HOLOMORPHIC PROJECTIVE MAPPINGS (*)

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Sommario. - *Si caratterizzano le trasformazioni di Moebius olomorfe fra tutte le applicazioni olomorfe di dischi aperti di spazi di Banach di funzioni continue.*

Summary. - *Holomorphic Moebius transformations are characterized within the set of all holomorphic maps acting on open balls in Banach spaces of continuous functions.*

Let B be a bounded domain in a complex Banach space $\mathcal{E}$, and let d be the Carathéodory distance in B. By definition [6], a complex geodesic for d is a holomorphic map ϕ of the open unit disc Δ of $\mathbf{C}$ into B such that, for all $\zeta_1, \zeta_2 \in \Delta$, $d(\phi(\zeta_1), \phi(\zeta_2))$ equals the Poincaré distance of ζ_1 and ζ_2 in Δ. The closed set $\phi(\Delta) \subset B$ will be called the support of the complex geodesic ϕ.

Since holomorphic maps contract the Carathéodory distance, the set G of all complex geodesics for d is not preserved under the action of the semigroup $\mathrm{Hol}(B, B)$ of all holomorphic maps of B into B. Although G is mapped into itself by the semigroup $\mathrm{Iso}(B)$ of all holomorphic isometries for d, the fact that the set G may be empty (as, for instance, when B is a non-simply connected bounded domain in $\mathbf{C}$) already indicates that the condition whereby $F \in \mathrm{Hol}(B, B)$ maps G into itself does not necessarily imply that $F \in \mathrm{Iso}(B)$. This implication holds, however, for domains of particular type: for instance, when $B = B(\mathcal{H})$ is the open unit ball of a complex Hilbert space $\mathcal{H}$. In this case the conclusion follows rather easily from the fact that the complex geodesics in $B(\mathcal{H})$ are the restrictions to Δ of affine maps of $\mathbf{C}$ into $\mathcal{H}$, implying that any two distinct points

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of $B(\mathcal{H})$ are contained in the support of an essentially unique complex geodesic for d. Actually a more subtle argument shows that $F \in \mathrm{Hol}(B(\mathcal{H}), B(\mathcal{H}))$ is an isometry under the weaker hypothesis whereby there exists some point $x \in B(\mathcal{H})$ such that $F \circ \phi$ is a complex geodesic for every complex geodesic ϕ whose support contains x.

A further example - where $B = B(T)$ is the open unit ball of the complex Banach space $C(T)$ of all complex-valued continuous functions on a compact Hausdorff space T (containing more than one point) - presents a radically different geometric situation. First of all, the set G is very large, and the behavior of the complex geodesics in $B(T)$ is quite different from the essential uniqueness one encounters in the case of $B(\mathcal{H})$. However, given any compact Hausdorff space T, a proper subset $G' \subset G$ can be intrisecally defined in terms of the extreme points of the closure $\overline{B(T)}$ of $B(T)$, such that the inclusion $F(G') \subset G'$ implies that $F \in \mathrm{Iso}(B(T))$ [8]. Examples show that being $F \in \mathrm{Iso}(B(T))$ does not imply the inclusion $F(G') \subset G'$.

For both the open unit balls $B(\mathcal{H})$ and $B(T)$, the elements of $\mathrm{Iso}(B)$ are (vector-valued) Moebius transformations mapping B into itself. Furthermore, $\mathrm{Iso}(B)$ turns out to be properly contained in the semigroup of all Moebius transformations mapping B into itself. This fact raises the problem of characterizing this larger semigroup within $\mathrm{Hol}(B, B)$. The answer to this question seems to be strictly related to the following problem: which are the elements $F \in \mathrm{Hol}(B, B)$ that map into itself the set $\tilde{G}$ of supports of all complex geodesics of B?

In the case of $B(\mathcal{H})$, it was shown in [7] that, if $F(\tilde{G}) \subset \tilde{G}$ and if the restriction of F to the support of any complex geodesic is a (scalar-valued) Moebius transformation, then F is represented by a Moebius transformation $B(\mathcal{H}) \to B(\mathcal{H})$. Since conversely any such transformation satisfies both the above conditions, the Moebius transformations acting on $B(\mathcal{H})$ are thus completely characterized within $\mathrm{Hol}(B(\mathcal{H}), B(\mathcal{H}))$ in terms of their behaviour on the supports of the complex geodesics.

The present paper is devoted to the solution of the problem for the ball $B = B(T)$. As in the case of $\mathrm{Iso}(B(T))$, the role of the supports of the elements of G will now be played by the supports of the complex geodesics in G'. The main result will be more easily stated in the case in which $F(0) = 0$. It will be shown that, if:

$dF(0) \in \mathcal{L}(C(T), C(T))$ is injective;

for any extreme point u of $\overline{B(T)}$, $F(\zeta u)$ is collinear to some extreme point of $\overline{B(T)}$ for all ζ in a neighborhood of 0 in Δ,

then there is a continuous linear form λ on $C(T)$, with $\|\lambda\| < 1$, such that

$$F(x) = \frac{1}{1 - \lambda(x))} dF(0)(x)$$

for all $x \in B(T)$.

Remark. It is worth noticing at this point that similar questions to the ones investigated in [7] and in the present paper arise also in differential geometry, when the domain B is replaced by a (finite dimensional) differentiable manifold M endowed with a symmetric linear connection in its tangent bundle. A diffeomorphism $F : M \to M$ preserving the geodesics and inducing affine transformations of the affine parameters of the geodesics is called an affine transformation. More in general, F is called a projective transformation, or a projective collineation, if it is only required to preserve the geodesics (without any restriction on its behavior on the parameters). The projective transformations have first come to the attention of H.Weyl in 1921, and were intensely investigated in the thirties; see [3] and [2] also for a comprehensive list of references. Among the articles listed in [2], a paper by L.Berwald [1] seems to be close to the topics of [7] and of the present article. In that paper an intrinsic *normal projective parameter* is defined on the geodesics, on which the projective transformations of M induce (real-valued) Moebius transformations.

1. Let T and S be compact Hausdorff spaces, let $B(T)$ and $B(S)$ be the open unit balls of the complex Banach spaces $C(T)$ and $C(S)$ of all continuous, complex-valued functions on T and on S. Denoting by $\Gamma(T)$ and $\Gamma(S)$ the sets of all complex extreme points of the closures $\overline{B(T)}$ and $\overline{B(S)}$, it is easily seen that

$$\Gamma(T) = \{v \in \overline{B(T)} : |v(t)| = 1 \text{ for all } t \in T\},$$

$$\Gamma(S) = \{u \in \overline{B(S)} : |u(s)| = 1 \text{ for all } s \in S\},$$

and that any point of $\Gamma(T)$ or of $\Gamma(S)$ is a real extreme point of $\overline{B(T)}$ and $\overline{B(S)}$ respectively.

Let $F \in \operatorname{Hol}(B(S), B(T)) \setminus \{0\}$ be such that $F(0) = 0$, and that, if $u \in \Gamma(S)$, then there is an open neighborhood U of 0 in the open unit disc Δ of $\mathbf{C}$ such that, if $\zeta \in U$, $F(\zeta u)$ is (0 or) collinear to some point of $\Gamma(T)$. This condition is equivalent to requiring that

$$|F(\zeta u)(t_1)| = |F(\zeta u)(t_2)| \tag{1}$$

for all $t_1, t_2 \in T$ and all $\zeta \in U$. By Harris' maximum principle (Theorem 9 of [4]), $F(\zeta u) \neq 0$ for some $u \in \Gamma(S)$.

For $u \in B(S)$, the fact that $u \in \Gamma(S)$ amounts to the existence of a point $s_o \in S$ at which $|u(s_o)| = 1$, and of a continuous function $\theta : S \to \mathbf{R}$ such that $\theta(s_o) = 0$ and

$$u(s) = \mathrm{e}^{i\theta(s)} u(s_0) \tag{2}$$

for all $s \in S$. Similarly, condition (1) amounts to the existence, for every continuous function $\theta : S \to \mathbf{R}$ with $\theta(s_o) = 0$, and for any $t_o \in T$, of a continuous function $\phi : T \times \Delta \to \mathbf{R}$ such that $\phi(t_o, \Delta) = \{0\}$ and that, if $u \in \Gamma(S)$ is expressed by (2), then there is an open neighborhood U of 0 in Δ for which

$$F(\zeta u)(t) = \mathrm{e}^{i\phi(t,\zeta)} F(\zeta u)(t_o)$$

for all $t \in T$ and all $\zeta \in U$. This equality implies that $\zeta \mapsto \phi(t, \zeta)$ is holomorphic on U, and therefore - being $\phi(t, \zeta) \in \mathbf{R}$ - then $\phi(t, \zeta)$ does not depend on ζ. Thus, the above condition can be re-stated in the following way:

Assumption 1. *Choose $t_o \in T$. For every constant $c \in \mathbf{C}$ with $|c| = 1$, and for every continuous function $\theta : S \to \mathbf{R}$, there is a continuous function $\phi : T \to \mathbf{R}$, such that $\phi(t_o) = 0$, and*

$$F(\zeta \mathrm{e}^{i\theta} c)(t) = \mathrm{e}^{i\phi(t)} F(\zeta \mathrm{e}^{i\theta} c)(t_o) \tag{3}$$

for all $t \in T$ and all $\zeta \in U$.

Let

$$F(x) = P_1(x) + P_2(x) + \cdots \tag{4}$$

be the power-series expansion of F in $B(S)$, where $P_\nu : C(S) \to C(T)$ is a continuous, homogeneous polynomial of degree $\nu = 1, 2, \ldots$, and

$P_1 = dF(0) \in \mathcal{L}(C(S), C(T))$ is the differential of F at 0. Then (3) is equivalent to

$$P_\nu(\mathrm{e}^{i\theta})(t) = \mathrm{e}^{i\phi(t)} P_\nu(\mathrm{e}^{i\theta})(t_o) \tag{5}$$

for all $\nu = 1, 2, \ldots$, and all $t \in T$.

For every $t \in T$ the continuous linear form $x \mapsto (P_1 x)(t)$ is represented by a unique complex, regular Borel measure μ_t on S:

$$(P_1 x)(t) = \int_S x \, d\mu_t := (x, \mu_t).$$

For $\nu = 1$, (5) reads

$$(\mathrm{e}^{i\theta}, \mu_t) = \mathrm{e}^{i\phi(t)} (\mathrm{e}^{i\theta}, \mu_{t_o}) \tag{6}$$

for all $\theta : S \to \mathbf{R}$ continuous and such that $\theta(s_0) = 0$.

In particular, $|(\mathrm{e}^{i\theta}, \mu_t)|$ is independent of $t \in T$, and $P_1 = 0$ if, and only if, $\mu_t = 0$ for some, hence for all, $t \in T$.

Since, by the maximum principle,

$$\begin{aligned} |\mu_t|(S) &= \sup\{|(x, \mu_t)| : x \in \overline{B(S)}\} \\ &= \sup\{|(P_1 x)(t)| : x \in \overline{B(S)}\} \\ &= \sup\{|(P_1 u)(t)| : u \in \Gamma(S)\}, \end{aligned}$$

then, by (6), $\|P_1\| = |\mu_t|(S)$ for every $t \in T$. Note that the total variation $\|\mu_t\| = |\mu_t|(S)$ of μ_t is independent of $t \in T$.

Let $f_{\nu,t,t_o} \in \mathrm{Hol}(C(S), \mathbf{C})$ be defined on $x \in B(S)$ by

$$f_{\nu,t,t_o}(x) = (x, \mu_{t_o}) P_\nu(x)(t) - (x, \mu_t) P_\nu(x)(t_o),$$

for $\nu = 2, 3, \ldots,\ t, t_o \in T$, then, by (5) and (6), $f_{\nu,t,t_o}(u) = 0$ for all $u \in \Gamma(S)$. The maximum principle yields then $f_{\nu,t,t_o} = 0$. That proves

LEMMA 1. *If F satisfies Assumption 1 and if $x \in C(S)$, then*

$$(x, \mu_{t_o}) P_\nu(x)(t) = (x, \mu_t) P_\nu(x)(t_o) \tag{7}$$

for all $t, t_o \in T$, $\nu = 2, 3, \ldots$.

It will be assumed henceforth that $dF(0) = P_1$ is injective.

Let $t \in T$. For any $x \in C(S) \setminus \{0\}$ there is some $t' \in T$ such that $(x, \mu_{t'}) \neq 0$ and there is a neighborhood W of x such that $(z, \mu_{t'}) \neq 0$ for all $z \in W$. Then, by Lemma 1, the function

$$x \mapsto \frac{P_\nu(x)(t)}{(x, \mu_t)} \tag{8}$$

is a scalar-valued holomorphic function on $C(S)$. Since its restriction to the complex line $\zeta \mapsto \zeta x$ is a homogeneous polynomial of degree $\nu - 1$, all differentials of degree $\geq \nu$ of (8) at 0 vanish. Hence, the function (8) is a continuous homogeneous polynomial $Q_{t,\nu-1} : C(S) \to \mathbf{C}$, and

$$P_\nu(x)(t) = (x, \mu_t) Q_{t,\nu-1}(x).$$

By (7), $Q_{t,\nu-1}(x) = Q_{t_o,\nu-1}(x)$ for all $x \in C(S)$ and all $t \in T$, i.e. $Q_{t,\nu-1}(x)$ does not depend on $t \in T$. Hence there is a continuous homogeneous polynomial of degree $\nu - 1 = 1, 2, \ldots$, $R_{\nu-1} : C(S) \to \mathbf{C}$, such that $P_\nu(x)(t) = (x, \mu_t) R_{\nu-1}(x)$, i.e.

$$P_\nu(x) = R_{\nu-1}(x) dF(0)(x)$$

for all $x \in C(S)$ and $\nu = 2, 3, \ldots$. Then, by (4), F is represented by

$$F(x) = (1 + R_1(x) + R_2(x) + \cdots) dF(0)(x) \tag{9}$$

for all $x \in B(S)$. Suppose now that, for every $t \in T$, there is a Moebius transformation

$$\zeta \mapsto \frac{\alpha(t)\zeta}{1 - \beta(t)\zeta}$$

mapping Δ into Δ, such that, if $u \in \Gamma(S)$, then

$$F(\zeta u)(t) = \frac{\alpha(t) u(t) \zeta}{1 - \beta(t) u(t) \zeta} \tag{10}$$

for all $t \in T$ and all ζ in a neighborhood of 0 in Δ. Since, for every $t \in T$,

$$F(\zeta u)(t) = \alpha(t) u(t) \zeta \{1 + (\beta(t) u(t) \zeta) + (\beta(t) u(t) \zeta)^2 + \cdots\} \ ,$$

then (9) yields

$$\begin{aligned} (u,\mu_t) &= \alpha(t)u(t) \\ R_\nu(u) &= (R_1(u))^\nu \end{aligned}$$

for $\nu = 2,3\dots$ and all $u \in \Gamma(S)$, $t \in T$. By the maximum principle, $(x,\mu_t) = \alpha(t)x(t)$ and

$$R_\nu(x) = (R_1(x))^\nu$$

for all $x \in C(S)$, $t \in T$. Thus (9) reads now

$$F(x)(t) = (1 + R_1(x) + (R_1(x))^2 + \cdots)(x,\mu_t). \tag{11}$$

Furthermore

$$\|R_1\| < 1. \tag{12}$$

To prove this latter inequality, note first that, by the maximum principle,

$$\|R_1\| = \sup\{\|R_1(x)\| : x \in \overline{B(S)}\} = \sup\{\|R_1(u)\| : u \in \Gamma(S)\}.$$

If $\|R_1\| \geq 1$, for every $\sigma \in (0,1)$ there exists some $u \in \Gamma(S)$ such that $R_1(u) \geq 1-\sigma$. Since $(1-\sigma)u \in B(S)$, then $R_1((1-\sigma)u) \geq (1-\sigma)^2$. Hence, for any $t_o \in T$,

$$\begin{aligned} |F((1-\sigma)u)(t_o)| &\geq (1-\sigma)|(u,\mu_{t_o})|(1 + (1-\sigma) + (1-\sigma)^2 + \cdots) = \\ &= \frac{1-\sigma}{\sigma}|(u,\mu_{t_o})| \ . \end{aligned} \tag{13}$$

Since $dF(0)$ is injective, and $|dF(0)(u(t))| = |(u,\mu_t)|$ does not depend on $t \in T$, then $(u,\mu_{t_o}) \neq 0$. Letting $\sigma \to 0$, then, by (13),

$$\lim_{\sigma\to 0}|F((1-\sigma)u)(t_o)| = +\infty$$

contradicting the fact that $F(B(S)) \subset B(T)$, and thereby proving (12). If ω is the complex, regular Borel measure on S representing the continuous linear form R_1 on $C(S)$, then (11) becomes

$$F(x) = \frac{1}{1-(x,\omega)}dF(0)(x) \tag{14}$$

for all $x \in B(S)$. In conclusion, the following theorem has been established.

THEOREM 2. *Let* $F \in \mathrm{Hol}(B(S), B(T))$ *be such that* $F(0) = 0$ *and that* $dF(0)$ *is injective. If, for every* $u \in \Gamma(S)$, *there exists a neighborhood* V *of* 0 *in* Δ *such that:*

1) *for all* $\zeta \in V$, $F(\zeta u)$ *is collinear to some point in* $\Gamma(T)$*;*

2) *the restriction of* F *to* $Vu = \{\zeta u : \zeta \in V\}$ *is a scalar Moebius transformation given by (10),*

then, there is a complex, regular Borel measure ω *on* S, *with*

$$\|\omega\| < 1, \tag{15}$$

such that F *is expressed by* (14) *for all* $x \in B(S)$.

Conversely, if $R \in \mathcal{L}(C(S), C(T))$ is injective, with $\|R\| \leq 1$, if $|R(u)(t)|$ is independent of $t \in T$ for all $u \in \Gamma(S)$, and if ω is a complex, regular Borel measure on S satisfying (15), then the function F defined on $B(S)$ by

$$F(x) = \frac{1}{1 - (x, \omega)} R(x)$$

is a holomorphic map of $B(S)$ into $C(T)$. If $F(B(S)) \subset B(T)$, then F fulfils all the hypotheses of Theorem 1.

The function F has a (unique) continuous extension to $\overline{B(S)}$.

EXAMPLE. If $T = S$ consists of two points, $C(T) = C(S)$ can be canonically identified with $\mathbf{C}^2$ endowed with the norm $\|(z_1, z_2)\| = \max\{|z_1|, |z_2|\}$, and $B(T) = B(S)$ is the open bi-disc $\Delta \times \Delta$. The set $\Gamma(T)$ is the two-dimensional torus $\mathbf{T} = \{(\mathrm{e}^{i\alpha_1}, \mathrm{e}^{i\alpha_2}) : \alpha_1, \alpha_2 \in \mathbf{R}\}$.

Let $F \in \mathrm{Hol}(\Delta \times \Delta, \Delta \times \Delta)$ satisfy all the hypotheses of Theorem 2. The fact that $dF(0)$ is injective and that (6) holds is equivalent to $dF(0)$ being represented by the 2×2 complex matrix

$$dF(0) = \begin{pmatrix} a & b \\ \mathrm{e}^{i\gamma}\overline{b} & \mathrm{e}^{i\gamma}\overline{a} \end{pmatrix},$$

where $\gamma \in \mathbf{R}$ and $|a| \neq |b|$. Moreover, since $\|dF(0)\| = |a| + |b|$, then

$$|a| + |b| < 1. \tag{16}$$

The measure ω is represented by the linear form on $\mathbf{C}^2$, $(z_1, z_2) \mapsto cz_1 + dz_2$, and (15) becomes now

$$|c| + |d| < 1. \tag{17}$$

In conclusion, if F fixes the origin, if $\det(dF(0)) \neq 0$, and if, for every $u \in \mathbf{T}$ there is an open neighborhood V of 0 in Δ such that, for all $\zeta \in V$, $F(\zeta u)$ is collinear to some point in $\mathbf{T}$, then F is expressed by the Moebius transformation defined on $(z_1, z_2) \in \Delta \times \Delta$ by

$$F(z_1, z_2) = \frac{1}{(1-(cz_1+dz_2))}\begin{pmatrix} az_1+bz_2 \\ \mathrm{e}^{i\gamma}(\overline{b}z_1+\overline{a}z_2) \end{pmatrix}$$

where $\gamma \in \mathbf{R}$ and a, b, c, d are complex numbers satisfying conditions (16), (17) and $|a| \neq |b|$.

2. A complex geodesic in $B(S)$ is a holomorphic map ϕ of Δ into $B(S)$ such that, for all ζ_1, ζ_2 in Δ, the Poincaré distance of ζ_1 and ζ_2 in Δ equals the Carathéodory-Kobayashi distance of $\phi(\zeta_1)$ and $\phi(\zeta_2)$ in $B(S)$.

For $\tau \in \mathbf{C}$ and $\zeta \in \Delta$, let $<\tau>_\zeta$ be the lenght of τ for the Poincaré differential metric of Δ at the point ζ. For $x \in B(S)$, let $\kappa_{B(S)}(x;\cdot)$ be the Carathéodory-Kobayashi differential metric of $B(S)$ at the point x. The map $\phi \in \mathrm{Hol}(\Delta, B(S))$ is a complex geodesic if (and only if) [6] there is some $\zeta \in \Delta$ such that

$$\kappa_{B(S)}(\phi(\zeta);\phi'(\zeta)) = <1>_\zeta .$$

For $u \in \partial B(S)$, the maximum principle implies that the map $\phi_u : \zeta \mapsto \zeta u$ is a complex geodesic. The strong maximum principle [5] yields

LEMMA 3. *If $u \in \Gamma(S)$ and if $\psi \in \mathrm{Hol}(\Delta, B(S))$ is such that $\psi(0) = 0$ and $\psi'(0) = u$, then $\psi = \phi_u$.*

Since the group $\mathrm{Aut}(B(S))$ of all holomorphic automorphisms of $B(S)$ acts transitively, the condition $\psi(0) = 0$ can be removed. For $x \in B(S)$, the map $H \in \mathrm{Hol}(B(S), B(S))$ defined on $y \in B(S)$ and $s \in S$ by

$$H(y)(s) = \frac{y(s)+x(s)}{1+\overline{x(s)}y(s)}, \tag{18}$$

is a holomorphic automorphism of $B(S)$ such that $H(0) = x$.

Since $\kappa_{B(S)}$ is invariant under the action of H, the function $\phi \in \mathrm{Hol}(\Delta, B(S))$ defined by

$$\phi := H \circ \phi_u : \zeta \mapsto \frac{\zeta u + x}{1 + \overline{x}\zeta u}$$

is a complex geodesic for which $\phi(0) = x$ and $\phi'(0) = (1 - |\phi(0)|^2)u$. Hence

$$\frac{\|\phi'(0)\|}{1 - |\phi(0)|^2} = 1.$$

In conclusion, the following lemma holds.

Lemma 4. *Let $\phi \in \mathrm{Hol}(\Delta, B(S))$. If ϕ is a complex geodesic, then*

$$\frac{\|\phi'(\zeta)\|}{1 - |\phi(\zeta)|^2} = \frac{1}{1 - |\zeta|^2}$$

for all $\zeta \in \Delta$. If this equality holds for some $\zeta \in \Delta$, then ϕ is a complex geodesic in $B(S)$.

Moreover, if

$$(1 - |\zeta|^2)\frac{\phi'(\zeta)}{1 - |\phi(\zeta)|^2} \in \Gamma(B(S)) \tag{19}$$

for some $\zeta \in \Delta$, then (19) *holds for all $\zeta \in \Delta$, and, if $\psi \in \mathrm{Hol}\,(\Delta, B(S))$ is such that $\psi(\zeta) = \phi(\zeta)$ and $\psi'(\zeta) = \phi'(\zeta)$ for some $\zeta \in \Delta$, then $\psi = \phi$.*

Remark. If, for a given $x \in B(S)$, there is a function $\phi \in \mathrm{Hol}(\Delta, B(S))$ for which

$$\phi(\zeta) = x \tag{20}$$

and (19) holds for some $\zeta \in \Delta$, then, composing ϕ on the right by a suitable holomorphic automorphism of Δ, one obtains a complex geodesic $\lambda \in \mathrm{Hol}(\Delta, B(S))$ such that

$$\lambda(0) = x \tag{21}$$

and

$$\frac{\lambda'(0)}{1 - |x|^2} \in \Gamma(B(S)). \tag{22}$$

In other words, there is no restriction in assuming $\zeta = 0$ in (19).

For $x \in B(S)$, let $\Xi(x, B(S))$ be the set of all functions $\lambda \in \mathrm{Hol}(\Delta, B(S))$ satisfying (21) and (22). Any such λ is a complex geodesic in $B(S)$. Furthermore, denoting by $M_\zeta \in \mathrm{Aut}(\Delta)$ the Moebius transformation defined on $\tau \in \Delta$ by

$$M_\zeta(\tau) = \frac{\tau - \zeta}{1 - \overline{\zeta}\tau},$$

in view of the Remark, the correspondence which associates to any $\lambda \in \mathrm{Hol}(\Delta, B(S))$ the function $\phi = \lambda \circ M_\zeta$ defines a bijective map of $\Xi(x, B(S))$ onto the set of all $\phi \in \mathrm{Hol}(\Delta, B(S))$ satisfying (20) and (19). Thus the set $\Xi(x, B(S))$ can be defined also by requiring that (19) and (20) be fulfilled.

In view of Lemma 3, for $x = 0$ (21) and (22) establish a bijective correspondence between $\Xi(0, B(S))$ and $\Gamma(S)$. Any $L \in \mathrm{Aut}(B(S))$ maps bijectively $\Xi(x, B(S))$ onto $\Xi(L(x), B(S))$. In particular, the function $\phi \mapsto H \circ \phi$ maps bijectively $\Xi(0, B(S))$ onto $\Xi(x, B(S))$.

Let $G \in \mathrm{Hol}(B(S), B(T))$. The following theorem is a direct consequence of the above considerations and of Theorem 1. For $x \in B(S)$, let $K \in \mathrm{Aut}(B(T))$ be defined on $z \in B(T)$ by

$$K(z) = \frac{z - G(x)}{1 - \overline{G(x)}z}.$$

THEOREM 5. *If there is some $x \in B(S)$ such that:*

$dG(x) \in \mathcal{L}(C(S), C(T))$ is injective;

for every $\lambda \in \Xi(x, B(S))$, there is an open neighborhood V of 0 *in Δ such that, if $\zeta \in V$, $(K \circ G \circ H \circ \lambda)(\zeta)$ is collinear to some point of $\Gamma(T)$;*

for every $t \in T$, the holomorphic map $\zeta \mapsto G(\lambda(\zeta))(t)$ of V into Δ is a Moebius transformation given by (10),

then there exists a complex, regular Borel measure ω on S satisfying (15) *and such that the map $F = K \circ G \circ H \in \mathrm{Hol}(B(S), B(T))$ is expressed by* (14) *for all $x \in B(S)$.*

Since H, F and K have continuous extensions to $\overline{B(S)}$ and to $\overline{B(T)}$,the following proposition holds.

Proposition 6. *If* $G \in \mathrm{Hol}(B(S), B(T))$ *satisfies the hypotheses of Theorem 5 for some* $x \in B(S)$*, then* G *has a (unique) continuous extension to* $\overline{B(S)}$*.*

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