RIEMANNIAN MANIFOLDS WITH SPECIAL CURVATURE TENSOR (*)

by Fabio Podestà (in Parma)
and Franco Tricerri † (in Firenze) (**)

SOMMARIO. - *Lo scopo di questo lavoro è studiare varietà Riemanniane a curvatura omogenea il cui tensore di curvatura è della forma $aR_{S^n}+bK$, $a, b \in \mathbb{R}$, dove K è semisimmetrico, i.e. $K \cdot K = 0$, e di Einstein. Quando $a > 0$, si prova che la varietà deve avere curvatura sezionale costante, mentre il caso $a < 0$ rimane aperto.*

SUMMARY. - *This paper is dealing with the problem of characterizing those curvature homogeneous Riemannian manifolds whose Riemannian curvature tensor is of the form $aR_{S^n} + bK$, $a, b \in \mathbb{R}$, where K is semisymmetric, i.e. $K \cdot K = 0$, and Einsteinian. When $a > 0$, it is shown that the manifold must be of constant sectional curvature, while the case $a < 0$ still remains open.*

0. Introduction.

The main purpose of this note is to continue the study of Riemannian manifolds having a special assigned curvature tensor (see, for example, [KP], [KTV], [TV1], [TV2]).

An n-dimensional Riemannian manifold (M, g) is said to be *curvature homogenous* (see [TV1]) if, for each $p, q \in M$, there exists a linear isometry $F : T_p M \to T_q M$ such that

$$F^* R_q = R_p$$

where R denotes the curvature tensor of (M, g). This is also equivalent to saying that R_p and R_q have the same components with respect to suitable orthonormal bases in $T_p M$ and $T_q M$. A homogeneous Riemannian manifold is clearly curvature homogeneous, but there are many examples of curvature homogeneous Riemannian manifolds which are not even locally homogeneous (see [KTV] and [TV1] for more details and further references). To give an alternative description of curvature homogeneity, we consider the principal fibre bundle $O(M)$ of orthonormal frames on M; the curvature tensor R may be viewed as a C^∞ map

$$R : O(M) \to \mathfrak{R}(V)$$

where $\mathfrak{R}(V)$ denotes the space of algebraic curvature tensors on an n-dimensional vector space V. The map R is $O(n)$-equivariant in the sense that $R(ua) = a^{-1} R(u)$ for all $u \in O(M)$ and all $a \in O(n)$, the orthogonal group acting on $O(M)$ and $\mathfrak{R}(V)$ in a standard way. It turns out that curvature homogeneity is actually equivalent to saying that $R(O(M))$ is contained in a single $O(n)$-orbit in $\mathfrak{R}(V)$. Moreover, given any algebraic curvature tensor R_o, we will say that (M, g) has a curvature tensor of type R_o if $R(O(M)) \subset O(n) \cdot R_o$, that is, if the image of the map R is contained in the $O(n)$-orbit through R_o. Recently, Kowalski and Prüfer ([KP]) have constructed a family of algebraic curvature tensors in dimension four which do not belong to any curvature homogeneous space.

One starting point for the problem we will be dealing with in this paper is the following result proved in [TV2]:

THEOREM. *Let (M, g, J) be an almost Hermitian manifold with real dimension $2n \geq 6$ and Riemannian curvature tensor of the form*

$$R = f R_{S^{2n}} + h R_{\mathbb{CP}^n}$$

where f and h are C^∞ functions with h not identically zero. Then M is a complex space form.

Remark. If $2n = 4$ the previous theorem also holds when h is supposed to be constant (see [TV2]). But if not, counterexamples are given in [Ol].

In this note we give some first results about characterizing Riemannian curvature homogeneous manifolds whose curvature tensor is of type

$$R_o = aR_{S^n} + bK_o \qquad\qquad (*)$$

where a, b are real numbers and K_o is a symmetric curvature tensor, that is, satisfies $K_o \cdot K_o = 0$. We recall here that, by a classical theorem due to E. Cartan (see [He]), each symmetric algebraic curvature tensor is the curvature tensor of a Riemannian symmetric space.

As we will see, the problem of giving a full classification of curvature homogeneous spaces having curvature of type $(*)$ seems to be not trivial, at least when the curvature K_o corresponds to a reducible Riemannian symmetric space. Indeed, we have the following contrasting examples:

EXAMPLE 1. In [Ts], Tsukada gave an explicit example of a four-dimensional curvature homogeneous manifold (M, g) whose curvature tensor is of type

$$R_o = -R_{S^4} + R_{\mathbb{R}^2 \times S^2}.$$

The manifold (M, g) turns out to be not locally homogeneous. Moreover, it can be proved (see [KTV]) that there is no homogeneous model space, that is, there does not exist any homogeneous space $(\tilde{M}, \tilde{g})$ with $R(O(M)) \subset \tilde{R}(O(\tilde{M}))$, where $\tilde{R}$ is the curvature map for $(\tilde{M}, \tilde{g})$.

EXAMPLE 2. Cartan isoparametric hypersurfaces (M, g) with three distinct principal curvatures in the space form $S^4(c)$ of constant curvature $c > 0$ provide homogeneous examples whose curvature is of type $(*)$. Indeed, it is easy to see that, if $-\lambda, 0, \lambda$ denote the three different principal curvatures, with $\lambda = \sqrt{3c}$, the curvature tensor of (M, g) is of type

$$R_o = 4R_{S^3} + 4\lambda^2 R_{\mathbb{R} \times S^2}.$$

Our main result deals with the case when the Ricci tensor of (M, g) is parallel and gives a partial classification in this case.

THEOREM. *Let (M, g) be a Ricci parallel, curvature homogeneous Riemannian manifold whose curvature is of type $(*)$. Then the constant a is nonpositive or (M, g) has constant sectional curvature.*

Remark. Note that (M, g) is Ricci parallel if and only if it is locally isometric to the product of Einstein spaces. In particular, this hypothesis of the Theorem is satisfied if K_o is the curvature tensor of an irreducible or, more generally, an Einstein symmetric space. In fact, in such a case (M, g) itself is Einstein.

Up to now, the authors were not able to find any example of a manifold satisfying the conditions of the Theorem with a negative a and not of constant sectional curvature.

1. Proof of the Theorem.

We may suppose that $R = aR_{S^n} + bK_o$ with $b \neq 0$, otherwise our claim follows immediately. In such a case, since (M, g) is curvature homogeneous, M has a G-structure P, where G is the Lie subgroup of $O(n)$ defined by

$$G = \{a \in O(n) | aK_o = K_o\}$$

(see [KTV]). Then K_o induces a tensor field K on M, defined by

$$K_p(x, y, z, w) = K_o(u^{-1}x, u^{-1}y, u^{-1}z, u^{-1}w)$$

where u is an element of the reduced bundle P belonging to the fibre over the point $p \in M$. Hence the Riemann curvature tensor R of (M, g) can be written as

$$R = aR^o + bK,$$

where R^o is the tensor field on M given by

$$R^o(x, y, z, w) = g(x, z)g(y, w) - g(x, w)g(y, z).$$

Note that

$$\nabla R = b\nabla K$$

where ∇ denotes the Levi Civita connection of (M, g). It follows that $\nabla K = \frac{1}{b}\nabla R$ satisfies the second Bianchi identity.

Moreover, by the curvature homogeneity, the length of the tensor field K is constant on the whole of M, so that, if Δ denotes the Laplace operator on (M, g), we have

$$\frac{1}{2}\,\Delta(\|K\|^2) = \|\nabla K\|^2 + (\nabla^2_{mm} K_{ijkl})\,K_{ijkl} = 0 \qquad (2.1)$$

where repeated indices mean summation.

Now, using the second Bianchi identity and the Ricci commutation formulas, we have

$$\begin{aligned}
(\nabla^2_{mm} K_{ijkl})\,K_{ijkl} &= -(\nabla^2_{mj} K_{mikl} + \nabla^2_{mi} K_{jmkl})\,K_{ijkl} \\
&= [-\nabla^2_{jm} K_{mikl} \\
&\quad -\nabla^2_{im} K_{jmkl} + (R_{mj} \cdot K)_{mikl} \\
&\quad +(R_{mi} \cdot K)_{jmkl}]K_{ijkl}.
\end{aligned} \qquad (2.2)$$

Since (M, g) is Ricci parallel, also $\nabla S = 0$, where S is the Ricci tensor field of type (0,2) corresponding to K. Hence, we have

$$\nabla^2_{jm} K_{mikl} = \nabla^2_{im} K_{jmkl} = 0.$$

Further, $K \cdot K = 0$ yields that (2.2) may be rewritten in the following way:

$$\begin{aligned}
(\nabla^2_{mm} K_{ijkl})\,K_{ijkl} &= a[(R^o_{mj} \cdot K)_{mikl} + (R^o_{mi} \cdot K)_{jmkl}]K_{ijkl} \\
&= 2a(R^o_{mj} \cdot K)_{mikl}K_{ijkl}.
\end{aligned} \qquad (2.3)$$

We now write (2.3) as follows:

$$\begin{aligned}
(\nabla^2_{mm} K_{ijkl})\,K_{ijkl} &= 2a[R^o_{pmmj} K_{pikl} + R^o_{pimj} K_{mpkl} \\
&\quad +R^o_{pkmj} K_{mipl}+ \\
&\quad +R^o_{plmj} K_{mikp}]K_{ijkl} = 2a(n-1)\|K\|^2+ \\
&\quad 2a[-\delta_{pj}\delta_{im} K_{mpkl} + (\delta_{pm}\delta_{kj} - \delta_{km}\delta_{pj})K_{mipl}+ \\
&\quad +(\delta_{pm}\delta_{lj} - \delta_{lm}\delta_{pj})K_{mikp}]K_{ijkl} = \\
&= 2a[(n-2)\|K\|^2 - 2\|S\|^2 - 2K_{mipl}K_{ipml}].
\end{aligned} \qquad (2.4)$$

The first Bianchi identity yields

$$K_{mipl}K_{ipml} = -||K||^2 - K_{impl}K_{ipml} = -||K||^2 - K_{mipl}K_{ipml},$$

so that (2.4) becomes

$$(\nabla^2_{mm}K_{ijkl})K_{ijkl} = 2a[(n-1)||K||^2 - 2||S||^2].$$

Hence, (2.1) can be written as

$$\frac{1}{2}\,\Delta(||K||^2) = ||\nabla K||^2 + 2a[(n-1)||K||^2 - 2||S||^2] = 0. \qquad (2.5)$$

On the other hand, for an arbitrary algebraic curvature tensor K_o with Ricci tensor S_o, we have

$$||K_o||^2 \geq \frac{2}{n-1}||S_o||^2, \qquad (2.6)$$

with equality if and only if K_o has constant sectional curvature (see, for example, [Be]).

It follows from (2.5) and (2.6) that a has to be nonpositive unless

$$||K||^2 = \frac{2}{n-1}||S||^2$$

everywhere on M, that is, (M, g) has constant sectional curvature.

$$\diamondsuit$$

Note that, when $a = 0$, (M, g) is a curvature homogeneous semi-symmetric space. These spaces have been treated thoroughly in [BKV].

References

[Be]   Besse A., *Einstein manifolds*, Ergeb. Math. Grenzgeb. 3. Folge 10, Springer Verlag, Berlin, Heidelberg, New York (1987).

[BKV]   Boeckx E., Kowalski O. and Vanhecke L., *Non-homogeneous relatives of symmetric spaces*, Differential Geom. and its Appl. **4** (1994), 45-69.

[He]   HELGASON S., *Differential Geometry, Lie groups and symmetric spaces*, Academic Press, New York (1978).

[KP]   KOWALSKI O. and PRÜFER F., *Curvature tensors in dimension four which do not belong to any curvature homogeneous space*, preprint (1994).

[KTV]   KOWALSKI O., TRICERRI F. and VANHECKE L., *Curvature homogeneous spaces with a solvable Lie group as homogeneous model*, J. Math. Soc. Japan **44** (1992), 461-484.

[Ol]   OLSZAK Z., *On the existence of generalized complex space forms*, Israel J. Math. **65** (1989), 214-218.

[Ts]   TSUKADA K., *Curvature homogeneous hypersurfaces immersed in a real space form*, Tôhoku Math. J. **40** (1988), 221-244.

[TV1]   TRICERRI F. and VANHECKE L., *Curvature homogeneous Riemannian manifolds*, Ann. Sci. Ecole Norm. Sup. **22** (1989), 535-554.

[TV2]   TRICERRI F. and VANHECKE L., *Curvature tensors on almost Hermitian manifolds*, Trans. A.M.S. **267** (1981), 365-398.