POINTS OF CONTINUITY, QUASICONTINUITY AND CLIQUISHNESS (*)

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Sommario. - *In questo lavoro si esamina per una funzione f la terna $(C(f), E(f), A(f))$ ove $C(f)$, $E(f)$ ed $A(f)$ sono gli insiemi dei punti di continuità, quasicontinuità e cliquishness rispettivamente.*

Summary. - *The triplet $(C(f), E(f), A(f))$, where $C(f)$, $E(f)$ and $A(f)$ are sets of all continuity, quasicontinuity and cliquishness points of a function f, respectively, is investigated.*

In what follows X denote a topological space. For a subset A of a topological space denote by $\mathrm{Cl}A$, $\mathrm{Int}A$ and A^d the closure of A, the interior of A and the set of all accumulation points of A, respectively. The letters $\mathbb{N}$ and $\mathbb{R}$ stand for the set of natural and real numbers, respectively.

Let X be a topological space and (Y, d) a metric one. We recall that a function $f : X \to Y$ is quasicontinuous (cliquish) at a point $x \in X$ if for each $\epsilon > 0$ and each neighbourhood U of x there is a nonempty open set $G \subset U$ such that $d(f(y), f(x)) < \epsilon$ for each $y \in G$ ($d(f(y), f(z)) < \epsilon$ for each $y, z \in G$) ([3]-[5], [8], [9]).

Denote by $C(f)$, $E(f)$ and $A(f)$ the set of all continuity, quasicontinuity and cliquishness points of a function $f : X \to Y$, respectively. It is known that $C(f) \subset E(f) \subset A(f)$, $C(f)$ is a G_δ set, $A(f)$ is closed [8] and $A(f) \setminus C(f)$ is of the first category [9].

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Are these conditions also sufficient? This means, let A, E and C be subsets of X such that $C \subset E \subset A$, C is a G_δ set, A is closed and $A \setminus C$ is of the first category. Does there exist a function $f : X \to Y$ such that $C = C(f)$, $E = E(f)$ and $A = A(f)$?

If X, Y are normed linear spaces and X is a Baire space, a positive answer is given in [4]. Some sufficient conditions are given also in [3]. We shall show that this is true also if X is a Baire pseudometrizable space without isolated points or X is a Baire perfectly normal resolvable locally connected space.

The following statement is claimed in [5].

A. (See [5; Theorem 2]). Let X be a topological space which is a union of two dense disjoint sets and let (Y, d) be a metric space with at least one accumulation point. Then for each decreasing sequence $\{W_n : n \in \mathbb{N}\}$ of open subsets of X and each set E satisfying inclusions

$$C = \bigcap_{n=1}^{\infty} W_n \subset E \subset \bigcap_{n=1}^{\infty} \mathrm{Cl} W_n = A$$

there is a map $f : X \to Y$ such that $C = C(f)$, $E = E(f)$ and $A = A(f)$.

We shall show that conditions on X and Y in A are not sufficient. Examples 1 and 2 show that the conditions on X are not sufficient and Example 3 shows that the conditions on Y are not sufficient.

Example 1. Let $X = \mathbb{N}$ with the cofinite topology and $Y = \mathbb{R}$ with the usual metric. Put $W_n = X \setminus \{1, 2, , 3, \dots, n\}$, $C = \bigcap_{n=1}^{\infty} W_n = \emptyset$, $E = A = \bigcap_{n=1}^{\infty} \mathrm{Cl} W_n = X$. It is easy to see that every quasicontinuous function $f : X \to Y$ is constant and hence $E(f) = X$ implies $C(f) = X$.

Example 2. Let $X = \mathbb{R}$ with the topology $\mathcal{T} = \{\emptyset, X\} \cup \{(a, \infty) : a \in \mathbb{R}\}$, $Y = \mathbb{R}$ with the usual metric. Put $W_n = (n, \infty)$, $C = \bigcap_{n=1}^{\infty} W_n = \emptyset$, $E = A = \bigcap_{n=1}^{\infty} \mathrm{Cl} W_n = X$. Easy it is to see that every quasicontinuous function $f : X \to Y$ is constant.

Example 3. Let $X = \mathbb{R}^2$ with the usual topology, $Y = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ with the usual metric. Put $W_n = X \setminus \{(0, 0)\}$ for each

$n \in \mathbb{N}$, $C = \bigcap_{n=1}^{\infty} W_n = X \setminus \{(0,0)\}$, $E = A = \bigcap_{n=1}^{\infty} \mathrm{Cl} W_n = X$. Let $f : X \to Y$ be such that $C = C(f)$. Then f must be constant on C. However then $(0,0) \notin C(f)$ implies $(0,0) \notin E(f)$.

Therefore A does not hold. However, the following "partial" theorems are true. We recall that a space X is said to be resolvable [7] if it is a union of two dense disjoint sets.

THEOREM 1. *Let X be a resolvable topological space and let (Y,d) be a metric one with $Y^d \neq \emptyset$. Let E, A be subsets of X. Then there is a function $f : X \to Y$ such that $E = E(f)$ and $A = A(f)$ if and only if there is a nonincreasing sequence $(W_n)_n$ of open subsets of X such that $\bigcap\limits_{n=1}^{\infty} W_n \subset E \subset \bigcap\limits_{n=1}^{\infty} \mathrm{Cl} W_n = A$.*

THEOREM 2. *Let X and Y be as in Theorem 1. Let C, A be subsets of X. Then there is a function $f : X \to Y$ such that $C = C(f)$ and $A = A(f)$ if and only if there is a nonincreasing sequence $(W_n)_n$ of open subsets of X such that $\bigcap\limits_{n=1}^{\infty} W_n = C$ and $\bigcap\limits_{n=1}^{\infty} \mathrm{Cl} W_n = A$.*

Proof of Theorems 1 and 2. Necessity follows from [5; Theorem 1]. The function f in [5; Theorem 2] (i.e. $f(x) = y_0$ for $x \in E$, $f(x) = y_{2n}$ for $x \in (W_n \setminus W_{n+1}) \setminus E \cap X_1$ and $f(x) = y_{2n+1}$ for $x \in (W_n \setminus W_{n+1}) \setminus E \cap X_2$, where X_1 and X_2 are dense disjoint subsets of X such that $X = X_1 \cup X_2$, y_0 is an accumulation point of Y, $(y_n)_n$ is one-to-one sequence converging to y_0 with $y_n \neq y_0$ and $W_0 = X$) is such that $E = E(f)$ and $A = A(f)$. If we put $E = \bigcap\limits_{n=1}^{\infty} W_n = C$ in the definition of f, we obtain $C = C(f)$ and $A = A(f)$. ◊

It is easy to see that if $A = X$ in Theorem 1, then the assumption "X is resolvable" can be omitted. Hence we have

COROLLARY 1. *Let X be a topological space and let (Y,d) be a metric one with $Y^d \neq \emptyset$. Then the set M is the set of all discontinuity points of some cliquish function $f : X \to Y$ if and only if M is an*

F_σ set of the first category.

This corollary generalizes a result from [10], where it is assumed that X is Baire and $Y = \mathbb{R}$.

Let C, E and A be subsets of X. Denote by (A), (B) and (C) the following conditions:

$$\text{(A)} \quad \begin{cases} \text{there is a function } f : X \to Y \\ \text{such that } C = C(f),\ E = E(f) \text{ and } A = A(f); \end{cases}$$

$$\text{(B)} \quad \begin{cases} C \subset E \subset A,\ C \text{ is } G_\delta,\ A \text{ is closed} \\ \text{and } A \setminus C \text{ is of the first category;} \end{cases}$$

$$\text{(C)} \quad \begin{cases} \text{there is a nonincreasing sequence } (W_n)_n \\ \text{of open subsets of } X \text{such that} \\ \bigcap\limits_{n=1}^{\infty} W_n = C \subset E \subset A = \bigcap\limits_{n=1}^{\infty} \mathrm{Cl} W_n\ . \end{cases}$$

Then (A) implies (B) and by [5] (A) implies (C). By [5] (C) implies (B). In general, (B) does not imply (C). A topological space is called perfect if every closed subset of this space is G_δ [2]. In the sequel we use normal space but we do not suppose that they are T_1 spaces.

PROPOSITION 1. *Let X be a perfectly normal space. Then the conditions* (B) *and* (C) *are equivalent.*

Proof. Let (B) be satisfied. Then $A = \bigcap\limits_{n=1}^{\infty} H_n$, where H_n are open sets and $\mathrm{Cl} H_{n+1} \subset H_n$ for each $n \in \mathbb{N}$. Since $A \setminus C$ is an F_σ set of the first category, $A \setminus C = \bigcup\limits_{n=1}^{\infty} F_n$, where F_n are closed nowhere dense and $F_n \subset F_{n+1}$ for each $n \in \mathbb{N}$. Put $W_n = H_n \setminus F_n$. Then W_n are open sets and $W_{n+1} \subset W_n$ for each $n \in \mathbb{N}$. It is easy to see that $C = \bigcap\limits_{n=1}^{\infty} W_n$ and $A = \bigcap\limits_{n=1}^{\infty} \mathrm{Cl} W_n$. $\diamondsuit$

Remark 1. The assumptions on X in Proposition 1 cannot be omitted. The space X from Example 1 is perfect but not normal. If we put $C = E = A = \{1\}$, then C, E, A satisfy (B) but they do not satisfy (C). The space X from Example 2 is normal but not perfect. If we put $C = E = \emptyset$, $A = (-\infty, 0]$, then C, E and A satisfy (B) but they do not satisfy (C).

Denote by (D), (E) and (F) the following conditions on a topological space X (ω_g is the oscillation of a function $f : X \to Y$):

$$\text{(D)} \begin{cases} \text{for each nonincreasing sequence } (W_n)_n \text{ of open} \\ \text{subsets of } X \text{ such that } B = \bigcap\limits_{n=1}^{\infty} \mathrm{Cl} W_n \setminus \bigcap\limits_{n=1}^{\infty} W_n \text{ is} \\ \text{co} - \text{dense (i.e. } X \setminus B \text{ is dense) there is a bounded} \\ \text{function } g : X \setminus B \to \mathbb{R} \text{ such that} \\ (i) \quad \liminf\limits_{u \to x} \omega_g(u) > 0 \text{ for each } x \in X \setminus \bigcap\limits_{n=1}^{\infty} \mathrm{Cl} W_n, \\ (ii) \quad \omega_g(x) = 0 \ \text{ for each } x \in \bigcap\limits_{n=1}^{\infty} W_n, \\ (iii) \quad \liminf\limits_{u \to x} \omega_g(u) = 0 < \omega_g(x) \ \text{ for each } x \in B; \end{cases}$$

$$\text{(E)} \begin{cases} \text{for each co} - \text{dense set } F \text{ of the first category} \\ \text{and } F_\sigma \text{ there is a continuous bounded function} \\ g : X \setminus F \to \mathbb{R} \text{ such that } \omega_g(x) > 0 \\ \text{for each } x \in F; \end{cases}$$

$$\text{(F)} \begin{cases} \text{for each nowhere dense set } F \text{ there is a} \\ \text{continuous function } g : X \setminus F \to [0,1] \\ \text{such that } \omega_g(x) = 1 \text{ for each } x \in F\ . \end{cases}$$

Example 3 shows that Y must contain large components. In the sequel we shall assume that $Y = \mathbb{R}$.

LEMMA 1. *The condition* (A) *implies* (D).

Proof. Let $(W_n)_n$ be a nonincreasing sequence of open subsets of X such that the set $B = \bigcap\limits_{n=1}^{\infty} \mathrm{Cl} W_n \setminus \bigcap\limits_{n=1}^{\infty} W_n$ is co-dense. Put $C = \bigcap\limits_{n=1}^{\infty} W_n$, $E = A = \bigcap\limits_{n=1}^{\infty} \mathrm{Cl} W_n$. Let $f : X \to \mathbb{R}$ be such that $C = C(f)$, $E = E(f)$ and $A = A(f)$. Put $h = arctg \circ f$. Then h is bounded and $C(h) = C$, $E(h) = E = A = A(h)$. Denote $g = h|_{X \setminus B}$. We shall show that g satisfies (i), (ii), (iii).

(i) Let $x \in X \setminus \bigcap_{n=1}^{\infty} \mathrm{Cl} W_n = X \setminus A$. Since $X \setminus A$ is open and $x \notin A(h)$ so $x \notin A(g)$. Therefore there is an $\epsilon > 0$ and an open neighbourhood U of x such that $U \subset X \setminus A$ and $d(h(V \setminus B)) \geqq \epsilon$ for each open subset V of U. This yields $\omega_g(y) \geqq \epsilon$ for each $y \in U$ and hence $\liminf_{u \to x} \omega_g(u) \geqq \epsilon > 0$.

(ii) If $x \in \bigcap_{n=1}^{\infty} W_n = C$, then $x \in C(h)$. Hence $x \in C(g)$ and $\omega_g(x) = 0$.

(iii) Let $x \in B$. Then $x \notin C(h)$ and hence $\omega_h(x) = \alpha > 0$. Let U be an open neighbourhood of x. Since $x \in E(h)$ there is an open nonempty set $U_1 \subset U$ such that $|h(x) - h(y)| < \frac{\alpha}{16}$ for each $y \in U_1$. Since $d(h) > \frac{\alpha}{2}$ there is $z \in U$ such that $|h(z) - h(x)| > \frac{\alpha}{4}$.

If $z \in A (= E(h))$ then there is an open nonempty set $U_2 \subset U$ such that $|h(z) - h(y)| < \frac{\alpha}{16}$ for each $y \in U_2$. Since B is co-dense, there are $u \in U_1 \setminus B$, $v \in U_2 \setminus B$. Then we have

$$\frac{\alpha}{4} < |h(z) - h(x)| \leqq |h(z) - h(v)| + |h(v) - h(u)| +$$

$$+ |h(u) - h(x)| < |h(u) - h(v) + \frac{\alpha}{8}.$$

This yields $|h(u) - h(v)| > \frac{\alpha}{8}$ and hence $d(h(U \setminus B)) > \frac{\alpha}{8}$ and therefore $\omega_g(x) \geqq \frac{\alpha}{8} > 0$.

If $z \notin A$, the set $U \setminus A$ is an open neighbourhood of z. Then $z \notin C(h)$ and $\omega_h(z) = \beta > 0$. This yields $d(h(U \setminus B)) \geqq d(h(U \setminus A)) > \frac{\beta}{2}$ and hence $\omega_g(x) \geqq \frac{\beta}{2} > 0$.

Further, let $\epsilon > 0$ and U be a neighbourhood of x. Then $x \in A(h)$ and hence there is an open nonempty $V \subset U$ such that $d(h(V)) < \epsilon$, i.e. $\omega_h(y) < \epsilon$ for each $y \in V$. Therefore $\liminf_{u \to x} \omega_g(u) \leqq \liminf_{u \to x} \omega_h(u) < \epsilon$. However this means that $\liminf_{u \to x} \omega_g(u) = 0$. $\diamondsuit$

Theorem 3. *Let X be a Baire space, $Y = \mathbb{R}$. Let C, E, A be subsets of X. Then* (A) *is equivalent with* (C) *and* (D).

Proof. By [5] we have (A) $\Rightarrow$ (C) and by Lemma 1 we have (A) $\Rightarrow$ (D).

(C) & (D) $\Rightarrow$ (A): Let $(W_n)_n$ be a nonincreasing sequence of open subsets of X such that $\bigcap_{n=1}^{\infty} W_n = C \subset E \subset A = \bigcap_{n=1}^{\infty} \mathrm{Cl} W_n$ and let $B = \bigcap_{n=1}^{\infty} \mathrm{Cl} W_n \setminus \bigcap_{n=1}^{\infty} W_n$. Since X is Baire, so B is co-dense. Let $g : X \setminus B \to \mathbb{R}$ be a bounded function satisfying (i), (ii), (iii). Let $c \in \mathbb{R}$ be such that $g(X \setminus B) \subset [-c, c]$.

Let $x \in B$ and U be a neighbourhood of x. Denote by

$$C(x,U) = \{y \in Y : \text{for each neighbourhood } V \text{ of } y \text{ there is}$$
$$\text{an open nonempty } G \subset U \text{ such that } g(G \setminus B) \subset V\}.$$

We can easy prove that $C(x,U)$ is a closed set and $C(x,U) \subset [-c,c]$. We shall show that $C(x,U)$ is nonempty. Let $n \in \mathbb{N}$. Since $\liminf_{u \to x} \omega_g(u) < \frac{1}{n}$ there is $u_n \in U \setminus B$ such that $\omega_g(u_n) < \frac{1}{n}$. Therefore there is an open neighbourhood G_n of u_n such that $d(g(G_n \setminus B)) < \frac{1}{n}$. Let $y \in [-c,c]$ be an accumulation point of the sequence $(g(u_n))_n$. Let V be a neighbourhood of y, let $\epsilon > 0$ be such that $(y - \epsilon, y + \epsilon) \subset V$ and let $m \in \mathbb{N}$ be such that $m > \frac{2}{\epsilon}$ and $g(u_n) \in (y - \frac{\epsilon}{2}, y + \frac{\epsilon}{2})$. Then for each $t \in G_m \setminus B$ we have

$$|y - g(t)| \leqq |y - g(u_m)| + |g(u_m) - g(t)| < \frac{\epsilon}{2} + \frac{1}{m} < \epsilon.$$

Therefore $g(G_m \setminus B) \subset V$ and $y \in C(x,U)$.

Now let $x \in B$ and let $\mathcal{U}_x$ be the neighbourhood system of x. Then $(C(x,U))_{U \in \mathcal{U}_x}$ is a family of closed subsets of $[-c,c]$ with the finite intersection property. Hence $C(x) = \bigcap_{U \in \mathcal{U}_x} C(x,U) \neq \emptyset$. Therefore $C(x)$ is a nonempty, closed and bounded set in $\mathbb{R}$. Denote by

$$D(x) = \max C(x).$$

The set B is a F_σ set of the first category so $B = \bigcup_{n=1}^{\infty} F_n$, where F_n are closed nowhere dense sets and $F_n \subset F_{n+1}$ for each $n \in \mathbb{N}$. Put $F_0 = \emptyset$. Now define a function $f : X \to \mathbb{R}$ by

$$f(x) = \begin{cases} g(x), & \text{if } x \in X \setminus B, \\ D(x), & \text{if } x \in B \cap E, \\ D(x) + \frac{1}{n}, & \text{if } x \in (F_n \setminus F_{n-1}) \setminus E. \end{cases}$$

We shall show that $C(f) = C$, $E(f) = E$ and $A(f) = A$.

1) Let $x \in X \setminus A$. Then $\liminf_{u \to x} \omega_g(x) = \alpha > 0$. Thus there is an open neighbourhood U of x such that $U \subset X \setminus A$ and $\omega_g(u) > \frac{\alpha}{2}$ for each $u \in U$. This yields $d(f(V \setminus B)) > \frac{\alpha}{2}$ for each neighbourhood V of u, i.e. $x \in A(f)$. We have thus

$$X \setminus A \subset X \setminus A(f). \tag{1}$$

2) Let $x \in B$. Then $\omega_f(x) \geqq \omega_g(x) > 0$ and

$$B \subset X \setminus C(f). \tag{2}$$

3) Let $x \in C = A \setminus B$ and let $\epsilon > 0$ and $\frac{1}{n} < \frac{\epsilon}{3}$. Then $\omega_g(x) = 0$ and hence there is an open neighbourhood U of x such that $d(g(U \setminus B)) = d(f(U \setminus B)) < \frac{1}{n}$. Since $x \notin F_n$, the set $U \setminus F_n$ is an open neighbourhood of x.

Let $y \in U \setminus F_n$.

If $y \in U \setminus B$, we have $|f(y) - f(x)| < \epsilon$.

If $y \in B$, there is $m > n$ such that $y \in F_m \setminus F_{m-1}$. Since $D(y) \in C(y, U)$ and B is co-dense there is $z \in U \setminus B$ such that $|f(z) - D(y)| = |g(z) - D(y)| < \frac{1}{n}$.

If $y \in B \cap E$, we have

$$\begin{aligned} |f(y) - f(x)| = |D(y) - f(x)| \leqq\ & |D(y) - f(z)| \\ & + |f(z) - f(x)| < \frac{1}{n} + \frac{1}{n} < \epsilon. \end{aligned}$$

If $y \in B \setminus E$, we have

$$\begin{aligned} |f(y) - f(x)| = |D(y) + \frac{1}{m} - f(x)| \leqq\ & |D(y) - f(z)| \\ & + |f(z) - f(x)| + \frac{1}{m} < \epsilon. \end{aligned}$$

Therefore for each $y \in U \setminus F_n$ we have $|f(y) - f(x)| < \epsilon$ and

$$C \subset C(f). \tag{3}$$

4) Let $x \in B \cap E$. Let U be a neighbourhood of x, let $\epsilon > 0$ and $\frac{1}{n} < \frac{\epsilon}{3}$. Then $f(x) = D(x) \in C(x, U)$ and hence there is an open nonempty $G \subset U$ such that $g(G \setminus B) = f(G \setminus B) \subset (D(x) - \frac{1}{n}, D(x) +$

$\frac{1}{n}$). The set $G \setminus F_n \subset U$ is open nonempty. Let $y \in G \setminus F_n$.
If $y \in G \setminus B$, we have $|f(y) - f(x)| < \frac{1}{n} < \epsilon$.
If $y \in B$, $y \in F_m \setminus F_{m-1}$ for some $m > n$. Then there is $z \in G \setminus B$ such that $|f(z) - D(y)| < \frac{1}{n}$. This yields $|f(y) - f(x)| \leqq |f(y) - D(y)| + |D(y) - f(z)| + |f(z) - f(x)| < \frac{1}{m} + \frac{1}{n} + \frac{1}{n} < \epsilon$. Therefore

$$B \cap E \subset E(f). \tag{4}$$

5) Let $x \in B \setminus E$. Then $x \in F_n \setminus F_{n-1}$ for some $n \in \mathbb{N}$. Therefore $f(x) = D(x) + \frac{1}{n}$. Since $f(x) \notin C(x)$ there are a neighbourhood V of $f(x)$ and a neighbourhood U of x such that for each nonempty open set $G \subset U$ there is $t \in G \setminus B$ such that $f(t) \notin V$. Therefore $x \notin E(f)$ and

$$B \setminus E \subset X \setminus E(f). \tag{5}$$

6) Let $x \in B$. Let $\epsilon > 0$ and let U be a neighbourhood of x. Since $\liminf\limits_{u \to x} \omega_g(u) = 0$, there is $u \in U \setminus B$ such that $\omega_g(u) < \frac{\epsilon}{5}$. Thus there is an open neighbourhood $V \subset U$ of u such that $d(f(V \setminus B)) < \frac{\epsilon}{5}$. Let $\frac{1}{n} < \frac{\epsilon}{5}$. Then $V \setminus F_n$ is an open nonempty set. Let $y, z \in V \setminus F_n$.
If $y, z \in V \setminus B$, we have $|f(y) - f(z)| < \epsilon$.
If $y, z \in B$, there are $k, m \in \mathbb{N}$, $k, m > n$ such that $y \in F_m \setminus F_{m-1}$, $z \in F_k \setminus F_{k-1}$. Then there are $y_1, z_1 \in V \setminus B$ such that $|f(y_1) - D(y)| < \frac{\epsilon}{5}$, $|f(z_1) - D(z)| < \frac{\epsilon}{5}$. Hence $|f(y) - f(z)| \leqq |f(y) - D(y)| + |D(y) - f(y_1)| + |f(y_1) - f(z_1)| + |f(z_1) - D(z)| + |D(z) - f(z)| < \frac{1}{m} + \frac{\epsilon}{5} + \frac{\epsilon}{5} + \frac{\epsilon}{5} + \frac{1}{k} < \epsilon$.
Similarly for $y \in B$ and $z \in V \setminus B$ we have $|f(y) - f(z)| < \epsilon$. Therefore we have

$$B \subset A(f). \tag{6}$$

Combining (1), (2) and (3) we get $C = C(f)$.
Combining (1), (3), (4) and (5) we obtain $E = E(f)$.
Finally, (1), (3) and (6) imply $A = A(f)$. ◊

What topological spaces do satisfy the condition (D)?

LEMMA 2. *The condition* (D) *implies* (E).

Proof. Let $F = \bigcup\limits_{n=1}^{\infty} F_n$, where F_n are closed nowhere dense, $F_n \subset F_{n+1}$ for each $n \in \mathbb{N}$ and F is co-dense. Put $W_n = X \setminus F_n$.

Then $(W_n)_n$ is a nonincreasing sequence of open subsets of X and $B = \bigcap_{n=1}^{\infty} \mathrm{Cl}W_n \setminus \bigcap_{n=1}^{\infty} W_n = F$ is co-dense. Let $g : X \setminus B \to \mathbb{R}$ be a bounded function satisfying (i), (ii) and (iii). Then evidently g is continuous and $\omega_g(x) > 0$ for each $x \in F$.

Proposition 2. *Let X be a resolvable perfectly normal topological space. Then the conditions* (D) *and* (E) *are equivalent.*

Proof. Let $(W_n)_n$ be a nonincreasing sequence of open subsets of X such that $B = \bigcap_{n=1}^{\infty} \mathrm{Cl}W_n \setminus \bigcap_{n=1}^{\infty} W_n$ is co-dense. Then B is a F_σ set of the first category and hence there is a bounded continuous function $k : X \setminus B \to \mathbb{R}$ such that $\omega_k(x) > 0$ for each $x \in B$. Let $X = X_1 \cup X_2$, where X_1 and X_2 are disjoint dense. Further there is a continuous function $h : X \to [0,1]$ such that $h^{-1}(0) = \bigcap_{n=1}^{\infty} \mathrm{Cl}W_n$. Now define a function $g : X \setminus B \to \mathbb{R}$ by

$$g(x) = \begin{cases} h(x) + k(x), & \text{if } x \in X_1, \\ k(x), & \text{if } x \in X_2. \end{cases}$$

Then g is bounded and we shall show that it satisfies (i), (ii) and (iii).

(i) Let $x \in X \setminus \bigcap_{n=1}^{\infty} \mathrm{Cl}W_n$. Then $h(x) = \alpha > 0$. Since h is continuous there is a neighbourhood $U \subset X \setminus \bigcap_{n=1}^{\infty} \mathrm{Cl}W_n$ of x such that $h(U) \subset (\frac{\alpha}{2}, \frac{3\alpha}{2})$. Put $\eta = \frac{\alpha}{8}$. Let $G \subset U$ be an arbitrary open set. Since k is continuous at x we can assume that $k(U) \subset (k(x) - \eta, k(x) + \eta)$. Let $y \in G \cap X_1$, $z \in G \cap X_2$. Then

$$\frac{\alpha}{2} < h(y) \leqq |h(y)+k(y)-k(z)|+|k(y)-k(z)| < |g(y)-g(z)|+2\eta.$$

This yields $|g(y) - g(z)| > \frac{\alpha}{4}$ and $d(g(V)) = d(g(V \setminus B)) \geqq \frac{\alpha}{4}$. Therefore
$\liminf_{u \to x} \omega_g(u) \geqq \frac{\alpha}{4} > 0$.

(ii) Let $x \in \bigcap_{n=1}^{\infty} W_n$. Let $\epsilon > 0$ and U be a neighbourhood of x such that $d(k(U \setminus B)) < \frac{\epsilon}{2}$. Then $d(g(U \setminus B) \cap h^{-1}([0, \frac{\epsilon}{2}))) < \epsilon$ and hence $\omega_g(x) = 0$.

(iii) Let $x \in B$. Then $\omega_k(x) = \alpha > 0$. Hence for each neighbourhood U of x we have $d(k(U \setminus B)) > \frac{3\alpha}{4}$. Further there is a neighbourhood H of x such that $d(h(H)) < \frac{\alpha}{4}$. Let U be an arbitrary neighbourhood of x. Then there are $y, z \in (H \cap U) \setminus B$ such that $|k(y) - k(z) > \frac{3\alpha}{4}$.

If $y, z \in X_1$, we have

$$\frac{3\alpha}{4} < |k(y) - k(z)| = |k(y) - g(y) + g(y) - g(z) + g(z) - k(z)|$$

$$\leqq |h(y) - h(z)| + |g(y) - g(z)| < \frac{\alpha}{4} + |g(u) - g(z)|$$

and hence $|g(y) - g(z)| > \frac{\alpha}{2}$.
Similarly for $y \in X_1$, $z \in X_2$ or $y, z \in X_2$ we have $|g(y) - g(z)| > \frac{\alpha}{2}$.
Therefore $\omega_g(x) \geqq \frac{\alpha}{2} > 0$.

Now let $\eta > 0$. Then for each $y \in h^{-1}([0, \frac{\eta}{2})) \setminus B$ we have $\omega_g(y) < \eta$ and hence $\liminf_{u \to x} \omega_g(u) = 0$. $\diamondsuit$

From Proposition 2 and Theorem 3 we obtain

THEOREM 4. *Let X be a Baire resolvable perfectly normal space. Then* (A) *is equivalent with* (B) *and* (E).

What topological spaces do satisfy the condition (E)?

LEMMA 3. *Let $f, g, f_n : X \to \mathbb{R}$ be functions ($n \in \mathbb{N}$). Then*

α) $\omega_{f+g}(x) \leqq \omega_f(x) + \omega_g(x)$,

β) $\omega_g(x) \leqq \omega_f(x) + \omega_{f+g}(x)$,

γ) if $\omega_f(x) = 0$, $\omega_{f+g}(x) = \omega_g(x)$,

δ) if $f_n \rightrightarrows f$ then $\omega_{f_n} \rightrightarrows \omega_f$.

Proof. We omit the standard proof. ♢

LEMMA 4. *The condition* (F) *implies* (E).

Proof. Let $F = \bigcup_{n=1}^{\infty} F_n$, where F_n are closed nowhere dense and $F_n \subset F_{n+1}$ for each $n \in \mathbb{N}$. Let $g_n : X \setminus F_n \to [0,1]$ be a continuous function such that $\omega_{g_n}(x) = 1$ for each $x \in F_n$ and $n \in \mathbb{N}$. Define a function $g : X \setminus F \to \mathbb{R}$ by

$$g(x) = \sum_{n=1}^{\infty} 4^{-n} g_n(x).$$

Then g is a continuous function.
We shall show that $\omega_g(x) > 0$ for each $x \in F$. Let $x \in F$. Then $x \in F_n \setminus F_{n-1}$ for some $n \in \mathbb{N}$ (where $F_0 = \emptyset$). Thus $\omega_{g_k}(x) = 0$ for each $k \leqq n-1$ and $\omega_{g_k}(x) = 1$ for each $k \geqq n$. If we denote by $t = \sum_{i=n+1}^{\infty} 4^{-i} g_i$ and $t_j = \sum_{i=n+1}^{n+j} 4^{-i} g_i$ then $t_j \rightrightarrows t$ and hence by Lemma 3 δ) and α) we get $\omega_t(x) \leqq \sum_{i=n+1}^{\infty} 4^{-i} = \frac{1}{3} 4^{-n}$. By Lemma 3 γ) we have $\omega_g(x) = \omega_{t+4^{-n}g_n}(x)$ and therefore by Lemma 3 β) we obtain $\omega_g(x) \geqq 4^{-n}\omega_{g_n}(x) - \omega_t(x) \geqq \frac{2}{3} 4^{-n} > 0$. ♢

LEMMA 5. *Let X be a perfectly normal locally connected topological space. Then we have* (F).

Proof. Let F be a closed nowhere dense set in X. Then there is a continuous function $h : X \to [0,1]$ such that $h^{-1}(0) = F$. Define $g : X \setminus F \to [0,1]$ by

$$g(x) = \left| \sin \frac{1}{h(x)} \right|.$$

Then evidently g is continuous.

Let $x \in F$ and let U be a neighbourhood of x We can assume that U is open and connected. Then $U \setminus F$ is nonempty open. Let T be a component in $U \setminus F$.

We shall show that $\mathrm{Cl}T \cap F \neq \emptyset$. Suppose, by contrary, that $\mathrm{Cl}T \cap F = \emptyset$. Being a component of a subspace $U \setminus F$, the set T is closed in $U \setminus F$, hence $T \cap (U \setminus F) = \mathrm{Cl}T \cap (U \setminus F)$. Since $\mathrm{Cl}T \cap F$ is empty, the set T is closed also in the subspace U. Simultaneously, T is a component of an open set $U \setminus F$ in a locally connected space, T is open in X. Therefore T is non-empty, closed and open in U. As U is connected, we have $T = U$. So $x \in U \cap F = T \cap U \cap F = \mathrm{Cl}T \cap U \cap F \subset \mathrm{Cl}T \cap F = \emptyset$, a contradiction.

Therefore $\mathrm{Cl}T \cap F \neq \emptyset$ and hence there is $\beta > 0$ such that $(0, \beta) \subset h(T)$. This yields that there are points $u, v \in U \setminus F$ such that $g(u) = 0$ and $g(v) = 1$. Therefore $\omega_g(x) = 1$. ♢

LEMMA 6. *Let X be a pseudometrizable space. Then we have* (F).

Proof. Let F be a closed nowhere dense set in X and let d pseudometrize X. We shall construct sets S_n in this way. Let $S_1 = \emptyset$. Assume that we have S_i for $i < n$. Denote

$$T_n = \{x \in X \setminus (F \cup \bigcup_{i<n} S_i) : d(x, F) < \frac{1}{n}\}$$

and

$$\mathfrak{P}_n = \{P \subset T_n : d(x, y) \notin (0, \frac{1}{n}] \text{ for each } x, y \in P\}.$$

According to Zorn lemma there is a maximal element S_n of $\mathfrak{P}_n$. Denote by

$$A = \bigcup_{n=1}^{\infty} S_{2n}, \qquad B = \bigcup_{n=1}^{\infty} S_{2n+1}.$$

We shall show that

$$(*) \qquad \mathrm{Cl}A = A \cup F.$$

1) Let $x \in F$. Let U be a neighbourhood of x Then there is an even number n such that $S(x, \frac{2}{n}) \subset U$ ($S(u, \epsilon)$ is the open sphere of radius $\epsilon > 0$ about u). If $S(x, \frac{2}{n}) \cap S_n = \emptyset$, then for $i < n$ we have $d(u, v) = 0$ for each $u, v \in S_i \cap S(x, \frac{1}{2n})$. Therefore there is $y \in S(x, \frac{1}{2n}) \cap T_n$. However then $d(y, z) > \frac{1}{n}$ for each $z \in S_n$ and hence S_n is not maximal in $\mathfrak{P}_n$, a contradiction. Therefore $S(x, \frac{2}{n}) \cap S_n \neq \emptyset$ and hence $x \in \mathrm{Cl}A$.

2) Now let $x \in \mathrm{Cl}A \setminus F$. Then $d(x, F) > \frac{1}{n}$ for some $n \in \mathbb{N}$. Since $x \in \mathrm{Cl}A$ so there is a sequence $(z_k)_k$ in $S(x, \frac{1}{2n})$ converging to x. However $z_k \notin S_i$ for $i \geqq n$ and hence we may assume that $z_k \in S_j$ for some even j, $j < n$ and each $k \in \mathbb{N}$. Then $d(z_s, z_t) = 0$ for each $s, t \in \mathbb{N}$ and hence $d(z_t, x) = 0$ for each $t \in \mathbb{N}$. Since S_j is a maximal element in $\mathfrak{P}_j$, we have $x \in S_j \subset A$. Therefore we have $(*)$.

Similarly we can prove that $\mathrm{Cl}B = B \cup F$. The sets S_i are mutually disjoint and hence $A \cap B = \emptyset$. With respect to $(*)$ we obtain $\mathrm{Cl}A \cap \mathrm{Cl}B = F$. Therefore A and B are disjoint and closed sets in $X \setminus F$ and hence there is a continuous function $g : X \setminus F \to [0, 1]$ such that $f(x) = 0$ for each $x \in A$ and $f(x) = 1$ for each $x \in B$. This yields $\omega_g(x) = 1$ for each $x \in F$. ♢

It is easy to see that all assertions (slightly modificated) are true if instead of $Y = \mathbb{R}$ we assume that Y contains a subspace isometric with $\mathbb{R}$ (e.g. if Y is a (nontrivial) normed linear space). By [7] every first countable topological space without isolated points is resolvable. Hence by Lemma 5, Lemma 6, Proposition 1 and Theorem 4 we obtain

Theorem 5. *Let X be a Baire resolvable perfectly normal locally connected space (or let X be a Baire pseudometrizable space without isolated points) and let Y be a metric space containing a subspace isometric with $\mathbb{R}$. Let C, E A be subsets of X. Then there is a function $f : X \to Y$ such that $C = C(f)$, $E = E(f)$ and $A = A(f)$ if and only if $C \subset E \subset A$, C is G_δ, A is closed and $A \setminus C$ is of the first category.*

One can see that if $E = X$ in Theorem 5, then the assumption "X is resolvable" can be omitted. Hence we have

Corollary 2. *Let X be a Baire perfectly normal locally connected space or let X be a Baire pseudometrizable space. Then the set M is the set of all discontinuity points of some quasicontinuous function $f : X \to \mathbb{R}$ if and only if M is an F_σ set of the first category.*

In [10], the question to characterize the sets of discontinuity points of quasicontinuous functions $f : \mathbb{R}^2 \to \mathbb{R}$ (or even $f : X \to \mathbb{R}$,

X is a topological space) is posed. It was solved in [6] for $X = \mathbb{R}^2$. Our corollary further generalizes this result. Examples 1, 2, 4 show that this is not true for arbitrary X.

PROBLEM. Is Theorem 5 true for every Baire resolvable perfectly normal topological space?

The next example shows that the condition "X is normal" in Theorem 5 cannot be replaced with "X is T_1 completely regular".

LEMMA 7. *Let X be a topological space and let (Y, d) be a metric one. Let $f, g : X \to Y$ be quasicontinuous functions. If $f|_A = g|_A$ for some dense subset A of X, then $C(f) = C(g)$.*

Proof. Suppose that there is $x \in C(g) \setminus C(f)$.

1) Let $f(x) = g(x)$. Then there is $\eta > 0$ such that for each neighbourhood U of x there is $t_U \in U$ with $d(f(x), f(t_U)) \geqq \eta$. Further there is an open neighbourhood U of x such that $d(g(x), g(y)) < \frac{\eta}{2}$ for each $y \in U$. Since f is quasicontinuous at t_U, there is an open nonempty set $G \subset U$ such that $d(f(t_U), f(y)) < \frac{\eta}{2}$ for each $y \in G$. Let $z \in A \cap G$. Then $\eta \leqq d(f(x), f(t_U)) \leqq d(g(x), g(z)) + d(f(z), f(t_U)) < \eta$, a contradiction.

2) Let $f(x) \neq g(x)$. Put $\eta = d(f(x), g(x)) > 0$. Then there is a neighbourhood U of x such that $d(g(x), g(y)) < \frac{\eta}{2}$ for each $y \in U$. Further there is an open nonempty $G \subset U$ such that $d(f(x), f(y)) < \frac{\eta}{2}$ for each $y \in G$. Let $z \in A \cap G$. Then $\eta \leqq d(f(x), g(x)) \leqq d(f(x), f(z)) + d(g(z), g(x)) < \eta$, a contradiction. $\diamondsuit$

EXAMPLE 4. Let X be the Niemytzki plane. Namely, put $P = \{(x, y) \in \mathbb{R}^2 : y > 0\}$, $L = \{(x, y) \in \mathbb{R}^2 : y = 0\}$, $X = P \cup L$. Let $\mathcal{T}$ be the topology on X such that $\mathcal{T}$ restricted to P is the usual topology. If $x \in L$ and S is any open sphere in P tangent to L at x, then $\{x\} \cup S$ is an open set in X containing x (see [2]). The space X is Baire resolvable Tychonoff perfect locally connected. For each $D \subset L$ we put $W_n = P \cup D$, $C = \bigcap_{n=1}^{\infty} W_n = P \cup D$, $E = A = \bigcap_{n=1}^{\infty} \mathrm{Cl} W_n = X$. Then C, E, A satisfy (C). By Lemma 7, if f, g are quasicontinuous functions and $C(f) \neq C(g)$ then $f|_P \neq g|_P$.

However, the cardinality of all subsets of L is $2^{\mathfrak{c}}$ while the cardinality of all continuous functions on P is $\mathfrak{c}$ (P is separable). Hence for some $D \subset L$ there does not exist a function $f : X \to \mathbb{R}$ with

$C(f) = P \cup D$, $E(f) = A(f) = X$.

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