COINCIDENCE POINTS OF MAPS ON Z_{p^α}-SPACES (*)

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SOMMARIO. - *Sia X uno spazio con una azione libera del gruppo ciclico Z_{p^α} ed $f : X \to M$ una mappa continua. Lo scopo di questo articolo è stimare per mezzo dell'indice Z_{p^α} la cardinalità dell'insieme*

$$A_f = \{x \in X | f(gx) = f(x) \text{ for all } g \in Z_{p^\alpha}\}$$

quando l'indice dello spazio X è noto ed M verifica opportune proprietà.

SUMMARY. - *Let X be a space with a free action of the cyclic group Z_{p^α} and $f : X \longrightarrow M$ a continuous map. The purpose of this paper is to estimate by means of the Z_{p^α}-index the size of the set*

$$A_f = \{x \in X | f(gx) = f(x) \text{ for all } g \in Z_{p^\alpha}\}$$

when the index of the space X is known, and the space M satisfies certain conditions.

1. Introduction.

Let X be a space with a free action of Z_{p^α}, p an odd prime, and $f : X \to M$ a continuous map. The index of X is an invariant of the action, defined originally by Conner and Floyd [3] for Z_2 actions, and later generalized by various authors to actions of other compact (and also noncompact) Lie groups (one possible definition, due to Fadell and Husseini [4] is given in paragraph 2). We will be concerned with the following questions: if the index of X is known, under what conditions on the space M and on the map f does there necessarily exist an orbit in X which is mapped to a

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single point in M, and, more generally, what can be said about the size of the set A_f of such orbits.

In case $p = 2$ and $\alpha = 1$ this question is related to the classical Borsuk-Ulam theorem, which says that $A_f \neq \emptyset$ if X is the sphere S^n, and M is the euclidean space $R^m, m \leq n$. Conner and Floyd in [3] showed, that this is true if M is any differentiable manifold of dimension $m < n$. On the other hand, Jaworowski [5] gave an example of a complex Y of dimension $2n-1$ (which is not a manifold), and a map $f : S^n \to Y$ with $A_f = \emptyset$, which proves that the restriction that M is a manifold is neccessary in the estimate of Conner and Floyd.

If $p \neq 2$ and $\alpha = 1$, it is known that $\dim A_f \geq n - (p-1)m$ in the following cases. Newman [10], proved this for maps $f : S^n \to R^m$, where n is odd and $m(p-1) \leq n$. Munkholm and Nakaoka [7] extended this to the case where X is an n-dimensional (n odd) differentiable homotopy sphere with a free differentiable action of Z_p, and M is a differentiable manifold, Volovikov [12] to the case where X is a Z_p-space with index n, M is an m-manifold, and f^* is trivial (or, more generally, takes the Wu classes of X to 0), and Necochea [9] to the case where X and M are Poincaré duality complexes, and a certain cohomology class is trivial. Cohen and Lusk [2] studied the sets $A_{f,q} \subset X$ containing all orbits from which precisely q points, $2 \leq q \leq p$ are mapped to a single point, and estimated their dimension in terms of certain numbers depending on M, q and p (which they computed for $M = R^m$), and obtained the above estimate for the set $A_{f,p} = A_f$ in case X is a Z_p-orientable n-manifold with trivial cohomology in dimensions $\leq m(p-1) - 1$.

In this paper we prove several restrictions on the size of A_f which are valid under quite general conditions. We also generalize our results to the case $\alpha > 1$. The case $\alpha > 1$ was considered by Munkholm and Nakaoka in [7] and they proved that

$$\dim A_f \geq n - (p^\alpha - 1)m - (m(\alpha - 1)p^\alpha - (m\alpha + 2)p^{\alpha - 1} + m + 3)$$

whenever X is an n-dimensional (n odd) homotopy sphere, and M is a differentiable manifold. We prove a different estimate which is also valid under more general conditions.

2. The Z_{p^α}-index.

Throughout this paper, “a space” will be a finitistic (compare [1]) paracompact Hausdorff space. The cohomology will be Chech cohomology with coefficients in a field (usually Z_p). The Z_{p^α}-index which we will use is a special case of the ideal-valued G-index introduced by Fadell and Husseini [4], which is defined on spaces with an action of a compact Lie group G and is a generalization of the index of Yang [13], and the Z_2-index of Conner and Floyd [3].

Let $EG \longrightarrow BG$ be the classifying bundle for G, $X_G = X \times_G EG$ the twisted product, and $H_G^*(X) \cong H^*(X \times EG)$ the Borel equivariant cohomology. The index, $\text{index}^G X$, is the kernel of the homomorphism induced on ordinary cohomology by the projection $p_G : X_G \to BG$, so it is a homogenuous ideal in the cohomology ring $H^*(BG)$ of the classifying space of G. We will work only with free actions, and in this situation the index has a simpler description. Since X is free, $X \longrightarrow X/G$ is a G-bundle, (compare [1, p.88]) so there exists a classifying map $\bar{\varphi} : X/G \longrightarrow BG$, which is covered by an equivariant map $\varphi : X \longrightarrow EG$. The index of X is in this case simply the kernel:

$$\text{index}^G X = \ker(\bar{\varphi}^* : H^*(BG) \longrightarrow H^*(X/G))$$

It is easy to show, that this index has the following properties, which in a sense characterise the concept of an index function:

Monotone property: If $f : X \longrightarrow Y$ is an equivariant map between G-spaces, then

$$index^G(Y) \subseteq index^G(X)$$

Additivity: If $X = A \bigcup B$, where A and B are closed invariant subspaces of the G-space X, then

$$index^G(A) \cdot index^G(B) \subseteq index^G(X)$$

Continuity: Let $H^*(BG)$ be a Noetherian ring, and let $H^*(X/G)$ be finitely generated as a module over $H^*(BG)$. If X is an invariant subspace of a paracompact G-space Y, then there exists a closed invariant neighbourhood N of X in Y, such that $X \subset int(N)$, and

$$index^G(X) = index^G(N) .$$

Proofs of these properties are in this setting just simple exercises. They can be found in [4] in a much more general setting.

As an easy consequence of these properties we obtain the following statement (compare also [5]):

PROPOSITION 1. If $f : X \longrightarrow Y$ is an equivariant map, $A \subseteq Y$ is a closed invariant subset, and $A_f = f^{-1}(A) \subseteq X$, then

$$\text{index}^G(A_f) \cdot \text{index}^G(Y - A) \subseteq \text{index}^G(X) \; .$$

Proof. Let N be the closed invariant neighbourhood of A_f in X which exists by the continuity of the index. By the additivity,

$$\text{index}^G(N) \cdot \text{index}^G(X - int(N)) =$$

$$= \text{index}^G(A_f) \cdot \text{index}^G(X - int(N)) \subseteq \text{index}^G(X) \; .$$

Since f maps $X - int(N)$ equivariantly to $(Y - A)$, the monotone property implies that

$$\text{index}^G(Y - A) \subset \text{index}^G(X - int(N))$$

and thus the proposition follows. $\diamondsuit$

Obvioulsy, proposition 1 provides a lower bound for the set A_f in terms of its cohomological dimension. It can be viewed as a very general Borsuk-Ulam type theorem in the following way. Given a map f from a certain G-space X (for which the index is known) to a representation space Y for G, let

$$Avf = \int_G g^{-1} f(gx) \; dg$$

be its average, which is an equivariant map. For any invariant subspace $A \subseteq Y$ (for example the origin), such that the action of G is free outside A, the above result gives an upper bound for the size (in terms of the cohomological dimension) of the set $A_f = Avf^{-1}(A)$. For example, if $G = Z_2$, $X = S^n$, $Y = R^n$, and $A = 0 \subset R^n$ it follows that the set A_f is nonempty which is exactly the classical Borsuk-Ulam theorem (compare, [5], [4]).

Let p be an odd prime. A model for the classifying bundle of Z_{p^α} is $S^\infty \to S^\infty / Z_{p^\alpha} = L^\infty_{p^\alpha}$ where S^∞ is the direct limit of unit spheres $S^{2n-1} \subset C^n$ with the standard action of $\mathbb{Z}_{p^\alpha}$ (multiplication with the p^α-th roots of unity), and $L^\infty_{p^\alpha}$ is the lens space. The Z_{p^α}-index of a Z_{p^α}-space X is therefore an ideal in the cohomology ring

$$H^*(L^\infty_{p^\alpha}; Z_p) = Z_p[a,b]/(a^2 = 0) = Z_p[b] \otimes \Delta[a];$$

$$a \in H^1(L^\infty_{p^\alpha}),\ b \in H^2(L^\infty_{p^\alpha})\ .$$

This is a graded ring with one generator in each dimension and its homogenuous ideals are generated by either a power b^n, a product ab^n or by a pair (b^n, ab^k). If the space X is finite dimensional, the second case is not possible.

Let I_n denote the ideal

$$I_n = \begin{cases} (b^k) & \text{if} \quad n = 2k \\ (ab^k, b^{k+1}) & \text{if} \quad n = 2k = 1 \end{cases}$$

which contains precisely all cohomology groups $H^q(L^\infty_p)$, $q \geq n$.

EXAMPLES.

1. Consider the sphere $S^{2n-1} \subset C^n$ with the standard action of Z_{p^α}. This can be considered as the $(2n-1)$-skeleton of the universal space S^∞, thus the index is $\ker\bar{\varphi} = \ker i^*$, where i is the inclusion $i : L^{2n-1}_{p^\alpha} \hookrightarrow L^\infty_{p^\alpha}$, so

$$\text{index}^{Z_{p^\alpha}}\ S^{2n-1} = (b^n) = I_{2n}.$$

Actually this is true for any free action of Z_{p^α} on the sphere S^{2n-1} because, as can be shown by obstruction theory, there exists an equivariant map from the sphere with an arbitrary free action of Z_{p^α} to the sphere with the standard action. Therefore

$$I_{2n} \subseteq \text{index}^{Z_{p^\alpha}}(S^{2n-1})\ .$$

The opposite inclusion follows because the dimension of S^{2n-1}, and therefore also of the quotient space is $2n-1$, and so $(b^n) \subset \ker\varphi^*$.

2. Let X_{2n} be the space obtained by glueing p^α copies of the $2n$-disk along the boundary S^{2n-1}. If one describes the disks as cones over the boundary sphere, then

$$X_{2n} = \{(x,t;i), x \in S^{2n-1}, t \in [0,1],$$

$$i = 1, ..., p^\alpha\}/(x,0;i) \sim (x,0;j), (x,1;i) \sim (y,1;i),$$

and any free action of Z_{p^α} on the boundary can be extended to a free action on X_{2n}, where the generator $\xi \in Z_{p^\alpha}$ acts in the following way:

$$\xi\,(x,t;i) = \begin{cases} (\xi x, 0; i), \\ (\xi x, t; (i+1) \bmod p^\alpha), \quad t > 0\ . \end{cases}$$

For any such action,

$$\operatorname{index}^{Z_{p^\alpha}} X_{2n} = (ab^n) = I_{2n+1}.$$

If the action on the sphere is the standard one, this is true because X_{2n} is the $(2n)$-skeleton of the lens space $L^\infty_{p^\alpha}$ in the standard cellular decomposition. For any other free action on the sphere, an equivariant map into the standard sphere can be extended in an obvious way to an equivariant map to the standard X_{2n}.

3. Maps from Z_{p^α}-spaces.

Let $f : X \longrightarrow M$ be a continuous map from a Z_{p^α}-space X to some space M, and let M^{p^α} denote the p^α-fold product, $M^{p^\alpha} = M \times \ldots \times M$, on which Z_{p^α} acts by cyclic permutations of the coordinates. The fixed point set of this action is the diagonal $\Delta(M)$. Every map $f : X \longrightarrow M$ induces an equivariant map $f^{p^\alpha} : X \longrightarrow M^{p^\alpha}$ defined by

$$f^{p^\alpha}(x) = (f(x), f(\xi x), \ldots f(\xi^{p^\alpha - 1}x)),$$

where ξ is a generator of the group Z_{p^α}. Then,

$$A_f = \{x \in X \mid f(gx) = f(x) \text{ for all } g \in Z_{p^\alpha}\} = (f^{p^\alpha})^{-1}(\Delta(M)).$$

If $\alpha = 1$, this action is free on $M^p - \Delta(M)$, and if $\operatorname{index}^{Z_p}(M^p - \Delta(M))$ is known, Proposition 1 can be directly applied to obtain a lower bound for the size of A_f. If $\alpha > 1$, the action on $M^p - \Delta(M)$ is not free, but M^{p^α} can be expressed as a union of a nested sequence od "diagonals":

$$\Delta(M) = \Delta_0 \subseteq \Delta_1 \subseteq \ldots \subseteq \Delta_{\alpha-1} \subseteq \Delta_\alpha = M^{p^\alpha}$$

where $\Delta_i = \Delta(M^{p^i}) \subseteq (M^{p^i})^{p^{\alpha-i}} = M^{p^\alpha}$. There is an obvious isomorphism $\Delta_i \cong M^{p^i}$ which is an isomorphism of Z_{p^j}-spaces for all $j \leq i$, and all these actions are free on the complement $\Delta_i - \Delta_{i-1}$, while the groups $Z^{p_i+1}, \ldots Z_{p^\alpha}$ all act on Δ_i trivially.

On the other hand, A_f can be expressed as the intersection of a descending sequence

$$A_{\alpha-1} = (f^{p^\alpha})^{-1}(\Delta_{\alpha-1}) \supseteq A_{\alpha-2} =$$

$$(f^{p^{\alpha}-1}|_{A_{\alpha-1}})^{-1}(\Delta_{\alpha-2}) \supseteq \ldots \supseteq A_0 = (f^p|_{A_1})^{-1}(\Delta_0) = A_f,$$

where $f^{p^{\alpha-i}} : A_{\alpha-i} \to \Delta_{\alpha-i} \cong M^{p^{\alpha-i}}$.

PROPOSITION 2. Let M be a Hausdorf space. For any $i \leq \alpha$:

1. If M is finite dimensional of dimension m,

$$I_{mp^\alpha+1} \subseteq \text{index}^{Z_{p^i}} (M^{p^\alpha} - \Delta_{\alpha-1}).$$

2. If M is compact and satisfies the following duality property: there exists an "orientation class " $U \in H^m(M \times M, M \times M - \Delta(M))$, such that the map $\gamma_U : H^{m-q}(M) \to H_q(M)$ defined as the slant product with U is an isomorphism (compare [11]), then

$$I_{mp^\alpha} \subseteq \text{index}^{Z_{p^i}} (M^{p^\alpha} - \Delta_{\alpha-1}).$$

If in addition to the assumptions above $H_j(M) = 0$ for all j, $0 < j \leq k < m$, (for example if M is k-connected), then

$$I_{mp^\alpha-k} \subseteq \text{index}^{Z_{p^i}} (M^{p^\alpha} - \Delta_{\alpha-1}).$$

3. If M is a differentiable manifold of dimension m, then in addition to the above,

$$\text{index}^{Z_{p^i}} (M^{p^\alpha} - \Delta_{\alpha-1}) \subseteq I_{m(p^\alpha-p^{\alpha-1})}.$$

LEMMA 1. Let X be a free Z_{p^α}-space with $H^i(X; Z_p) = 0$ for all $i \geq n_0$. Then also $H^i(X/Z_{p^\alpha}; Z_p) = 0$ for all $i \geq n_0$.

Proof. We will prove the lemma by induction on α. If $\alpha = 1$, it follows from the Smith exact sequence that

$$\text{rank } H^i_\sigma(X) \leq \sum_{j \geq i} \text{rank } H^j(X),$$

where $H^i_\sigma(X)$ are the special Smith cohomology groups of X (compare [1]). Since X is free, $H^i_\sigma(X) \cong H^i(X/Z_p)$. It obviously follows that rank $H^i(X/Z_p) =$ rank $H^i_\sigma(X) = 0$, and so $H^i(X/Z_p) = 0$ for all $i \geq k$. If $\alpha > 1$, there is a canonical homeomorphism $X/Z_{p^\alpha} \cong (X/Z_{p^{\alpha-1}})/Z_p$, and the statement immediately follows from $\alpha - 1$ to α. ♢

REMARK. The lemma can be proved in a similar way also if the action is not free.

PROOF OF PROPOSITION 2.

1. This is obvious: $\dim M^{p^\alpha} = mp^\alpha$, and so $H^i(M^{p^\alpha} - A) = 0$ for all $A \subseteq M^{p^\alpha}$ and all $i > mp^\alpha$.

2. By the lemma it is enough to show that $H^i(M^{p^\alpha} - \Delta_{\alpha-1}) = 0$ for all $i \geq mp^\alpha - k$. Since coefficients are in a field, $H^i(M^{p^\alpha} - \Delta_{\alpha-1}) \cong H_i(M^{p^\alpha} - \Delta_{\alpha-1})$. Let $U \in H^m(M \times M, M \times M - \Delta(M))$ be the "orientation class" and let $U^{p^\alpha} = U \otimes \ldots \otimes U \in (H^m(M \times M, M \times M - \Delta(M)))^p \cong H^{mp^\alpha}(M^{p^\alpha} \otimes M^{p^\alpha}, M^{p^\alpha} \otimes M^{p^\alpha} - \Delta(M))$. The slant product with the restriction of U^{p^α} defines an isomorphism $H^{mp^\alpha - i}(M^{p^\alpha}, \Delta_{\alpha-1}) \cong H_i(M^{p^\alpha} - \Delta_{\alpha-1})$. On the other hand, by the Künneth formula, $H^q(\Delta_{\alpha-1}) \cong H^q(M^{p^{\alpha-1}}) \cong H_q(\Delta_{\alpha-1}) = 0$ for all q, $0 < q \leq k < m$, and it follows from the exact sequence of the pair $(M^{p^\alpha}, \Delta_{\alpha-1})$ that $H^q(M^{p^\alpha}, \Delta_{\alpha-1}) = 0$ for all $q \leq k$, so $H_i(M^{p^\alpha} - \Delta_{\alpha-1})) = 0$ for all $i \geq mp - k$.

3. The last inclusion follows from the fact, that there exists an equivariant embedding $S^{m(p^\alpha - p^{\alpha-1})-1} \hookrightarrow M^{p^\alpha} - \Delta_{\alpha-1}$ of the sphere into a regular neighborhood of $\Delta_{\alpha-1}$ as the fiber of the normal sphere bundle.

$\diamondsuit$

THEOREM 1. If for some n, $\operatorname{index}^{Z_{p^i}} X \subseteq (b^n) = I_{2n}$ for all $i \leq \alpha$, then

1. If M is finite dimensional and $\dim M = m$,

$$\operatorname{index}^{Z_p} A_f \subseteq I_{2n-mp^\alpha} \text{ and}$$

$$\dim A_f \geq 2n - mp^\alpha - 1.$$

It follows, that the set A_f is necessarily nonempty, if $2n > mp^\alpha$.

2. If M is compact and satisfies the duality property stated in Proposition 2 and $H^q(M) = 0$ for all $0 < q \leq k$ (for example if M is q-connected), then

$$\operatorname{index}^{Z_p} A_f \subseteq I_{2n - mp\frac{p^\alpha - 1}{p-1} + k\alpha} \text{ and}$$

$$\dim A_f \geq 2n - mp\frac{p^\alpha - 1}{p - 1} + k\alpha - 1.$$

In this case, A_f is necessarily nonempty, if

$$2n > mp\frac{p^\alpha - 1}{p-1} + k\alpha.$$

Proof. We will prove only the second statement since the first follows rather obviously by the same kind of arguments. First let $\alpha = 1$. By Proposition 1,

$$(\operatorname{index}^{Z_p} A_f)(\operatorname{index}^{Z_p}(M^p - \Delta(M))) \subseteq \operatorname{index}^{Z_p} X.$$

Let $\alpha \neq 0 \in H^{2n-mp-k-1}(L_p^\infty)$, and $\beta \neq 0 \in H^{mp-k}(L_p^\infty)$. By Proposition 2, $\beta \in \operatorname{index}^{Z_p}(M^{p^\alpha} - \Delta(M))$. Notice that one of the two degrees $2n - mp - k - 1$ and $mp - k$ must be even, since their sum is odd, so either α or β is a multiple of some power b^k, and therefore their product is nonzero. If $\alpha \in \operatorname{index}^{Z_p} A_f$, it would follow from Proposition 1 that $\alpha \cup \beta \in \operatorname{index}^{Z_p} X$, which is impossible, since $\operatorname{index}^{Z_p} X$ contains no nonzero element of order $q < 2n$. Therefore, $\varphi^*(\alpha) \in H^{2n-mp-k-1}(A_f)$ is nonzero.

If $\alpha > 1$, then by Proposition 2,

$$I_{mp^\alpha - k} \subseteq \operatorname{index}^{Z_{p^i}}(M^{p^\alpha} - \Delta_{\alpha-1}) \subseteq I_{mp^\alpha}\ ;\ \ 1 \le i \le \alpha,$$

and by Proposition 1

$$\operatorname{index}^{Z_{p^i}} A_{\alpha-1}(f) \subseteq I_{2n-mp^\alpha+k}\ .$$

Now consider the restriction $f : A_{\alpha-1}(f) \to \Delta_{\alpha-1}$. By Proposition 2,

$$I_{mp^\alpha - k} \subseteq \operatorname{index}^{Z_{p^i}}(\Delta_{\alpha-1} - \Delta_{\alpha-2}) \subseteq I_{mp^{\alpha-1}},$$

and by Proposition 1

$$\operatorname{index}^{Z_{p^i}} A_{\alpha-2}(f) \subseteq I_{(2n-mp^\alpha+k)-mp^{\alpha-1}+k} = I_{2n-m(p^\alpha+p^{\alpha-1})+2k}.$$

After $\alpha - 1$ steps of this procedure, we obtain the estimate in the theorem.

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