A NOTE ON THE CLASSIFICATION OF $\mathcal{F}$-FIBRATIONS (*)

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SOMMARIO. - *Il concetto di una $\mathcal{F}$-fibrazione viene definito richiedendo che le fibre appartengano ad una fissata categoria $\mathcal{F}$ di spazi topologici. In questa nota sono descritte alcune costruzioni sulle $\mathcal{F}$-fibrazioni che vengono usate per ridurre la classificazione, a meno di equivalenza omotopica fibrata, delle $\mathcal{F}$-fibrazioni alla classificazione delle loro fibrazioni principali associate.*

SUMMARY. - *The concept of an $\mathcal{F}$-fibration is defined by requiring that the fibres belong to a fixed category of spaces $\mathcal{F}$. We describe several constructions on $\mathcal{F}$-fibrations and then show how to reduce, under suitable conditions, the classification up to fibre homotopy equivalence of $\mathcal{F}$-fibrations to the classification of their associated principal fibrations.*

1. Preliminaries on $\mathcal{F}$-notation.

We recall some standard terminology used in the theory of $\mathcal{F}$-fibrations. All spaces and maps will belong to **kTop**, the category of compactly generated spaces. In particular, let $\mathcal{F}$ be a non-empty subcategory of **kTop**. A triple $(E, p : E \to B, B)$ is an $\mathcal{F}$-*arrow* if p is onto and all fibres are objects of $\mathcal{F}$. A map $(g, \overline{g}) : (D, q, A) \to (E, p, B)$ between $\mathcal{F}$-arrows is an $\mathcal{F}$-*map* if the restriction of g to every fiber is in $\mathcal{F}$. Two $\mathcal{F}$-maps are $\mathcal{F}$-*homotopic* if there exists a homotopy $(G, \overline{G}) : (D \times I, p \times 1_I, A \times I) \to (E, p, B)$ between them, which is an $\mathcal{F}$-map. Two $\mathcal{F}$-arrows are $\mathcal{F}$-*homotopy equivalent* if there is an $\mathcal{F}$-map between them, that has an $\mathcal{F}$-map as $\mathcal{F}$-homotopy inverse.

An $\mathcal{F}$-arrow is an $\mathcal{F}$-*fibration* if it has the covering homotopy property with respect to all $\mathcal{F}$-arrows, i.e. if for every $\mathcal{F}$-arrow (D, q, A), every $\mathcal{F}$-map $(g, \overline{g})$ from p to q and every homotopy $\overline{G} : A \times I \to B$ of $\overline{g}$, there is an $\mathcal{F}$-homotopy $G : D \times I \to E$ of g covering $\overline{G}$. Because of the

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PROPOSITION 1.1 ([2], Lemma 7.2.7) *$\mathcal{F}$-fibrations are fibrations*

all fibres of an $\mathcal{F}$-fibration over a connected base have the same homotopy type. To reflect this structure we will require in the sequel that in $\mathcal{F}$ all maps are $\mathcal{F}$-homotopy equivalences and that there exists a distinguished object $F \in \mathcal{F}$ such that $\mathcal{F}(F, X) \neq \emptyset$ for all $X \in \mathcal{F}$. We call such an $\mathcal{F}$ a *category of fibres.* Clearly, the pullback of an $\mathcal{F}$-fibration is again an $\mathcal{F}$-fibration.

An $\mathcal{F}$-fibration is *numerable* if there exists a numerable covering for its base space such that the restriction of the fibration to every element of the base is $\mathcal{F}$-homotopy equivalent to the trivial $\mathcal{F}$-fibration with fibre F. We denote by $\mathcal{B}_{\mathcal{F}}$ the category of all numerable $\mathcal{F}$-fibrations over connected spaces in **kTop** and of all $\mathcal{F}$-maps between them.

THEOREM 1.2 (Dold-May [2], Th.7.2.4) *If a map between two $\mathcal{F}$-fibrations over the same base is an $\mathcal{F}$-homotopy equivalence when restricted to every element of some numerable covering of the base, then this map is an $\mathcal{F}$-homotopy equivalence.*

PROPOSITION 1.3 ([2], Th. 7.2.13) *The pullbacks of an $\mathcal{F}$-fibration along two homotopic maps are $\mathcal{F}$-homotopy equivalent.*

We denote by $k_{\mathcal{F}}$ the cofunctor from the homotopy category of **kTop** to **Set** that assigns to every $B \in$ **kTop** the set of $\mathcal{F}$-homotopy equivalence classes of numerable $\mathcal{F}$-fibrations over B, and to every homotopy class of maps in **kTop** the pullback along it.

Just as for fibre bundles, associated principal fibrations play a fundamental role in the study of $\mathcal{F}$-fibrations. Let $\mathcal{H}$ be the image of the category $\mathcal{F}$ via the functor $\mathcal{F}(F, -) :$ **kTop** $\to$ **kTop** (where **kTop** is considered as enriched over itself). It's easy to check that $\mathcal{H}$ is again a category of fibres with distinguished object $H := \mathcal{F}(F, F)$. Note that H is a topological monoid with the property that all translations in H are homotopy equivalences. We call $\mathcal{H}$ the *associated principal category of fibres* for $\mathcal{F}$. A *principal $\mathcal{H}$-fibration* is an $\mathcal{H}$-fibration together with a continuous fibrewise right action of the distinguished fibre H on the total space.

We will use the following construction to pass from principal $\mathcal{H}$-fibrations to $\mathcal{F}$-fibrations. Given a right H-space A and a left H-space B over a topological monoid H form the amalgamated product $A \times_H B$ as the quotient of the product $A \times B$ modulo the equivalence relation, generated by the requirement that $(a \cdot h, b)$ is equivalent to $(a, h \cdot b)$ for all $a \in A, b \in B$ and $h \in H$.

In particular if $\mathcal{F}$ is a category of fibres with the distinguished fibre F, X an object in $\mathcal{F}$ and $H = \mathcal{F}(F, F)$ then $\mathcal{F}(F, X)$ is right H-space and F is left H-space. Their amalgamated product $\mathcal{F}(F, X) \times_H F$ is of the same homotopy type as F. In fact, take $\varphi \in \mathcal{F}(F, X) \neq \emptyset$ and let $\overline{\varphi}$ be its $\mathcal{F}$-homotopy inverse. The maps

$$f : F \to \mathcal{F}(F, X) \times_H F, \quad f : x \mapsto [(\varphi, x)]$$

and

$$\bar{f} : \mathcal{F}(F, X) \times_H F \to F, \quad \bar{f}\,[(\alpha, x)] \mapsto \overline{\varphi}(\alpha(x))$$

are both well-defined. Moreover the compositum

$$f\bar{f} : [(\alpha, x)] \mapsto [(\varphi, \overline{\varphi}\alpha(x))] = [(\varphi\overline{\varphi}\alpha, x)]$$

is homotopic to the identity, while

$$\bar{f}f : x \mapsto \overline{\varphi}\varphi(x)$$

is even $\mathcal{F}$-homotopic to the identity. As we want to use this construction to obtain $\mathcal{F}$-fibrations from $\mathcal{H}$-fibrations, we must assure that these amalgamated products are in $\mathcal{F}$ so we define a category of fibres $\mathcal{F}$ to be a *complete category of fibres* if the space $\mathcal{F}(F, X) \times_H F$ is in $\mathcal{F}$ for every $X \in \mathcal{F}$, the map $\mathcal{F}(F, f) \times_H F$ is in $\mathcal{F}$ for every $\mathcal{F}$-map f and moreover, the evaluation map

$$ev : \mathcal{F}(F, X) \times_H F \to X, \quad ev[(\alpha, x)] \mapsto \alpha(x)$$

is an $\mathcal{F}$-morphism. It's easy to check that all examples given in [1] are indeed complete categories of fibres.

2. Associated Principal Fibrations.

As a first step toward the homotopy classification of $\mathcal{F}$-fibrations we reduce it to the classification of their associated principal fibrations.

First, we define a functor $\Phi : \mathcal{B}_{\mathcal{F}} \to \mathcal{B}_{\mathcal{H}}$ as follows: given (E, p, B) in $\mathcal{B}_{\mathcal{F}}$ let $\overline{E}$ be the subspace of all maps $g \in \mathbf{kTop}(F, E)$, such that $(g, \overline{g}) : (F, pr, *) \to (E, p, B)$ is an $\mathcal{F}$-map, that is, the image of g is contained in a fibre of p. Define $\overline{p} : \overline{E} \to B$ as $\overline{p}(g) := p(g(x))$ for any $x \in F$.

Proposition 2.1 ([1], Lemma 5.8) *$(\overline{E}, \overline{p}, B)$ is a numerable $\mathcal{H}$-fibration.*

Moreover, for $(g, \overline{g}) : (E, p, B) \to (E', p', B')$ we put $\Phi(g, \overline{g}) := (g \circ -, \overline{g})$ which is obviously an $\mathcal{H}$-map and that yields the functoriality of Φ.

Now we construct a functor $\Psi : \mathcal{B}_{\mathcal{H}} \to \mathcal{B}_{\mathcal{F}}$ using the amalgamated products. Given an arrow (D, p, B) in $\mathcal{B}_{\mathcal{H}}$ there is the fiberwise right action of H on D so we have the space $D \times_H F$ and the projection $\widehat{p} : D \times_H F \to B$ is determined by $\widehat{p}[(d, x)] := p(d)$. For an $\mathcal{H}$-arrow map $(g, \overline{g}) : (D, p, B) \to (D', p', B')$ we put $\Psi(g, \overline{g}) := (\widehat{g}, \overline{g})$ where $\widehat{g}[(d, x)] := [(g(d), x)]$.

Proposition 2.2 *Suppose the category of fibres $\mathcal{F}$ is complete. Then $(D \times_H F, \widehat{p}, B)$ is a numerable $\mathcal{F}$-fibration, and $\Psi(g, \overline{g})$ is a map of $\mathcal{F}$-arrows.*

Proof. That $(D \times_H F, \widehat{p}, B)$ is an $\mathcal{F}$-arrow and that $\Psi(g, \overline{g})$ is a map of $\mathcal{F}$-arrows follows from the completeness of the category of fibres $\mathcal{F}$. Assume now that the arrow (D, p, B) is $\mathcal{F}$-homotopy trivial, with $(g, \overline{g}) : (D, p, B) \to (B \times H, pr, B)$ the $\mathcal{H}$-homotopy equivalence.

Then $\Psi(g, \overline{g})$ is clearly an $\mathcal{F}$-homotopy equivalence between $(D \times_H F, \widehat{p}, B)$ and $((B \times H) \times_H F, \widehat{p}, B)$. On the other side the maps

$$(B \times H) \times_H F \to B \times F, \quad [(b, h), x] \mapsto (b, h(x))$$

and

$$B \times F \to (B \times H) \times_H F, \quad (b, x) \mapsto [(b, e), x]$$

(where e is the unit in H) are mutually inverse fiber-homeomorphisms.

In $D \times_H F$ the amalgamated product is fiberwise so it obviously commutes with restrictions. It follows that $(D \times_H F, \widehat{p}, B)$ is locally $\mathcal{F}$-homotopy trivial with respect to the same covering as (D, p, B).

$\diamondsuit$

For a category of fibres $\mathcal{F}$ there is the homotopy category of the category $\mathcal{B}_{\mathcal{F}}$ with the same objects, but with maps the equivalence classes of maps in $\mathcal{B}_{\mathcal{F}}$ with respect to $\mathcal{F}$-homotopy and similarly for $\mathcal{B}_{\mathcal{H}}$. We can now formulate our main result.

THEOREM 2.3 *If the category of fibres $\mathcal{F}$ is complete then the functors Φ and Ψ induce an equivalence between the homotopy categories of $\mathcal{B}_{\mathcal{F}}$ and $\mathcal{B}_{\mathcal{H}}$.*

Proof. The image of the $\mathcal{F}$-fibration (E, p, B) by the functor $\Psi \circ \Phi$ is the $\mathcal{F}$-fibration $(\overline{E} \times_H F, \widehat{\overline{p}}, B)$. The maps

$$\mu_E : \overline{E} \times_H F \to E, \quad \mu_E[(\alpha, x)] := \alpha(x)$$

determine a natural transformation $\mu : \Psi \circ \Phi \to \mathrm{Id}$. If (E, p, B) is $\mathcal{F}$-homotopy trivial it's easy to check that the map μ_E is $\mathcal{F}$-homotopy equivalence. On the other side, both functors Φ and Ψ clearly commute with restrictions so, for an arbitrary numerable $\mathcal{F}$-fibration, the map μ_E is $\mathcal{F}$-homotopy equivalence on all members of the numerable covering for B hence, by Theorem **??** μ_E is an $\mathcal{F}$-homotopy equivalence. We conclude that μ is a natural equivalence between $\Psi \circ \Phi$ and the identity on the homotopy category of $\mathcal{B}_{\mathcal{F}}$.

Applying the same method we obtain that for (D, p, B) in $\mathcal{B}_{\mathcal{H}}$ the maps

$$\nu_D : D \to \overline{D \times_H F}, \quad (\nu_D(d))(x) := [(d, x)]$$

determine the natural equivalence between the identity and $\Phi \circ \Psi$ on the homotopy category of $\mathcal{B}_{\mathcal{H}}$. $\diamondsuit$

COROLLARY 2.4 *Let $\mathcal{F}$ be a complete category of fibres. Then the functors $k_{\mathcal{F}}$ and $k_{\mathcal{H}}$ are naturally equivalent.*

Proof. We define a natural transformation $\varphi : k_{\mathcal{F}} \to k_{\mathcal{H}}$ as follows: for every $B \in$ **kTop** and a class $[(E, p, B)]$ in $k_{\mathcal{F}}(B)$ let $\varphi_B : [(E, p, B)] \mapsto [(\overline{E}, \overline{p}, B)]$. The previous theorem implies that this is indeed a well-defined transformation. Similarly we construct, using the functor Ψ, a natural transformation $\psi : k_{\mathcal{H}} \to k_{\mathcal{F}}$ and it's easy to check that ψ is the inverse of φ. $\diamondsuit$

By this corollary, if $\mathcal{F}$ is a complete category fibres, we have reduced the classification of numerable $\mathcal{F}$-fibrations to the classification of their associated principal fibrations. For these it is possible to build a classifying space by exploiting the principal structure given by the action of the toplogical monoid H.

References

[1] Morgan C. and Piccinini R.A., *Fibrations*, Expo. Math. **4** (1986), 217-242.

[2] Piccinini R.A., *Lectures on Homotopy Theory.* North-Holland, Amsterdam, 1992.