CONSTANT OBSTACLE PROBLEM AND REARRANGMENTS (*)

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Sommario. - *Problema con ostacolo costante e riordinamenti. Si dà una maggiorazione della misura dell'insieme di contatto della soluzione di una disequazione variazionale con ostacolo costante, con un operatore ellittico del secondo ordine contenente i termini di ordine inferiore.*

Summary. - *We give an upper bound for the measure of the coincidence set of the solution of a variational inequality with constant obstacle, related to an elliptic second order operator with lower-order terms.*

1. Introduction.

We consider the obstacle problem

$$u \in K, (\forall v \in K) \int_\Omega (a \operatorname{grad} u | \operatorname{grad}(v-u)) + \int_\Omega (b|\operatorname{grad} u)(v-u) + \\ + \int_\Omega cu(v-u) \geq \int_\Omega f(v-u),$$

where the operator is elliptic and

$$K = \{v \in H_0^1(\Omega); v \leq k\}, \qquad k > 0\ .$$

Our aim is to establish some properties of a solution of the problem, by Schwarz symmetrization.

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This technique has been developped first by Talenti [31] to compare the solution of a homogeneous Dirichlet problem, whose equation does not include first order terms, by the solution of a suitable homogeneous Dirichlet problem with spherical symmetrical data; afterwards this method has been fitted to more general cases: see Alvino-Trombetti [5, 6], Bandle [9], Chiti [14], P. L. Lions [27], Talenti [32], Alvino-Lions-Trombetti [1, 3, 4], Ferone-Posteraro [19], Giarruso-Trombetti [22], Trombetti-Vasques [33]; in particular Alvino-Lions-Trombetti [3] establish comparison results concerning equations with all lower terms.

Comparison results for solution to variational inequalities were first established by Bandle-Mossino [10], who studied an obstacle problem with an elliptic operator without fist order terms and with the obstacle vanishing on the boundary (i.e. the obstacle $\psi \in H_0^1(\Omega)$. Results for a complete second order elliptic inequality have been achieved by Alvino-Matarasso-Trombetti [7]. Other results have been established by Posteraro-Volpicelli [30]. In the mentioned papers the authors always suppose that the obstacle vanishes on the boundary; with this hypothesis they can choose in the variational inequality, as test functions, the two functions $u \pm (u-t)^+$, obtaining from the inequality an equality and then applaying the methods used for the equations. Besides the comparison results, they obtain in a particular case a lower bound for the measure of the coincidence set.

The obstacle problem, when the obstacle does not vanish on the boundary, has been studied by Maderna-Salsa [28] for a variational inequality containing only second order terms, with the obstacle constant on the boundary and with regular data. By a replacement of the unknown function, they first obtain the constant obstacle problem

$$u \in K, \quad (\forall v \in K) \int_\Omega (a \operatorname{grad} u | \operatorname{grad}(v-u)) \geq \int_\Omega f(v-u) \ ,$$

where

$$K = \{v \in H_0^1(\Omega); v \leq 1\} \ ;$$

then, introducing a function Φ (*the contact funtion*) depending only on the measure of Ω and on f, they found that the unique solution of the equation $\Phi(\lambda) = 1$ is an upper bound of the measure of the coincidence set.

For a parabolic obstacle problem see Diaz-Mossino [18].

2. Hypothesis and results.

Let Ω an open boundet set of $\mathbf{R}^N$ $(N \geq 2)$, with regular boundary. If S is a (Lebesque) measurable subset of $\mathbf{R}^N$, $|S|$ is the measure of S. If $r > 0$ and $a \in \mathbf{R}^N$, $\mathrm{B}(a,r)$ (resp. $\mathrm{B}'(a,r)$) is the open (resp. closed) ball of $\mathbf{R}^N$ centered in a and with radius r. V_N is $|\mathrm{B}(0,1)|$. If $\phi : \Omega \longrightarrow \overline{\mathbf{R}}$ is measurable the (decreasing) distribution function of ϕ is the function

$$\mu_\phi : \overline{\mathbf{R}} \longrightarrow [0,|\Omega|], t \longrightarrow |\{x \in \Omega; \phi(x) > t\}| \ ;$$

the decreasing rearrangement of ϕ is the function

$$\phi_* : [0,|\Omega|] \longrightarrow \overline{\mathbf{R}}, s \longrightarrow \sup(\{t \in \overline{\mathbf{R}}; \mu_\phi(t) \geq s\}) \ .$$

For simplicty, we put ϕ_*^+ for the decreasing rearrangement $(\phi^+)_*$ of the positive part ϕ^+ of ϕ.

For an exhaustive statement of the proprieties of rearrangements, see [2], [9], [15], [23], [24], [32], [29] for exemple.

We recall that if $\phi \in L^1(\Omega)$, then $\phi_* \in L^1([0,|\Omega|])$ and that

$$\int_A f \leq \int_0^{|A|} f_* \ , \tag{1}$$

for a measurable subset A of Ω. As a conseguence of Polya-Szego theorem, we recall that if $\phi \in H_0^1(\Omega)$ and if $\phi \geq 0$, then $\phi_* \in C(]0,|\Omega|])$, with $\phi_*(|\Omega|) = 0$.

Let L the differential second order operator

$$L : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega), v \longrightarrow -\mathrm{div}\,(a\mathrm{grad}\,v) + (b|\mathrm{grad}\,v) + cv \ ,$$

where $a = (a_{i,j})_{i,j=1,2,\ldots,N}$, $(b_i)_{i=1,2,N}$, and where we suppose the coefficients $a_{ij}, b_i, c \in L^\infty(\Omega)$ and satisfying the following conditions

$$(\forall h \in \mathbf{R}^N)\,(a(x)h|h) \geq M|h|^2, \qquad M > 0 \tag{2}$$

$$|b(x)| \leq B, \qquad B \geq 0 \tag{3}$$

$$c(x) \geq 0 \tag{4}$$

for almost all $x \in \Omega$.

Let $k > 0$ the costant we choose as obstacle and let

$$K = \{v \in H_0^1(\Omega); v \leq k\} \tag{5}$$

the related closed, convex subset of $H_0^1(\Omega)$.

Let $u \in K$ a solution of the variational inequality

$$(\forall v \in K) \int_\Omega (a \operatorname{grad} u | \operatorname{grad}(v-u)) + \int_\Omega (b|\operatorname{grad} u)(v-u) + \\ + \int_\Omega cu(v-u) \geq \int_\Omega f(v-u) \ , \tag{6}$$

with $f \in L^2(\Omega)$, such that $f^+ \neq 0$.

Let I the coincidence set of u.

We need some regularity condition on u.

We suppose $u \in C^1(\overline{\Omega})$; for this, it is sufficient to suppose the coefficients $a_{ij} \in C^1(\overline{\Omega})$, the bilinear form associated to L coercive and $f \in L^p(\Omega)$ with $p > N$ [12].

We also suppose that ∂I is a regular hypersurface of $\mathbf{R}^N$; this condition is more delicate; see [25], [13], [20], [8] for exemple, for this topic.

From the above hypotheses it follows at once:

$$(\forall v \in H_0^1(\Omega)) \int_\Omega (a \operatorname{grad} u | \operatorname{grad} v) + \int_\Omega (b|\operatorname{grad} u)v = \int_{\Omega - I} (f - cu)v \ . \tag{7}$$

In reality, as the reader will be able to observe, we need the above regularity conditions only in order that the equality in (7) is true for particular test functions.

So we could replace these hypotheses with (7).

The basic result of the paper, of which the other ones are direct consequence, is the following inequality on u_*^+:

$$u_*^+(s) \leq N^{-2} V_N^{-\frac{2}{N}} M^{-1} \int_s^{|\Omega|} \sigma^{\frac{2}{N}-2} \exp\left(V_N^{-\frac{1}{N}} M^{-1} B \sigma^{\frac{1}{N}}\right) \cdot \\ \cdot \left(\int_{|I|}^{\sigma} f_*^+(r - |I|) \exp\left(-V_N^{-\frac{1}{N}} M^{-1} B r^{\frac{1}{N}}\right) dr\right) d\sigma \ , \tag{8}$$

for every $s \in]|I|, |\Omega|]$.

This inequality leads on the consideration of the following functions:

$$\Psi : \{(t,\lambda) \in [0,|\Omega|] \times [0,|\Omega|]; t \geq \lambda\} \longrightarrow \overline{\mathbf{R}},\ t \longrightarrow$$
$$N^{-2} V_N^{-\frac{2}{N}} M^{-1} \int_t^{|\Omega|} \sigma^{\frac{2}{N}-2} \exp\left(V_N^{-\frac{1}{N}} M^{-1} B \sigma^{\frac{1}{N}}\right) \cdot$$
$$\cdot \left(\int_\lambda^\sigma f_*^+(r-\lambda) \exp\left(-V_N^{-\frac{1}{N}} M^{-1} B r^{\frac{1}{N}}\right)\, dr\right)\, d\sigma\ ,$$

and

$$\Phi :]0,|\Omega|] \longrightarrow \mathbf{R},\ t \longrightarrow \Psi(t,t)\ .$$

We call Φ *contact function*. For $B = 0$, Φ is the function considerated in [28].

As a direct consequence of (8) we obtain

$$k \geq \sup(\Phi) \Rightarrow |I| = 0 \qquad \text{and} \qquad k < \sup(\Phi) \Rightarrow |I| \leq \Phi^{-1}(k)\ .$$

From this, we give a sufficient condition for $|I| = 0$ in terms of L^p norms of f^+.

We compare u_*^+ with the decreasing rearrangements of the solutions of homogeneous Dirichlet problems with spherical symmetrical data.

Lastly we extend the results to an obstacle problem where the obstacle is constant only on the boundary of Ω.

3. An inequality on u_*^+.

The inequalty stated by the following theorem is the base for all other results. In the proof we applay to this case the methods of Talenti [32] and Alvino-Lions-Trombetti [4].

THEOREM 1. *Let u solution of (6), where K is fixed by (5); let u satisfy (7); then for every $s \in]|I|,|\Omega|]$ we have*

$$u_*^+(s) \leq N^{-2} V_N^{-\frac{2}{N}} M^{-1} \int_s^{|\Omega|} \sigma^{\frac{2}{N}-2} \exp\left(V_N^{-\frac{1}{N}} M^{-1} B \sigma^{\frac{1}{N}}\right) \cdot$$
$$\cdot \left(\int_{|I|}^\sigma f_*^+(r-|I|) \exp\left(-V_N^{-\frac{1}{N}} M^{-1} B r^{\frac{1}{N}}\right)\, dr\right)\, d\sigma\ . \quad (9)$$

Proof. Let $t \in]0,k[$. We have $(u-t)^+ \in H_0^1(\Omega)$. Choosing in (7) $v = (u-t)^+$, we find

$$\int_{\Omega,u>t} (a \operatorname{grad} u | \operatorname{grad} u) \quad + \quad \int_{\Omega,u>t} (b|\operatorname{grad} u)(u-t) + \\ + \quad \int_{\Omega-I,u>t} cu(u-t) = \int_{\Omega-I,u>t} f \cdot (u-t) \; .$$

Then for almost all $t \in]0,k[$ we have

$$-\frac{d}{dt} \int_{\Omega,u>t} (a \operatorname{grad} u | \operatorname{grad} u) = \\ = -\int_{\Omega,u>t} (b|\operatorname{grad} u) - \int_{\Omega-I,u>t} cu + \int_{\Omega-I,u>t} f \; . \qquad (10)$$

From the ellipticy condition (2), using the incremental ratios, we obtain

$$-M \frac{d}{dt} \int_{\Omega,u>t} |\operatorname{grad} u|^2 \leq -\frac{d}{dt} \int_{\Omega,u>t} (a \operatorname{grad} u | \operatorname{grad} u) \; . \qquad (11)$$

for almost all $t \in]0,k[$.

Now we find upper bounds for the terms of the second side of (10).

From (3), using the coarea fomula [21], we find

$$-\int_{\Omega,u>t} (b|\operatorname{grad} u) \leq B \int_{\Omega,u>t} |\operatorname{grad} u| \leq \\ \leq B \int_t^k \left(-\frac{d}{ds} \int_{\Omega,u>s} |\operatorname{grad} u| \right) ds \; . \qquad (12)$$

Using the incremental ratios, by Schwarz inequality, we find

$$-\frac{d}{ds} \int_{\Omega,u>s} |\operatorname{grad} u| \leq \left(-\frac{d}{ds} \int_{\Omega,u>s} |\operatorname{grad} u|^2 \right)^{\frac{1}{2}} (-\mu'_{u^+}(s))^{\frac{1}{2}} \; . \qquad (13)$$

Then from (12) we find

$$-\int_{\Omega,u>t} (b|\operatorname{grad} u) \leq \\ \leq B \int_t^k \left(-\frac{d}{ds} \int_{\Omega,u>s} |\operatorname{grad} u|^2 \right)^{\frac{1}{2}} (-\mu'_{u^+}(s))^{\frac{1}{2}} \, ds \; . \qquad (14)$$

By coarea formula [21] we find

$$-\frac{d}{ds}\int_{\Omega,u>s}|\mathrm{grad}\,u|\,ds = P_\Omega(u>s) = P_{\mathbf{R}^N}(u>s) \tag{15}$$

for almost all $s\in]0,k[$, where $P_\Omega(u>s)$ (resp. $P_{\mathbf{R}^N}(u>s)$ is De Giorgi perimeter of $\{x\in\Omega; u(x)>s\}$ relative to Ω (resp. $\mathbf{R}^N$).

By the isoperimetric inequality we have

$$NV_N^{\frac{1}{N}}(\mu_{u^+}(s))^{1-\frac{1}{N}} \le P_{\mathbf{R}^N}(u>s) \tag{16}$$

for almost all $s\in]0,k[$.

From (13), (15), (16) we find

$$NV_N^{\frac{1}{N}}(\mu_{u^+}(s))^{1-\frac{1}{N}} \le \left(-\frac{d}{ds}\int_{\Omega,u>s}|\mathrm{grad}\,u|^2\right)^{\frac{1}{2}}(-\mu'_{u^+}(s))^{\frac{1}{2}} \tag{17}$$

and therefore

$$\left(\frac{d}{ds}\int_{\Omega,u>s}|\mathrm{grad}\,u|^2\right)^{\frac{1}{2}}(-\mu'_{u^+}(s))^{\frac{1}{2}} \le$$
$$\le N^{-1}V_N^{-\frac{1}{N}}(\mu_{u^+}(s))^{\frac{1}{N}-1}(-\mu'_{u^+}(s))\left(-\frac{d}{ds}\int_{\Omega,u>s}|\mathrm{grad}\,u|^2\right)$$

for almost all $s\in]0,k[$.

Then from (14) we have

$$-\int_{\Omega,u>t}(b|\mathrm{grad}\,u) \le N^{-1}V_N^{-\frac{1}{N}}B\int_t^k(\mu_{u^+}(s))^{\frac{1}{N}-1}(-\mu'_{u^+}(s))\cdot$$
$$\cdot\left(-\frac{d}{ds}\int_{\Omega,u>s}|\mathrm{grad}\,u|^2\right)ds\,. \tag{18}$$

By (1) we have

$$\int_{\Omega-I,u>t}f \le \int_{\Omega-I,u>t}f^+ \le \int_0^{\mu_{u^+}(t)-|I|}f_*^+\,. \tag{19}$$

From (10), by (11), (18), (19) and (4), we find

$$-\tfrac{d}{dt}\textstyle\int_{\Omega,u>t}|\mathrm{grad}\,u|^2 \le N^{-1}V_N^{-\frac{1}{N}}M^{-1}B\cdot$$
$$\textstyle\int_t^k(\mu_{u^+}(s))^{\frac{1}{N}-1}(-\mu'_{u^+}(s))\cdot$$
$$\cdot\left(-\frac{d}{ds}\int_{\Omega,u>s}|\mathrm{grad}\,u|^2\right)ds + M^{-1}\int_0^{\mu_{u^+}(t)-|I|}f_*^+ \tag{20}$$

for almost all $t \in]0, k[$.

Applying to (20) Gronwall lemma, we find

$$-\frac{d}{dt}\int_{\Omega,u>t} |\mathrm{grad}\, u|^2 \le M^{-1}\int_0^{\mu_{u^+}(t)-|I|} f_*^+ +$$
$$+N^{-1}V_N^{-\frac{1}{N}}M^{-2}B\exp\left(V_N^{-\frac{1}{N}}M^{-1}B(\mu_{u^+}(t))^{\frac{1}{N}}\right)\cdot$$
$$\cdot\int_{|I|}^{\mu_{u^+}(t)}\left(\int_0^{\sigma-|I|} M^{-1}f_*^+\right)\sigma^{\frac{1}{N}-1}\exp\left(-V_N^{-\frac{1}{N}}M^{-1}B\sigma^{\frac{1}{N}}\right)d\sigma \quad (21)$$

for almost all $t \in]0, k[$.

Setting in the last integral $r = \sigma - |I|$ and integrating by parts, (21) becomes

$$-\frac{d}{dt}\int_{\Omega,u>t} |\mathrm{grad}\, u|^2 \le M^{-1}\exp\left(V_N^{-\frac{1}{N}}M^{-1}B(\mu_{u^+}(t))^{\frac{1}{N}}\right)\cdot$$
$$\cdot\int_0^{\mu_{u^+}(t)-|I|} f_*^+(r)\exp\left(-V_N^{-\frac{1}{N}}M^{-1}B(r+|I|)^{\frac{1}{N}}\right)dr\,.$$

Multiplying by $-\mu'_{u^+}(t)$ and using (17) we find

$$N^2V_N^{\frac{2}{N}}(\mu_{u^+}(t))^{2-\frac{2}{N}} \le M^{-1}\exp\left(V_N^{-\frac{1}{N}}M^{-1}B(\mu_{u^+}(t))^{\frac{1}{N}}\right)\cdot$$
$$\cdot\left(\int_0^{\mu_{u^+}(t)-|I|} f_*^+(r)\exp\left(-V_N^{-\frac{1}{N}}M^{-1}B(r+|I|)^{\frac{1}{N}}\right)dr\right)(-\mu'_{u^+}(t)) \quad (22)$$

for almost all $t \in]0, k[$.

Hence we have

$$-(u_*^+)'(s) \le N^{-2}V_N^{-\frac{2}{N}}M^{-1}s^{\frac{2}{N}-2}\exp\left(V_N^{-\frac{1}{N}}M^{-1}Bs^{\frac{1}{N}}\right)\cdot$$
$$\cdot\int_{|I|}^{s} f_*^+(r-|I|)\exp\left(-V_N^{-\frac{1}{N}}M^{-1}Br^{\frac{1}{N}}\right)dr \quad (23)$$

for almost all $s \in]|I|, |\Omega|]$.

By $u^+ \in H_0^1(\Omega)$ we have $u_*^+ \in C(]0, |\Omega|)$ and $u_*^+(|\Omega|) = 0$.

From (23), integrating on $[s, |\Omega|]$, we obtain (9).

Remark 1. We remark that in regular hypotheses, for exemple if $a_{i,j} \in C^1(\Omega)$, $u \in C^2(\Omega - I)$ and ∂I is a regular hypersurface, (9) may be obtained in a faster manner. In fact, setting for all $t \in [0, k]$

$$E_t = \{x \in \Omega; u(x) > t\} ,$$

on $E_t - I$ we have $L(u) = f$. For almost all $t \in [0, k]$, ∂E_t is a regular hypersurface of $\mathbf{R}^N$; so, integrating on $E_t - I$ and using $\operatorname{grad} u = 0$ on ∂I, we find

$$\int_{\partial E_t} (a \operatorname{grad} u | \frac{\operatorname{grad} u}{|\operatorname{grad} u|}) \, dH + \int_{E_t} (b | \operatorname{grad} u) + \int_{E_t - I} cu = \int_{E_t - I} f ,$$

where H is the canonic measure on ∂E_t.

From this we find

$$M \int_{\partial E_t} |\operatorname{grad} u| \, dH \leq \int_0^{\mu_{u^+}(t) - |I|} f_+^*(r) \, dr + B \int_{E_t} |\operatorname{grad} u| . \quad (24)$$

For almost all $s \in [t, k]$, we have

$$\mu'_{u^+}(s) = \int_{\partial E_s} \frac{1}{|\operatorname{grad} u|} \, dH ;$$

so we find

$$\int_{E_t} |\operatorname{grad} u| = \int_t^k P_{\mathbf{R}^N}(u > s) \, ds \leq$$

$$\leq \int_t^k \left(\int_{\partial E_s} |\operatorname{grad} u| \, dH \right)^{\frac{1}{2}} (-\mu'_{u^+}(s))^{\frac{1}{2}} \, ds . \quad (25)$$

By the isoperimetric inequality we have

$$N V_N^{\frac{1}{N}} (\mu_{u^+}(s))^{1 - \frac{1}{N}} \leq P_{\mathbf{R}^N}(u > s)$$

$$\leq \left(\int_{\partial E_s} |\operatorname{grad} u| \, dH \right)^{\frac{1}{2}} (-\mu'_{u^+}(s))^{\frac{1}{2}} . \quad (26)$$

From (25) and (26) we find

$$\int_{E_t} |\operatorname{grad} u| \leq N^{-1} V_N^{-\frac{1}{N}} \int_t^k \left(\int_{\partial E_s} |\operatorname{grad} u| \, dH \right)$$

$$(\mu_{u+}(s))^{\frac{1}{N}-1}(-\mu'_{u^+}(s))\,ds\,. \tag{27}$$

So from (24) and (27) we find

$$M\int_{\partial E_t}|\text{grad}\,u|\,dH \le \int_0^{\mu_{u^+}(t)-|I|} f_+^*(r)\,dr+$$
$$+N^{-1}V_N^{-\frac{1}{N}}B\int_t^k(\mu_{u+}(s))^{\frac{1}{N}-1}\left(\int_{\partial E_s}|\text{grad}\,u|\,dH\right)$$
$$(-\mu_{u^+})'(s))\,ds\,, \tag{28}$$

Applying to (28) Gronwall lemma, we find

$$M\int_{\partial E_t}|\text{grad}\,u|\,dH \le M^{-1}\exp\left(V_N^{-\frac{1}{N}}M^{-1}B(\mu_{u^+}(t))^{\frac{1}{N}}\right)\cdot$$
$$\cdot\int_0^{\mu_{u^+}(t)-|I|} f_*^+(r)\exp\left(-V_N^{-\frac{1}{N}}M^{-1}B(r+|I|)^{\frac{1}{N}}\right)\,dr \tag{29}$$

Multiplying by $-\mu'_{u^+}(t)$ and using again the isoperimetric inequality from (29) we find (22).

4. Consequences.

The results of this sections express relations between k and $|I|$; we find in particular an upper bound for $|I|$.

The results are generalisations of those of [28], that we obtain for $B = 0$.

COROLLARY 1. Let $|I| \neq 0$; then we have:

$$k \le N^{-2}V_N^{-\frac{2}{N}}M^{-1}\int_{|I|}^{|\Omega|}\sigma^{\frac{2}{N}-2}\exp\left(V_N^{-\frac{1}{N}}M^{-1}B\sigma^{\frac{1}{N}}\right)\cdot$$
$$\cdot\left(\int_{|I|}^{\sigma}f_*^+(r-|I|)\exp\left(-V_N^{-\frac{1}{N}}M^{-1}Br^{\frac{1}{N}}\right)\,dr\right)\,d\sigma\,. \tag{30}$$

Proof. By $\mu_{u^+}(|I|-) = k$ and continuity of u_*^+ at $|I|$, it follows that $u_*^+(|I|) = k$; then, choosing in (9) $s = |I|$, we obtain (30).

If we consider in the second side of (30) $|I|$ as a variable, we obtain the function Φ, that we have called *contact funtion* [28]. Hence relation (30) becomes

$$k \leq \Phi(|I|) \, . \tag{31}$$

From the hypothesis that $f^+ \neq 0$, it follows that the contact function is strictly decreasing.

In fact for every $t \in]0, |\Omega|]$, setting

$$D_t = \{(\sigma, s); t \leq \sigma \leq |\Omega|, 0 \leq s \leq \sigma - t\} \, ,$$

we have

$$\Phi(t) = N^{-2} V_N^{-\frac{2}{N}} M^{-1} \int \int_{D_t} \sigma^{\frac{2}{N}-2} \exp\left(V_N^{-\frac{1}{N}} M^{-1} B \sigma^{\frac{1}{N}}\right) f_*^+(s) \cdot$$

$$\cdot \exp\left(-V_N^{-\frac{1}{N}} M^{-1} B (s+t)^{\frac{1}{N}}\right) \, d\sigma ds \, ;$$

let $t', t \in]0, |\Omega|]$, $t' < t$; we have $D_t \subset D_{t'}$; so we have

$$\Phi(t') - \Phi(t) \geq N^{-2} V_N^{-\frac{2}{N}} M^{-1} \int \int_{D_{t'}} \sigma^{\frac{2}{N}-2} \exp\left(V_N^{-\frac{1}{N}} M^{-1} B \sigma^{\frac{1}{N}}\right) \cdot$$

$$\cdot f_*^+(s) \cdot$$

$$\cdot \left(\exp\left(-V_N^{-\frac{1}{N}} M^{-1} B (s+t')^{\frac{1}{N}}\right) - \exp\left(-M^{-1} B V_N^{-\frac{1}{N}} (s+t)^{\frac{1}{N}}\right)\right)$$

$$d\sigma ds \geq 0 \, ;$$

since $f^+ \neq 0$, the integral is not 0; so we have $\Phi(t') - \Phi(t) > 0$; this proves that Φ is strictly decreasing.

Assumed $\Phi :]0, |\Omega|] \longrightarrow [0, \sup(\Phi)[$, we can consider Φ^{-1}.

From (31) we obtain the following corollary.

COROLLARY 2. *We have:*

1. $k \geq \sup(\Phi) \Rightarrow |I| = 0$;

2. $k < \sup(\Phi) \Rightarrow |I| \leq \Phi^{-1}(k)$.

Proof. Let $k \geq \sup(\Phi)$; suppose $|I| \neq 0$; since Φ is strictly decreasing, by (31) we have

$$k \leq \Phi(|I|) < \lim_{t \to 0} \Phi(t) = \sup(\Phi) \, ;$$

so we have $k < \sup(\Phi)$, contradiction. This proves (1).

Let $k < \sup(\Phi)$. If $|I| = 0$, we have $|I| \leq \Phi^{-1}(k)$. Supposed $|I| \neq 0$, by (31) we have $k \leq \Phi(|I|)$, and then $|I| \leq \Phi^{-1}(k)$. This proves (2).

As in [28], by Hölder inequality, the condition $k \geq \sup(\Phi)$, may be easily obtained for $p > \frac{N}{2}$ as conseguence of inequalities on L^p norms of f_+.

COROLLARY 3. *Let $p > \frac{N}{2}$; let $f_+ \in L^p(\Omega)$; let*

$$\|f_+\|_p \leq kNV_N^{\frac{2}{N}} Mp^{-1}(2p-N) \exp\left(-V_N^{-\frac{1}{N}} M^{-1} B|\Omega|^{\frac{1}{N}}\right) |\Omega|^{\frac{Np}{2p-N}} \ ;$$

then we have $|I| = 0$.

Remark 2. The second side of (9) may be related to the solution of a homogeneous Dirichlet problem with spherical symmetrical data.

For this, we set

$$R = \sqrt[N]{|\Omega|/V_N} \qquad \text{and} \qquad \tilde{\Omega} = \mathrm{B}(0,R) \ ,$$

so that $|\Omega| = |\tilde{\Omega}|$.

For every $\lambda \in [0, |\Omega|]$, we set

$$U_\lambda : \tilde{\Omega} \longrightarrow \overline{\mathbf{R}}, x \longrightarrow \begin{cases} \Psi(\lambda,\lambda) & V_N|x|^N \leq \lambda \\ \Psi(V_N|x|^N,\lambda) & \lambda < V_N|x|^N < R \end{cases}$$

and

$$f_\lambda : \tilde{\Omega} \longrightarrow \overline{\mathbf{R}}, x \longrightarrow \begin{cases} 0 & V_N|x|^N \leq \lambda \\ f_*^+(V_N|x|^N - \lambda) & \lambda < V_N|x|^N < R \end{cases}$$

For every $\lambda \in [0, |\Omega|]$ we have $U_\lambda \in H_0^1(\tilde{\Omega})$ and U_λ is the solution of the homogeneous Dirichlet problem

$$(\forall v \in H_0^1(\tilde{\Omega}))\ M \int_{\tilde{\Omega}} (\operatorname{grad} U_\lambda | \operatorname{grad} v) + \\ + B \int_{\tilde{\Omega}} (\frac{x}{|x|} | \operatorname{grad} U_\lambda(x)) v(x)\, dx = \int_{\tilde{\Omega}} f_\lambda v$$

[32].

Furthermore the decreasing rearrangement of U_λ is related to Φ; precisely we have

$$U_{\lambda *}(s) = \begin{cases} \Phi(\lambda) & 0 < s \leq \lambda \\ \Phi(s) & \lambda < s < |\Omega| \end{cases} .$$

As $U_\lambda \in H^2(\tilde{\Omega})$, we also note that, for $\lambda \neq 0$, U_λ is solution of the constant obstacle problem with spherical symmetrical data

$$\tilde{K} = \{v \in H_0^1(\tilde{\Omega}); v \leq \tilde{\Phi}(\lambda)\} ,$$
$$(\forall v \in \tilde{K})\ M \int_{\tilde{\Omega}} (\operatorname{grad} U | \operatorname{grad}(v - U)) +$$
$$+ B \int_{\tilde{\Omega}} \left(\frac{x}{|x|} | \operatorname{grad} U(x)\right)(v(x) - U(x))\, dx \geq \int_{\tilde{\Omega}} g_\lambda (v - U) ,$$

where $g_\lambda \in L^2([0, |\Omega|])$, $g_\lambda \geq 0$, $g_\lambda(x) = f_\lambda(x)$ for $\lambda < V_N |x|^N < R$, $g_\lambda(x)$ arbitrary for $V_N |x|^N \leq \lambda$.

As $f^+ \neq 0$, the coincidence set of U_λ is $\mathrm{B}'(0, \sqrt[N]{\lambda / V_N})$, whose measure in λ.

It is now possible to interpret (9) as an inequality between decreasing rearrangements of solutions of constant obstacle problems.

In fact for $\lambda = |I|$, (9) gives at once

$$(\forall s \in]0, |\Omega|]\ u_*^+(s) \leq U_{|I|*}(s) .$$

We remark that we compare the decreasing rearrangements of the solutions of two constant obstacle problems with different value of the constants.

We have

$$0 \leq \lambda_1 \leq \lambda_2 \leq |\Omega| \Rightarrow (\forall s \in]0, |\Omega|)\ U_{\lambda_2 *}(s) \leq U_{\lambda_1 *}(s) .$$

In fact, let $s \in [\lambda_2, |\Omega|]$; let us prove $U_{\lambda_2 *}(s) \leq U_{\lambda_1 *}(s)$ i. e.

$$\int_{\lambda_1}^{\sigma} \exp\left(-V_N^{-\frac{1}{N}} M^{-1} B r^{\frac{1}{N}}\right) f_*^+(r - \lambda_1)\, dr \geq$$
$$\geq \int_{\lambda_2}^{\sigma} \exp\left(-V_N^{-\frac{1}{N}} M^{-1} B r^{\frac{1}{N}}\right) f_*^+(r - \lambda_2)\, dr \qquad (32)$$

for every $\sigma \in [s, |\Omega|]$; let

$$\psi :]0, \sigma[\longrightarrow \mathbf{R}, \lambda \longrightarrow \int_{\lambda}^{\sigma} \exp\left(-V_N^{-\frac{1}{N}} M^{-1} B r^{\frac{1}{N}}\right) f_*^+(r - \lambda)\, dr ;$$

from

$$\psi(\lambda) = \int_0^{\sigma-\lambda} \exp\left(-V_N^{-\frac{1}{N}} M^{-1} B(r'+\lambda)^{\frac{1}{N}}\right) f_*^+(r')\, dr'$$

it follows at once that ψ is decreasing; this means (32). Since $U_{\lambda_1 *}$ (resp. $U_{\lambda_2 *}$) is equal to $\Phi(\lambda_1)$ (resp. $\Phi(\lambda_2)$) on $]0, \lambda_1]$ (resp. $]0, \lambda_2]$), we have $U_{\lambda_2 *}(s) \leq U_{\lambda_1 *}(s)$ for all $s \in]0, |\Omega|]$.

So for $0 \leq \lambda \leq |I|$ we have

$$(\forall s \in]0, |\Omega|]) \; u_*^+(s) \leq U_{\lambda *}(s) \; .$$

In particular we have

$$(\forall s \in]0, |\Omega|]) \; u_*^+(s) \leq U_{0 *}(s) \; .$$

Remark 3. We may extend the previous results to an obstacle problem where the obstacle is constant only on the boundary of Ω.

Let $\psi : \Omega \longrightarrow \mathbf{R}$ such that ψ is continuous, $\psi - k \in H_0^1(\Omega)$ and $L(\psi - k) \in L^2(\Omega)$; let

$$\hat{K} = \{v \in H_0^1(\Omega); v \leq \psi\}$$

the related closed, convex subset of $H_0^1(\Omega)$.

Let $\hat{u} \in \hat{K}$ a solution of the variational inequality obtained from (6) replacing K with $\hat{K}$. Denoting by $\hat{I}$ its coincidence set, we suppose

$$(\forall v \in H_0^1(\Omega)) \int_\Omega (a \operatorname{grad}(u - \psi + k) | \operatorname{grad} v) +$$

$$+ \int_\Omega (b | \operatorname{grad}(u - \psi + k)) v = \int_{\Omega - I} (f - L(\psi - k) - cu) v \; .$$

Let

$$f_0 = f - L(\psi - k) \qquad \text{and} \qquad u_0 = \hat{u} - \psi + k \; .$$

Then $u_0 \in K$ and u_0 is solution of the variational inequality obtained from (6) replacing f with f_0; moreover $\hat{u}$ and u_0 have the same coincidence set.

So, denoted $\hat{\Phi}$ the contact function corresponding to $\hat{f}^+$, we have

1. $|\hat{I}| \neq 0 \Rightarrow k \leq \hat{\Phi}(|\hat{I}|)$;
2. $k \geq \sup(\hat{\Phi}) \Rightarrow |\hat{I}| = 0$;
3. $k < \sup \hat{\Phi} \Rightarrow |\hat{I}| \leq \Phi^{-1}(k)$.

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