

THE WORK OF PROFESSOR MARIO DOLCHER (*)

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SOMMARIO. - *Sono messi in evidenza alcuni importanti risultati in Topologia presenti nell'opera del Professor Mario Dolcher e il suo contributo allo sviluppo degli studi di Matematica a Trieste.*

SUMMARY. - *Some important results in Topology which are present in the work of Professor Mario Dolcher and his contribution to the development of mathematical studies in Trieste are pointed out.*

1. Introduction.

It is a great honour and pleasure for me to have the opportunity to recall the work of Professor Mario Dolcher. As it was announced, this conference takes place in the framework of the celebrations of the fifty years of the Faculty of Engineering of the University of Trieste; this anniversary is also strictly tied with the scientific life of Professor Mario Dolcher, who, after having completed his studies at the Scuola Normale Superiore in Pisa, was one of the young mathematicians (another, well known to many participants to this conference, was Professor Stefano Guazzone) who began their career in the University of Trieste in 1945 – 46, devoting their teaching and research work first to the Faculty of Engineering, since the Faculty of Sciences was activated some years later. He was one of the first collaborators of the late Professor Bruno de Finetti, the first chairman, in 1947, of

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the Institute of Mathematics (formerly Seminar of Mathematics, founded in 1946 by Professor Ugo Morin and directed from 1947 by de Finetti).

The scientific work of Professor Mario Dolcher has had a great influence for many years on many mathematicians grown in Trieste. He suggested lots of interesting arguments in Topology, but also in different fields of Mathematics. Several of the present or recently past members of the Department of Mathematics had their mathematical activity inspired by him, as invaluable teacher in courses of Analysis, Algebra, Topology and Elementary Mathematics from an Higher Standpoint or as supervisor of their dissertations (I can remember Alfredo Bellen, Sergio Invernizzi, Romano Isler, Andrea Sgarro, Maurizio Trombetta, Aljoša Volčič, Fabio Zanolin and myself). Some of them, after a “topological” beginning, have changed their mathematical interests and are now working in fields ranging from Numerical Analysis to Measure Theory, to Differential Equations, to Mathematical Economics, to Theoretical Informatics, but all preserved the love for the rigour and honesty of the proof and for the mathematical fantasy and invention that is a typical character of the work of Mario Dolcher. He has always been for us a model to follow, sometimes difficult to imitate (at least for me).

Almost all Dolcher’s papers were written in Italian; in addition some results, even important ones, had a very limited circulation (in fact they were published as “Quaderni matematici”, i.e. internal reports of the Trieste Mathematical Institute). So it is really a pleasure for me to try to expose to a wider mathematical audience some achievements of Professor Mario Dolcher.

2. The First Period 1942–1959.

- 1) *Geometria delle trasformazioni continue, I: Sopra un teorema di Radó.* Annali della Scuola Normale Superiore di Pisa, Serie II, Vol XIV (1945), 1–18.
- 2) *Nozione generale di struttura per un insieme.* Rend. Sem. Matem. Univ. Padova, XVIII (1949), 265 – 291.
- 3) *Questioni di minimo per insiemi chiusi sconnettenti uno spazio topologico.* Rend. Sem. Matem. Univ. Padova, XIX (1950), 159 – 171.

- 4) *Geometria delle trasformazioni continue. Un teorema sulle trasformazioni di una corona circolare.* Rivista Matem. Università di Parma, 5 (1954), 339–361.
- 5) *Determinazione del massimo numero di eccezioni alla n -vocià dell'inversa di una trasformazione piana continua.* Annali Univ. Ferrara (Nuova Serie), Vol VII, 3 (1959), 53–64.

Some papers of Professor Mario Dolcher, Part A.

Initially Dolcher studied the properties of continuous transformations of a plane (a complex plane) into itself. The main idea was to single out those properties of analytic functions depending only on the topology of the Riemann surface and hence depending only on the continuity of the transformations of a sphere into itself and not on the metric properties (monogeneity) of the function, following the methods of T. Radó [R] and L. Cesari [C]. To this study are dedicated papers (1), (3), (4) and (5) of Part A. Paper A (2) is concerned with the general definition of a structure for a set; the argument will be treated in a separate article by Professor Dolcher. In paper A (1) a generalization is proved of a theorem of T. Radó. Precisely, let Φ be a continuous transformation from a region \overline{C} (bounded by a Jordan curve C^*) in the plane π into the plane π' ; let us denote by $O(Q; \Phi(C^*)) = n(Q)$ the topological index of the curve $\Phi(C^*)$ with respect to the point Q . Let $\mu(Q)$ be the number of pre-images in π of the point $Q \in \pi'$. If A' is an open connected set on which $n(Q) = n(A') > 0 (Q \in A')$ is constant, then, roughly speaking, A' is covered (at least) n times by $\Phi(C)$; but there can be some exceptional points, i.e. points $Q \in A'$ such that $\mu(Q) < n(Q)$. The following theorem is proved

THEOREM 2.1. *Suppose that for a point $Q_0 \in \Phi(\overline{C})$ there is a component G_0 of $G = \Phi^{-1}(Q_0)$ such that for any simple closed curve γ^* in \overline{C} , which does not meet G but circuits G_0 , $O(Q_0; \Phi(\gamma^*)) \geq n(Q_0)$ holds; if g_0 is the maximal neighborhood of Q_0 which does not meet $\Phi(C^*)$ and $Q (\neq Q_0) \in g_0$, then $\mu(Q) \geq n(Q_0)$.*

This theorem generalizes a theorem of Radó, which was proved under the additional assumption that $\mu(Q_0) = 1$. Cesari proved that every exceptional point is isolated, so that if B' is an open connected subset such that $\overline{B'} \subset A'$ then it cannot contain more than a finite number of exceptional points. In A (5) this number is precisely determined:

THEOREM 2.2. *If Φ and C are as above, the exceptional points are denoted by p'_i ($i = 1, 2, \dots, k$) and $m_i (< n)$ are the cardinalities of their pre-images under Φ , then*

$$\sum_{i=1}^k m_i \geq (k-1)n + 1 \quad .$$

Hence, no more than $n - 1$ exceptional points can be contained in A' .

In A (4) the same formula is proved to hold in the case of an annulus if $n = n_\alpha - n_\beta$, where $n_\alpha > n_\beta > 0$ are the topological indices of the two equioriented boundaries of the region and it is supposed that their images have the same orientation. It should be noted that all these results were proved without making use of the transformation $z = w^n$.

Finally in A (3) Dolcher solves problems concerning the existence: (1) of minimal closed subspaces separating a given pair of points in a topological space; (2) of minimal closed subspaces that disconnect the space; (3) of a minimal closed subset contained in any closed set separating a given pair of points. In the case of locally connected spaces, necessary and sufficient conditions are given, which are shown to be sufficient in the case of a general topological space. These results extend to a (locally connected) space similar results proved by Mazurkiewicz [M1, M2] and Kuratowski [K] in the case of the space \mathbb{R}^n .

3. The Period After 1955

- 1) *Topologia delle famiglie di filtri*. Rendiconti Sem. Matem. Univ. Padova, XXIV (1955), 443 – 473.
- 2) *Una proprietà delle famiglie numerabili di catene sopra un albero*. Ann. Univ. Ferrara (Nuova Serie), Vol. VIII, 2 (1959), 7 – 16.
- 3) *Topologie e strutture di convergenza*. Annali Scuola Normale Sup. di Pisa, Serie III, Vol. XIV, I (1960), 63 – 92.
- 4) *Esistenza di strutture di convergenza aventi grado successionale arbitrario*. Quaderni matematici, Istituto di Matematica, Univ. di Trieste.
- 5) *Condizioni per la deducibilità di una topologia da convergenza di successioni*. Quaderni matematici, Istituto di Matematica, Univ. di Trieste.
- 6) *Questioni topologiche collegate con la quasi-periodicità nel campo reale*. Rend. Sem. Matem. Univ. Padova, (1965), 1 – 17.
- 7) *Sul problema dell'approssimazione per le funzioni quasi-periodiche in un gruppo abeliano topologico*. Quaderni matematici, Istituto di Matematica, Univ. di Trieste, 1963-64.
- 8) *An alternative definition of pseudoradial spaces*. Convergence Structures 1984, Proc. of the Conference on Convergence, Ed. by J. Novák, W. Gähler, H. Herrlich, J. Mikusiński, Akademie-Verlag, Berlin 1985, 71 – 76.

Further papers of Topology, Part B.

In 1955 Dolcher began to be interested to aspects of General Topology, which eventually became the main interest of his research and the starting point of the research of many of his students.

In B (1), in order to study by a general method the problem of extensions of a topological space (in particular, compactifications and completions) he introduced the notion of a topological space associated with any family of filters. This method was first applied, with the aim of obtaining results independent from the Choice Axiom, to the family of compactifying filters by his student Gerolini [G].

Given a family Φ of filters, two binary relations \circ and \triangleleft on filters are introduced.

DEFINITION 3.1. Let α and β be filters. They are said to be *tied*

$$\alpha \circ \beta$$

if $A \in \alpha$ and $B \in \beta$ implies $A \cap B \neq \emptyset$.

DEFINITION 3.2. A family Γ of filters is said to be *tied* with a filter α , denoted

$$\Gamma \circ \alpha \text{ or } \alpha \circ \Gamma \quad ,$$

if for any $A \in \alpha$ there is a filter $\gamma \in \Gamma$ such that $\gamma \circ [A]$. Here, as usual, $[A]$ denotes the filter generated by the set A .

DEFINITION 3.3. A family Γ of filters is said to *enter* a filter α

$$\Gamma \triangleleft \alpha$$

if $\forall A \in \alpha, \exists \gamma \in \Gamma$ such that $A \in \gamma$.

The following propositions can be easily proved:

$$(3.4) \quad \{\beta\} \triangleleft \alpha \Leftrightarrow \beta \supset \alpha \quad ;$$

$$(3.5) \quad \{\beta\} \triangleleft [A] \Leftrightarrow A \in \beta \quad ;$$

$$(3.6) \quad \{[B]\} \triangleleft [A] \Leftrightarrow B \subset A \quad ;$$

$$(3.7) \quad \Gamma \triangleleft \alpha \Rightarrow \alpha \circ \Gamma \quad ;$$

$$(3.8) \quad \alpha \in \Gamma \Rightarrow \Gamma \triangleleft \alpha \quad ;$$

$$(3.9) \quad \Gamma \triangleleft \alpha, \Gamma_1 \supset \Gamma, \alpha_1 \subset \alpha \Rightarrow \Gamma_1 \triangleleft \alpha_1 \quad .$$

Now, given a set S , and a family Φ of filters on S , if $\Theta \subset \Phi$, define Θ' as follows

$$(3.10) \quad \Theta' = \{\varphi \in \Phi : \Theta \triangleleft \varphi\} \quad .$$

It can be shown that the following propositions hold

$$(K1) \quad \emptyset' = \emptyset \quad ;$$

$$(K2) \quad \forall \Theta (\subset \Phi) \quad \Theta \subset \Theta' \quad ;$$

$$(K3) \quad \forall \Theta_1 (\subset \Phi), \forall \Theta_2 (\subset \Phi) \quad (\Theta_1 \cup \Theta_2)' = \Theta_1' \cup \Theta_2' \quad ;$$

$$(K4) \quad \forall \Theta (\subset \Phi) \quad (\Theta')' = \Theta \quad .$$

Since properties (K1), ..., (K4) are the well known Kuratowski axioms we can conclude that a family of filters Φ over a set S becomes a topological space when the closure of $\Theta \subset \Phi$ is given by Θ' , the family of all filters in Φ in which Θ enters. This topology is called by Dolcher the **strong topology** and Φ with this topology is called the **strong space**. It is proved that any strong space of filters satisfies the T_0 separation property; conditions are given under which the strong space is T_1 , T_2 or regular. In particular a sufficient condition that the strong space Φ be regular is that

$$(3.11) \quad \varphi \circ \Theta (\varphi \in \Phi, \Theta \subset \Phi) \Rightarrow \Theta \triangleleft \varphi \quad .$$

When Φ satisfies the above condition (3.11) it is called a *regular family of filters*.

Let us define, for every $A \subset S$

$$(3.12) \quad \Phi_A^* = \{\varphi \in \Phi : A \in \varphi\} \text{ end}$$

$$(3.13) \quad \Phi_A^\circ = \{\varphi \in \Phi : \varphi \circ [A]\} \quad .$$

Then it can be shown that Φ_A^* is an open set and Φ_A° is a closed subset of the strong space for any subset $A \subset S$. In fact the family $\{\Phi_F^* : F \in \varphi\}$ is a base of open neighborhoods of the point $\varphi \in \Phi$ in the strong topology. Even more can be shown: the families of sets of filters given by $\{\Phi_F^* : F \in \varphi, \varphi \in \Phi\}$ and $\{\Phi_F^\circ : F \in \varphi, \varphi \in \Phi\}$ are a base of open and respectively of closed sets in the space Φ endowed with the strong topology. Finally the following *subordination theorem* holds

THEOREM 3.14. *Let S be a T_0 topological space and let $\Sigma = \{\sigma_p : p \in S\}$ be the family of the filters of neighborhoods of points of S . If Φ is a family*

of filters on S such that $\Sigma \subset \Phi$, then the subspace Σ of the strong space Φ is homeomorphic to S , according to the correspondence

$$p(\in S) \mapsto \sigma_p(\in \Sigma) \quad .$$

Dolcher defines also the notion of an *absolutely closed* family $\Gamma \subset \Phi$.

DEFINITION 3.15. A subset Γ of Φ is said to be *absolutely closed* if it contains every filter to which it is tied; i.e. if

$$\varphi \in \Phi, \varphi \circ \Gamma \Rightarrow \varphi \in \Gamma \quad .$$

The family of absolutely closed subsets of the set Φ satisfies the properties of the closed sets in a topological space and then defines a topology on Φ , which is called the **weak topology** in the family Φ of filters over a set S . The space Φ endowed with this topology is called the **weak space**. It is proved that, in general, the strong topology is finer than the weak one. The two topologies coincide if and only if the family Φ is a regular family of filters.

As an easy application, it is shown that the real line can be defined as a filter space starting from the *cut-filters* on the rational numbers \mathbb{Q} . The cut-filters on \mathbb{Q} are those filters φ that have a base of intervals, i.e. such that for any $A \in \varphi$ there is an open rational interval in φ for which $(a_1, a_2) \subset A$ holds; moreover, if $(a_1, a_2) \in \varphi$, there is in φ an open interval (a_3, a_4) with $a_1 < a_3 < a_4 < a_2$; finally $\cap \varphi$ contains at most one rational point. The strong and weak topologies coincide on the family of cut-filters and give rise to a model of the real line.

I must say that I like very much this paper, that, perhaps, did not receive the due attention for a long time even by Dolcher's students.

In B (2) it is shown that given an r -tree A , if \mathcal{C} and \mathcal{C}' are two countable families of chains that cover A then there is an automorphism of A which transforms the family \mathcal{C} in the family \mathcal{C}' . As a consequence, it is proved that

THEOREM 3.16. *There is a countable, totally disconnected Hausdorff space without isolated points, which is not isomorphic to the rational numbers \mathbb{Q} .*

The most widely known paper of Dolcher and the one that have had more success among his students is B (3): *Topologie e strutture di convergenza*.

Given a set E a sequence S is a function $S : \mathbb{N}^+ \rightarrow E$. A sequence is denoted by $\{p_n\}$ or by $\{p_n\}_n$; the sequence $\{p_{r,s}\}_r$ is the sequence $\{p_{1,s}, p_{2,s}, \dots, p_{r,s}, \dots\}$. If S is a sequence, $|S|$ is the underlying set of points in the sequence. If S, S' are sequences Dolcher writes $S' \subset S$ if S' is a subsequence of S . Three operations, called α, β and γ are introduced on the set ${}^{\mathbb{N}^+}E$ of all sequences on a set E .

- α) Given $\{p_n\}$ consider deduced from it every sequence for which the given one is a *remainder* (i.e. every $\{p'_n\}$ is such that there is $k \in \mathbb{N}$ for which $p'_{k+n} = p_n$).
- β) Given $\{p_n\}$ consider deduced from it every sequence $\{p_{r_n}\}$ such that $\lim r_n = \infty$; in particular, every subsequence.
- γ) Given a finite number of sequences $S^i = \{p_n^i\}$ ($i = 1, 2, \dots, k$) consider deduced from them every sequence S that has $h (\leq k)$ subsequences $S'^j = \{p_{r_n^j}^j\}$ equal respectively to h of the S^i and such that the natural numbers r_n^j ($j = 1, 2, \dots, h; n = 1, 2, \dots$) are all distinct, and every positive natural number appears among them.

Let us denote by $(\alpha\beta\gamma)$ the operation through which, given a family \mathcal{S} of sequences, every possible sequence is obtained by successive applications of the operations (α) , (β) and (γ) in arbitrary number and order: every sequence so obtained is said to be *deduced from the family* \mathcal{S} . The totality of sequences deduced in such a way is called the $(\alpha\beta\gamma)$ -closure of the family \mathcal{S} and denoted by \mathcal{S}^* .

It is proved that

$$(3.17) \quad (\mathcal{S}^*)^* = \mathcal{S}^* \quad .$$

An application $\Phi : {}^{\mathbb{N}^+}E \rightarrow \mathcal{P}(E)$ is a *convergence structure* (or simply a *convergence*) on E provided the following conditions are satisfied

- (FK1) every constant sequence $\{p\}$ converges to p .
- (FK2) if a sequence converges to p , then every subsequence converges to p .
- (FK3) if a sequence does not converge to p , there is a subsequence, no subsequence of which converges to p .

These are the usual Fréchet – Kuratowski axioms for a convergence, without mentioning any unicity of the limit.

Dolcher denotes by λ a set with a convergence; \mathcal{L} is the category of all convergences with the convergence maps. A natural order is given among the convergences having the same underlying set E ;

$$\lambda_1 \prec \lambda_2 (\lambda_1 \text{ is less fine than } \lambda_2)$$

means that any sequence converging in λ_2 to a point p also converges in λ_1 to p .

Important subcategories of \mathcal{L} are those satisfying the following extra conditions

(FKT'₀) For every pair of distinct points, there is a sequence converging only to one of the two.

(FKT''₀) For every pair of distinct points p and q one of the two constant sequences $\{p\}$, $\{q\}$ does not converge to both points.

(FKT₁) Every constant sequence converges to a unique limit point.

(FKT₂) No sequence converges to two different points.

Such subcategories will be denoted by \mathcal{L}'_0 , \mathcal{L}''_0 , \mathcal{L}_1 , \mathcal{L}_2 and they will be thought as complete subcategories of \mathcal{L} .

The category of topological spaces and continuous maps will be denoted by \mathcal{T} . The class of topologies on a given underlying set E will be considered endowed with the usual order. \mathcal{T}_0 , \mathcal{T}_1 , \mathcal{T}_2 will denote the subcategories of topological spaces which satisfy the separation axioms T_0 , T_1 , T_2 .

Two functors $L : \mathcal{T} \rightarrow \mathcal{L}$ and $T : \mathcal{L} \rightarrow \mathcal{T}$ are defined. L associates to every topological space (E, τ) a convergence on E as follows: A sequence in E $L(\tau)$ -converges to p if it is eventually in every neighborhood of p . Given a convergence structure λ on a set E a topology $T(\lambda)$ on E is defined as follows: A set $A \subset E$ is open in $T(\lambda)$ if and only if for every $p \in A$ every sequence converging to p is eventually in A (i.e. A is *sequentially open*). The functors L and T are defined in the obvious way on the maps. The two functors are shown to be monotone

$$(3.18) \quad \tau \prec \tau' \Rightarrow L(\tau) \prec L(\tau') \quad ,$$

$$(3.19) \quad \lambda \prec \lambda' \Rightarrow T(\lambda) \prec T(\lambda') \quad ,$$

For the composite functors, the following equations are proved

$$(3.20) \quad (\forall \tau \in \mathcal{T}) \quad (TL(\tau) \succ \tau) \quad ;$$

$$(3.21) \quad (\forall \lambda \in \mathcal{L}) \quad (LT(\lambda) \prec \lambda) \quad .$$

It is easily seen that a topology τ is deducible from a convergence (i.e. $\tau = T(\lambda)$) if and only if

$$(3.22) \quad TL(\tau) = \tau \quad .$$

Analogously a convergence λ is deducible from a topology if and only if

$$(3.23) \quad LT(\lambda) = \lambda \quad .$$

There are convergences which are not deducible from a topology, but all convergences in \mathcal{L}_2 are; if they are deducible from a topology, in general, they are deducible from infinite topologies and $T(\lambda)$ is the finest one. For topologies, it is proved that not all topologies, even not all Hausdorff topologies, can be deduced from a convergence. However if τ is an Hausdorff deducible topology it can be deduced from a unique convergence.

The behaviour of the sequential closure is also examined. If λ is a convergence structure on a set E and $A \subset E$ denote by \overline{A} the $T(\lambda)$ -closure of A and by \hat{A} its sequential closure. Usually $A \subset \hat{A} \subset \overline{A}$ holds, and $\hat{A} = \overline{A}$ if and only if $\hat{\hat{A}} = \hat{A}$. If $\hat{A}^{(0)} = A$ and for every ordinal $\eta < \omega_1$, $\hat{A}^{(\eta)} = \cup_{\alpha < \eta} \hat{A}^{(\alpha)}$ then Dolcher proves that

$$(3.24) \quad \hat{A}^{(\omega_1)} = \overline{A} \quad ,$$

where $\hat{A}^{(\omega_1)} = \cup_{\alpha < \omega_1} \hat{A}^{(\alpha)}$.

If (E, λ) is a convergence structure, the following ordinal number is defined

$$(3.25) \quad \eta = \min\{\alpha : \hat{A}^{(\alpha)} = \hat{A}^{(\alpha+1)}, \forall A \subset E\} \quad .$$

It is called the *sequential degree* of (E, λ) and, as it was shown, we have $\eta \leq \omega_1$ (in fact whenever $\hat{A}^{(\eta)} = \hat{A}^{(\eta+1)}$, $\hat{A}^{(\eta)} = \overline{A}$ holds). In B (4), answering a problem raised by J. Novák [N], Dolcher shows that for any $\eta \leq \omega_1$ there is a space having sequential degree η . A model E_η of such a space is produced for each η and the cardinality of such model is \aleph_0 for $0 < \eta < \omega$; it is 2^{\aleph_0} for $\omega \leq \eta < \omega_1$; it is 2^{\aleph_1} for $\eta = \omega_1$. The problem of characterizing the topologies deducible from a convergence (i.e. the sequential topologies) is examined once more in B (5) where the following theorem is proved

THEOREM 3.26. *A topology τ on a set E satisfies condition the $TL(\tau) = \tau$ (i.e. it is a sequential topology) if and only if for every $x \in E$ the enlarged envelope $[x]^{\Phi, \omega^*}$ coincides with the neighborhood filter σ_x of x .*

Here the following notion were used. If $\Phi = \{\varphi_i\}_{i \in I}$ is an indexed family of filters on set E and ν is a filter on the set I , for every $J \in \nu$ consider the filter $\psi_J = \bigwedge_{i \in J} \varphi_i$. Define $\Psi = \bigvee_{J \in \nu} \psi_J$. Then this filter Ψ is not trivial; it will be called the *envelope filter of the family Φ along the filter ν on I* . This filter is denoted by ν^Φ and its members are sets of the type $\cup_{i \in J} F_i$, $J \in \nu, F_i \in \varphi_i$. In particular it is considered here the case that $I = E$, a topological space, and Φ is Σ , the family of all neighborhood filters, or the family of filters generated by the “queues” of sequences converging to the points of E . The construction of the envelope can be iterated according to the following inductive scheme. Let $\nu^{\Phi, 0} = \nu$ and $\nu^{\Phi, n+1} = (\nu^{\Phi, n})^\Phi$. Finally the ω -envelope of the family Φ along ν is the filter $\nu^{\Phi, \omega} = \bigwedge_{n < \omega} \nu^{\Phi, n}$. Now if $\Phi = \{\varphi_x\}_{x \in E}$ is a family of filters indexed on E , every set of the form $\cup_{x \in A} \{F_x : F_x \in \varphi_x\}$ is said to be Φ -deduced by $A \subset E$. A sequence A_n of subsets of E such that $A_0 = A$ is said to be Φ -deduced by A if A_{n+1} is Φ -deduced from A_n . A set is said to be (Φ, ω) -deduced from A if it is the union of a sequence Φ -deduced from A . If $\varphi_x \prec [x]$ a sequence of sets Φ -deduced from A is increasing. If moreover ν is a filter on E for every $X \in \nu$ let $\Phi(X)$ be the family of all sets (Φ, ω) -deduced from X . The union of all such sets for X ranging in ν is a filter base and the filter generated by it is called *the enlarged ω -envelope of the family Φ along the filter ν* . It is denoted by ν^{Φ, ω^*} . The following holds

$$(3.27) \quad \nu^{\Phi, \omega^*} \prec \nu^{\Phi, \omega} \quad .$$

With the above discussion, the statement of theorem (3.27) should be clear.

In B (6) a topology on \mathbb{R} (deduced by a precompact uniformity) is described which is compatible with the addition of real numbers and is such that the class of almost-periodic functions coincides with that of uniformly continuous ones; it is the topology which is the trace on \mathbb{R} of the Bohr compactification of the real line. This study is continued in B (7) where the functions considered are real-valued functions, defined on a topological abelian group. However, since the Bochner theorem does not hold in this more general context, it is shown that there is a locally compact group on which the Banach algebra of the almost-periodic functions is larger than the algebra generated by the functions chosen to be considered as periodic.

In B (8) the pseudoradial spaces are characterized in an interesting way.

It is known that pseudoradial topological spaces are a generalization of sequential spaces (see for example [H], [A], [DIT]). Here a sequence means any application from an ordinal number η to a set E . If $\eta = \omega$ then the usual sequential case is obtained. A topological space is said to be pseudoradial if for any non-closed subset $C \subset E$ there is a point $x \in \overline{C} \setminus C$ and a sequence $(x_\alpha)_{\alpha < \eta}$ such that $x_\alpha \in C$ for any $\alpha < \eta$ and $x_\alpha \rightarrow x$. Here Dolcher introduces the following definition of a *colimit of a filter \mathcal{G} according to a family of filters Φ* .

Given a set E , a filter \mathcal{G} and a family Φ of filters \mathcal{F}_x indexed by the elements of E , for $M \subset E$ consider the filter $\mathcal{F}_M = \bigwedge_{x \in M} \mathcal{F}_x$, generated by sets $\cup_{x \in M} F_x$, $F_x \in \mathcal{F}_x$. The family of sets

$$(3.28) \quad \{\cup_{n=0}^{\infty} A_n : A_0 \in \mathcal{G}, A_{n+1} \in \mathcal{F}_{A_n}, n \in \mathbb{N}\}$$

is a filter base. The filter generated by it is said to be the colimit of \mathcal{G} according to Φ and is denoted by

$$(3.29) \quad \text{Col}_{\Phi} \mathcal{G} .$$

The case of interest here is the following: E is a topological space, \mathcal{G} is the filter $[A]$, with $\emptyset \neq A \subset E$, possibly $A = \{x\}$, \mathcal{F}_x is the intersection of all sequence filters converging to x . The following theorem is proved

THEOREM 3.30. *A topological space X is pseudoradial if and only if for every point x the filter of its neighborhoods is the colimit of the ultrafilter $[x]$.*

4. Concluding Remarks.

I would like to remark only some points concerning constructions invented by Dolcher concerning filters and convergence structures. The definition of strong topology, starting with sets as in (3.11), is strictly related to the topology of the Stone space of a Boolean Algebra. But here, due to the concern of Dolcher on constructibility, only filters are used, not ultrafilters. Analogously his construction, reminded after theorem 3.26, of the envelope and ω -envelope filter of a family Φ along a filter ν on Φ (in fact on the set that indexes Φ) is a construction strictly related to that of the *sum of a family of ultrafilters X with respect to a ultrafilter y* given by Frolík [F]. This well known paper of Frolík, was communicated in 1966

and published in 1967. Although no date is given for paper B (5), it is certainly independent from Frolík's and should go back to 1966-67. Here again the construction is designed for filters in general, not for ultrafilters. The "philosophy" of this paper is to characterize sequential spaces in "local" terms, i.e. through the structure of the neighborhood filter of the points of the space. Other efforts in this direction were done in Trieste by Isler [I] and Volčič [V], and many new results in new directions were obtained, starting from B (3), by F. Zanolin (see, for example [FZ]). In B (4) a construction is given of a sequential structure having arbitrary sequential order $\eta \leq \omega_1$. These results should be compared with the ones of Arhangel'skiĭ and Franklin [AF], in which "sequential fans" are produced having sharper topological properties: in fact they are all countable, 0-dimensional, Hausdorff spaces, while Dolcher's are not countable for $\eta \geq \omega$. Finally, I should mention that, following the ideas of Dolcher in B (3), M. Trombetta investigated convergence structures determined by transfinite sequences (see for example [T]).

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