then any square integrable martingale X with values in $E \subset L(F, G)$ is 2-summable relative to (F, G).

Note that X is p-summable relative to (F, G) if and only if I_X has a σ-additive extension to $\mathcal{P}$ and I_X has bounded semivariation on $\mathcal{R}$ (rather than on $\mathcal{P}$) with respect to (F, L^P_G). Hence the problem of summability reduces to a great extent to that of obtaining a σ-additive extension of I_X from $\mathcal{R}$ to $\mathcal{P}$. Once the summability of X is assured, we can apply the theory of section 5 to the measure $m = I_X$, and define an integral with respect to I_X. This will lead to the stochastic integral.

Summability criteria.

Now we turn to the fundamental question concerning the stochastic measure. When is I_X summable? We arrive at the unexpected result that if $E \not\supset c_0$, then the mere boundedness of I_X on $\mathcal{R}$ implies that X is p-summable relative to $(\mathbb{R}, E)$! This is one of the main results in [B-D.4]. Otherwise stated, boundedness of I_X on $\mathcal{R}$ implies not only that the finitely additive I_X is σ-additive on $\mathcal{R}$, but that I_X has a σ-additive extension to $\mathcal{P}$. The key tools involved in the proof of this theorem involve properties of the pertinent quasimartingales induced by X. Let's state the theorem.

1. Theorem. (Summability extension theorem). *If $E \not\supset c_0$, then the following assertions (1) - (5) are equivalent. If E is any general Banach space, then assertions (2) - (5) are equivalent and (1) implies (2).*

(1) I_X can be extended to a σ-additive measure on $\mathcal{P}$.

(2) I_X is bounded on $\mathcal{R}$, the ring generated by predictable rectangles in $\mathbb{R}_+ \times \Omega$.

Let $Z \subset L^q_{E'}$ be any closed norming subspace for L^P_E. For $g \in Z$, let G denote the process defined by $G_t = D(g|\mathcal{F}_t)$.

(3) For each $g \in Z, XG$ is a quasimartingale on $(0, \infty)$.

(4) For each $g \in Z, < I_X, g >: \mathcal{R} \to \mathbb{R}$ is bounded.

(5) For each $g \in Z, < I_X, g >$ is σ-additive and bounded on $\mathcal{R}$.

The Stochastic integral.

Now we shall define the stochastic integral. We will study the class of predictable process H (i.e. processes measureable with respect to $\mathcal{P}$ such that $\int HdX = \int HdI_X$ exists.

X will be a p-summable process $X : \mathbb{R}_+ \times \Omega \to E \subset L(F, G)$, throughout this section, relative to the pair (F, G). Hence I_X is a σ-additive measure on $\mathcal{P}$ with values in $L^P_E \subset L(F, L^P_G)$. We shall apply the integration theory in section 5, with $\Sigma = \mathcal{P}$ or $\Sigma = \mathcal{P}[0, \infty], m = I_X$, E, replaced by L^P_E, G replaced by L^P_G and $Z \subset (L^P_G)'$, a norming subspace for L^P_G, where $\frac{1}{p} + \frac{1}{q} = 1$. For $z \in G$, we have the measure $m_z = (I_X)_z : \mathcal{P}[0, \infty] \to F'$ defined for $A \in \mathcal{P}[0, \infty]$ and $y \in F$ as follows:

$$< y, m_z(A) >=< m(A)y, z >= \int < I_X(A)(w)y, z(w) > dP(w).$$

Thus we have

$$(\tilde{I}_X)_{F,L^P_G} = \tilde{m}_{F,L^P_G} = \sup \{| m_z |: z \in Z, \| z \|_q \leq 1\}.$$

Note that $\{\infty\} \times \Omega$ is $| m_z |$-null, for each z. We shall write $I = I_X, \tilde{I}_{F,G} = \tilde{I}_{F,L^P_G}, I_{F,G} = \{| m_z |: z \in Z_1\}$.

We denote by $\mathcal{F}_F(I_{F,G})$ the space of all predictable processes $H : \mathbb{R}_+ \times \Omega \to F$ such that

$$\tilde{I}_{F,G}(H) = \sup\{\int | H | d | m_z |: z \in Z_1\} < \infty.$$

For any extension of H to $\overline{\mathbb{R}}_+ \times \Omega$, the value of $\tilde{I}_{F,G}(H)$ is the same. We know $\mathcal{F}_F(I_{F,G})$ is complete for this seminorm. Again for any set

$\mathcal{C} \subset \mathcal{F}_F(I_{F,G})$, we denote by $\mathcal{F}_F(\mathcal{C}, I_{F,G})$ the closure of $\mathcal{C}$ in $\mathcal{F}_F(I_{F,G})$. We can define the integral $\int H dI_X \in Z'$ for $H \in \mathcal{F}_F(I_{F,G})$ and the mapping $H \to \int H dI_X$ is a continuous linear mapping from $\mathcal{F}_F(I_{F,G})$ into Z'. The integral $\int H dI_X$ depends on the norming space Z, but the integral corresponding to Z is the restriction to Z of the integral corresponding to $(L_G^P)'$. To further simplify notation, we write

$$\mathcal{F}_{F,G}(X) = \mathcal{F}_{F,L_G^P}(X) = \mathcal{F}_F((I_X)_{F,L_G^P}).$$

If $H \in \mathcal{F}_{F,G}(X)$, then for every $t \geq 0$, we have $1_{[0,t]}H \in \mathcal{F}_{F,G}(X)$. We denote

$$\int_{[0,t]} H dI_X = \int 1_{[0,t]} H dI_X,$$

and define

$$\int_{[0,\infty]} H dI_X := \int_{[0,\infty]} H dI_X := \int H dI_X.$$

Thus for each $H \in \mathcal{F}_{F,G}(X)$, we obtain a family $(\int_{[0,t]} H dI_X)_{t \in [0,\infty]}$ of elements in Z'. We are interested in the subspace of $\mathcal{F}_{F,G}(X)$ which consists of processes H such that for every $t \in [0,\infty]$, the integral $\int_{[0,t]} H dI_X$ belongs to the subspace L_G^P of Z'. If in each equivalence class $[\int_{[0,t]} H dI_X]$ we choose a representative, also denoted by $\int_{[0,t]} H dI_X$, we obtain a process $(\int_{[0,t]} H dI_X)_{t \in [0,\infty]}$ with values in G such that $\int_{[0,t]} H dI_X$ belongs to L_G^P for each t. Now we need to be sure we have a cadlag modification for this process. We shall make the appropriate definition of our main Lebesgue space presently. Note that we have in place the Vitali and Lebesgue theorems at our disposal. A number of technical results are needed to establish the final stochastic integral. One useful tool is the following.

2. THEOREM. *Let $(H^n)_{0 \leq n < \infty}$ be a sequence of elements from $\mathcal{F}_{F,G}(X)$ such that $| H^n | \leq | H^0 |$ for each n, and assume that $H^n \to H$ pointwise.*

If $\int H^n dI_X \in L_G^P$ for each $n \geq 1$ and if the sequence $(\int H^n dI_X)_n$ converges pointwise on Ω, weakly in G, then $\int H dI_X \in L_G^P$ and $\int H^n dI_X \to \int H dI_X$ in the $\sigma(L_G^P, L_{G'}^q)$ topology of L_G^P, as well as pointwise, weakly in G. If $(\int H^n dI_X)$ converges pointwise strongly in G, then $\int H^n dI_X \to \int H dI_X$ strongly in L^1_G.

Note that since X is cadlag, then for simple processes H of the form

$$H = 1_{\{0\}\times A_0} x_0 + \Sigma_{1\le i\le n} 1_{(s_i,t_i]\times A_i} x_i,$$

where the sets in the definition of H are predictable, we have

$$\int_{[0,t]} H dI_X = 1_{A_0} X_0 x_o + \Sigma_{1\le i\le n} 1_{A_i} x_i (X_{t_i\wedge t} - X_{s_i\wedge t})$$

and the process $(\int_{[0,t]} H dI_X)$ is cadlag.

Now we define our Lebesgue space of processes.

3. Definition. We denote by $L^1_{F,G}(X)$ the space of processes $H \in \mathcal{F}_{F,G}(X)$ satisfying the following two conditions:

(1) $\int_{[0,t]} H dI_X \in L^P_G$ for every $t \in [0,\infty]$.

(2) The process $(\int_{[0,t]} H dI_X)_{t\in[0,\infty]}$ has a cadlag modification.

The processes $H \in L^1_{F,G}(X)$ are said to be <u>integrable</u> with respect to X. In this case, any cadlag modification of the process in (2) is called the STOCHASTIC INTEGRAL of H with respect to X and is denoted by $\int H dX$ or $H \cdot X$:

$$(H\cdot X)_t = (\int H dX)_t = \int_{[0,t]} H dI_X \text{ a.s.}$$

Note that $(H \cdot X)_\infty = \int H dI_X$.

If X is real valued, regard $\mathbb{R}$ as being embedded in $L(F, F)$, and then the space of F-valued integrable processes is denoted by $L^1_{F,F}(X)$.

One can prove that $L^1_{F,G}(X)$ is complete relative to the seminorm $\tilde{I}_{F,G}$ and that $L^1_{F,G}(X)$ contains all caglad (left continuous, right limits exist) processes of $\mathcal{F}_{F,G}(X)$. In particular, $L^1_{F,G}(X)$ contains the class $\mathcal{E}$ of predictable elementary processes of the form

$$H_0 1_{\{0\}} + \Sigma_{1\le i\le n} H_i 1_{(T_i,T_{i+1}]}$$

where $(T_i)_{0\le i\le n+1}$ is an increasing family of stopping times with $T_0 = 0$, and H_i is bounded, F-valued, and $\mathcal{F}_{T_i}$-measurable for each i.

The completeness of $L^1_{F,G}(X)$ follows from the next result.

4. THEOREM. *Let (H^n) be a sequence in $L^1_{F,G}(X)$ and assume that $H^n \to H$ in $\mathcal{F}_{F,G}(X)$. Then $H \in L^1_{F,G}(X)$. Moreover, for every t, we have $(H^n \cdot X)_t \to (H \cdot X)_t$ in L^P_G, and there exists a subsequence (n_r) such that $(H^{n_r} \cdot X)_t$ converges to $(H \cdot X)_t$ a.s. as $r \to \infty$, uniformly on every compact time interval.*

5. COROLLARY. $L^1_{F,G}(X)$ is complete.

6. COROLLARY. If $I_{F,G}$ is uniformly σ-additive, then $L^1_{F,G}(X)$ contains all the F-valued, bounded, predictable processes (in particular, this is true when $G \not\supset c_0$ or $F = \mathbb{R}$).

Observe that one can state the Vitali and Lebesgue theorems for stochastic integrals, in light of the theory in section 5. As an illustration of the general theory in section 5 applied to stochastic integration, we state the following theorem.

7. THEOREM. *Assume X is p-summable relative to (F, G). Assume F is reflexive and $I_{F,G}$ is uniformly σ-additive (for example if $c_0 \not\subset G$). Let $K \subset L^1_{F,G}(\mathcal{B}, X)$ be a set satisfying the following conditions:*

(1) *K is bounded in $L^1_{F,G}(\mathcal{B}, X)$.*

(2) *$H1_{A_n} \to 0$ in $L^1_{F,G}(\mathcal{B}, X)$, uniformly for $H \in K$, whenever $A_n \in \mathcal{P}$ and $A_n \downarrow \phi$.*

Then K is conditionally weakly compact in $L^1_{F,G}(\mathcal{B}, X)$. If, in addition, $c_0 \not\subset G$, then K is relatively weakly compact in $L^1_{F,G}(\mathcal{B}, X)$. In this last case, for every sequence (H^n) from K, there exists a subsequence (H^{n_r}) such that $(\int H^{n_r} dX)_t$ converges weakly in L^P_G, as $r \to \infty$, for every t.

REMARKS. Now that we have established an integration theory for stochastic processes, there are, as the reader can well imagine, a myriad of results to be established – for example, the relation of the integral to stopping times, stopped processes, limits, summability of the process $H \cdot X$, the associativity properties

$H \cdot (K \cdot X) = (HK) \cdot X$, jumps $\triangle(H \cdot X)$, local summability, semi-summable processes, the Hilbert space setting, and the ensuing isometries involving $(\tilde{I}_M)_{F,L^2_G}$, $\| I_M(\mathbb{R}_+ \times \Omega) \|_{L^2_E}$, where M is a square integrable martingale, semimartingales, and so on. These topics, and more, are treated in [B-D.4]. Applications to Ito's formula are given in [B-D.6]. We mention that in the infinite dimensional case, a summable process need <u>not</u> be a semimartingale, hence the Bichteler-Dellacherie theorem breaks down in this setting.

The stochastic integral, first developed by Ito, and its extension, in the real valued case, is given in [D-M]. The reader is urged to study the deep and beautiful theory masterfully presented in this book. The starting point for establishing the stochastic integral for the reals is the isometry theorem for square integrable martingales. This method was also used by Kunita [Ku] for Hilbert space valued processes. Pellaumail [P] had the wonderful idea to consider I_X in the real case and express the stochastic integral as a genuine integral with respect to an infinite dimensional measure; due to the lack of a satisfactory integration theory the project was not completed – even the establishment of a cadlag modification of $H \cdot X$ could not be obtained. Kussmaul, using this approach, treated the real case in the elegant book [K]. In [M.2] and [M-P], the Hilbert-valued case is discussed in detail. Two integrals are defined: the first is an isometric integral using Hilbert-Schmidt operators; the second uses the notion of a control process, which exists due to deep stopped inequalities established by Métivier and Pellaumail for square integrable martingales [M.1, section 19].

Several attempts in the Banach space setting have been made ([P], [Y.1], [Y.2], [M-P], [M.2], [K′], [Pr]) but either the Banach spaces were too restrictive, or the construction did not yield the convergence theorems necessary for a full development of the stochastic integral. Without this, Ito's formula cannot be properly established (see [G-P]). By using Pellaumail's approach involving the stochastic measure, coupled with the appropriate Lebesque space for the bilinear integral, the general stochastic integral is developed in [B-D.4]. The nuclear space case is treated in [U] and [B-D.7].

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