

FAMILIES OF SPACES HAVING PRESCRIBED EMBEDDABILITY ORDER-TYPE (*)

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SOMMARIO. - Sia $\mathcal{F} = \{X_i : i \in I\}$ una famiglia (quasi) ordinata di spazi topologici ove $X_i \leq X_j$ ogniqualvolta X_i è omeomorfo ad un sottospazio di X_j e si consideri il seguente problema: dato un insieme ordinato S è possibile determinare una famiglia di spazi $\mathcal{F}(S)$ tali che $(\mathcal{F}(S), \leq)$ è ordinatamente isomorfa ad S ? Si vede essere un esercizio non banale anche solo ottenere un esempio "concreto" di famiglia ordinata persino in una maniera semplice come gli interi negativi. Estendendo e modificando un argomento di Watson e Matier si mostra come l'induzione transfinita possa essere usata per costruire famiglie di spazi con prescritti tipi d'ordine. In particolare emerge che ogni insieme ordinato con la potenza del continuo può essere modellato (in questo senso) su una famiglia di sottospazi della retta reale.

SUMMARY. - Let a family $\mathcal{F} = \{X_i : i \in I\}$ of topological spaces be (quasi) ordered by writing $X_i \leq X_j$ whenever X_i is homeomorphic to a subspace of X_j , and consider the problem: given an ordered set S , can we exhibit a family $\mathcal{F}(S)$ of spaces such that $(\mathcal{F}(S), \leq)$ is order-isomorphic to S ? It appears to be a non-trivial exercise to obtain a 'concrete' example of a family ordered in even such a simple way as the negative integers. By extending and modifying an argument of Watson and Matier we show how transfinite induction can be used to construct families of spaces which have certain prescribed order-types. In particular it emerges that any ordered set on continuum-many elements can be modelled (in this sense) by a family of subspaces of the real line.

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Introduction.

When topological spaces are ordered by (homeomorphic) embeddability, it is rather a delicate task to devise a family of them whose interrelationships match a given ordered set. For instance, an unthinking attempt to model the order-type \mathbb{N}^* of the negative integers using subspaces of the real line \mathbb{R} might be to try $X_n = \mathbb{R} \setminus \{1, 2, 3, \dots, n\}$. Certainly X_{n+1} can be embedded in X_n here but, since also X_n is embeddable into X_{n+1} , what we have modelled is in fact not \mathbb{N}^* but a countable ordered set in which every two elements are comparable both ways round. The root of the difficulty, as this example suggests, is that the embeddability ordering is not a partial order but merely a quasi order: given only that X is embeddable into Y and not homeomorphic to it, we cannot deduce that Y is non-embeddable into X . The subset X_{n+1} of X_n needs to be selected in such a way that *every* attempt at embedding X_n into it must fail.

The literature contains at least one direct procedure for doing such a selection - but in the context of ordered sets rather than of topological spaces. It arose from an *impromptu* talk given by Professor Stephen Watson to the second author and Julie Matier, was elaborated in the latter's doctoral thesis [3] and an article [4] arising therefrom, and is further analysed in [1]. The first author has refined and extended it to deal with a much wider range of order structures [5]. Each time, the fundamental idea is to use well-ordering on the sets X_i being constructed *and* on the mappings f which could effect an embedding, so that whenever there is a risk of f 'inappropriately' embedding X_i into X_j , an element x is adjoined to X_i where $f(x)$ is not in X_j and will never be subsequently added in.

The purpose of the present note is to abstract the construction of [5] so as to allow its interpretation for topological spaces, thus constructing families as delineated in the title.

Construction.

Let α be an infinite cardinal number. Suppose that we can find

- a set C of cardinality α ,
- a non-empty subset Q of C of cardinality $\beta < \alpha$,
- a family \mathcal{I} of subsets of C and
- a family \mathcal{F} of partial mappings from subsets of C into C

such that conditions (i) to (vi) below are all satisfied.

Firstly, for each $f \in \mathcal{F}$ whose domain $\text{dom}(f) = Q$ we classify the elements of $C \setminus Q$ into four types as follows:

we call x in $C \setminus Q$ a

non-extension point of f if there is no f^* in \mathcal{F} , extending f , with $\text{dom}(f^*) = Q \cup \{x\}$,

multi-extension-point of f if there is more than one such f^* ,

Q-extension-point of f if there is exactly one such f^* and $f^*(x) \in Q$,

unique-extension-point of f if there is exactly one such f^* and $f^*(x) \in C \setminus Q$.

Now we postulate that

- i) if $K \subseteq C$ then the identity mapping $\text{id}_K \in \mathcal{F}$,
- ii) whenever $Q \subseteq \text{dom}(f)$, $f \in \mathcal{F}$ and $x \in \text{dom}(f) \setminus Q$ then the restrictions $f|_Q$ and $f|_{Q \cup \{x\}} \in \mathcal{F}$,
- iii) the set $\mathcal{F}^* = \{f \in \mathcal{F} : \text{dom}(f) = Q\}$ has cardinality not exceeding α ,
- iv) Q intersects every member of \mathcal{I} ,
- v) when $f \in \mathcal{F}$ and f has fixed points in every member of \mathcal{I} then $f = \text{id}_{\text{dom}(f)}$,
- vi) for each $f \in \mathcal{F}^*$,
either f has α non-extension-points
or each $I \in \mathcal{I}$ contains α unique-extension-points of f .

Next, let S be a given qoset with α elements. We denote by $P_Q(C)$ the family

$$\{X : Q \subseteq X \subseteq C\}$$

and impose on it a relation \leq thus:

$$X_1 \leq X_2 \Leftrightarrow \exists f : X_1 \rightarrow X_2 \text{ where } f \in \mathcal{F}.$$

Observe from (i) that \leq is at least reflexive. We shall obtain a mapping

$$\theta : S \rightarrow P_Q(C)$$

such that $(\theta(S), \leq)$ is a qoset order-isomorphic to S under θ . Since $\mathcal{F}^* \times S$

has cardinality α (see (iii) above) we can label its elements with those of the ordinal number α :

$$\mathcal{F}^* \times S = \{(f_\beta, s_\beta) : \beta \in \alpha\}.$$

Make an arbitrary choice of $q_o \in Q$. We inductively construct three α -sequences $(x_\beta), (y_\beta), (z_\beta)$ ($\beta \in \alpha$) such that

- a) $x_\beta, y_\beta \in (C \setminus Q) \cup \{q_o\}, z_\beta \in C \setminus Q$,
- b) apart from repetitions of q_o , all the terms are distinct,
- c) whenever $f_\beta = \text{id}_Q$ we have $x_\beta = y_\beta = q_o$ and
- d) whenever $f_\beta \neq \text{id}_Q$ then
either x_β is a non-extension-point of f_β and $y_\beta = q_o$
or x_β is a unique-extension-point of f_β and $y_\beta = f_\beta^*(x_\beta)$.

For a given γ in α we suppose that the elements $x_\beta, y_\beta, z_\beta$ for $\beta < \gamma$ have been chosen in a way that satisfies (a) to (d). In order to select x_γ, y_γ and z_γ we examine the map f_γ .

If f_γ is id_Q we choose $x_\gamma = q_o, y_\gamma = q_o, z_\gamma$ to be an element of $C \setminus Q$ which differs from all previous choices. (This is possible since the cardinality of $C \setminus Q$ is α whereas at most 3γ earlier choices have taken place.) Clearly (a) to (d) are now valid to the γ^{th} terms.

If f_γ is not id_Q we use (v) to yield I_γ in \mathcal{I} which contains no fixed point of f_γ . Then by (vi) one of the following cases arises:

(I) f_γ has α non-extension-points: here we select as x_γ one of these, ensuring that it differs from all preceding choices, put $y_\gamma = q_o$ and $z_\gamma =$ any 'unselected' point in $C \setminus Q$; or

(II) I_γ contains a set J of α unique-extension-points of f : now if

$$B = \{y \in J : y \text{ or } f_\gamma^*(y) \text{ has already been chosen as a term in one of the three sequences}\}$$

it is evidently possible to select as x_γ an element of $J \setminus B$; the choice of I_γ ensures that $y_\gamma = f_\gamma^*(x_\gamma)$ is distinct from x_γ , that of B guarantees that both are distinguishable from all earlier terms. Again pick z_γ as any unused member of $C \setminus Q$, and note that throughout (I) and (II) conditions (a) to (d) have been preserved. An appeal to transfinite induction establishes the existence of the three α -sequences satisfying (a) to (d) for all values of β

in α .

For each $s \in S$ put

$$\theta(s) = Q \cup \{x_\delta, z_\delta : s_\delta \leq s\}.$$

Since $r \leq s$ implies $\theta(r) \subseteq \theta(s)$, condition (i) shows that

$$r \leq s \Rightarrow \theta(r) \leq \theta(s).$$

Conversely, if $\theta(r) \leq \theta(s)$ we can find $f \in \mathcal{F}$ such that $f : \theta(r) \rightarrow \theta(s)$. The pair $(f|_Q, r)$ belongs to $\mathcal{F}^* \times S$ and was therefore enumerated as (f_β, s_β) in the transfinite listing of that set (for some $\beta \in \alpha$). There are again two cases to examine:

(I) If $f_\beta = f|_Q$ is id_Q , then (iv) and (v) combine to make $f = \text{id}_{\theta(r)}$ from which $\theta(r) \subseteq \theta(s)$ follows. Now since $s_\beta = r$, we do have $s_\beta \leq r$ whence z_β belongs to $\theta(r)$. Bearing in mind that z_β is distinct from every term of the ‘ x ’ sequence, its consequent membership of $\theta(s)$ entails that $s_\beta \leq s$, that is, $r \leq s$.

(II) If $f_\beta = f|_Q$ is not id_Q , we notice that x_β cannot be a non-extension-point for f_β since

$$x_\beta \in \theta(r) = \text{dom}(f)$$

whence (ii) shows that the restriction of f to $Q \cup \{x_\beta\}$ is an extension in \mathcal{F} of f_β . So x_β must be a unique-extension-point for f_β , and $y_\beta = f_\beta^*(x_\beta) = f|_{Q \cup \{x_\beta\}}(x_\beta) = f(x_\beta)$. Thus $y_\beta \in \theta(s)$ which, however, contradicts its membership of $C \setminus Q$ and its distinctness from the terms of the ‘ x ’ and ‘ z ’ sequences. Only (I) is therefore viable, and we have proved that

$$\theta(r) \leq \theta(s) \Rightarrow r \leq s.$$

In summary so far, we have:

PROPOSITION. *Under the stated assumptions on C, Q, \mathcal{I} and \mathcal{F} , every goset on α points can be embedded into $P_Q(C)$.*

Interpretations.

EXAMPLES. (i) The simplest instance of this arises by taking $\alpha = \mathbf{c}$, $C =$ the ordered set \mathbb{R} of real numbers, $Q =$ the set \mathbb{Q} of rationals,

\mathcal{I} the collection of open intervals and \mathcal{F} the family of strictly increasing real functions. In this context, non-extension-points do not occur, and conditions (i) to (v) are trivial. Matier, in effect, verified (vi) and deduced (taking S to be the positive integers in reverse order) that there is an infinite descending chain of subsets of \mathbb{R} , ordered by sub-chain embeddability.

(ii) The present authors extended Matier's argument to the situation where C is an infinite chain all of whose open intervals have cardinality α and which possesses an order-dense subset Q having cardinality β , where $\alpha = 2^\beta$: again taking \mathcal{I} as the family of open intervals in C and \mathcal{F} the collection of strictly increasing partial maps from C to C . They concluded that each poset on α points (or fewer) can then be embedded in $P_Q(C)$, and also pointed out that if the generalized continuum hypothesis is assumed, then for every successor cardinal α such a chain C may be found.

The main purpose of this note is to obtain versions of the above in which $P_Q(C)$ is a family of topological spaces (rather than ordered sets) and the ordering is characterized by homeomorphic (rather than order-isomorphic) embeddability. Notice first that a rather artificial form of this can be derived directly from the preceding: for suppose that

$$\theta : S \rightarrow P_Q(C)$$

has been contrived so that

$$r \leq s \Rightarrow \theta(r) \subseteq \theta(s) \text{ and}$$

$$r \not\leq s \Rightarrow \text{there is no strictly increasing map from } \theta(r) \text{ into } \theta(s).$$

Let each $\theta(s)$ be made into a topological space $\theta_t(s)$ by giving it the topology $\tau(\uparrow)$ of increasing subsets. The identity map on $\theta_t(r)$ (whenever $r \leq s$) continues to embed $\theta_t(r)$ homeomorphically into $\theta_t(s)$. Yet if $r \not\leq s$ and there were even a continuous one-to-one mapping $g : \theta_t(r) \rightarrow \theta_t(s)$, the choice of topology forces g to be strictly increasing - a contradiction. In other words, the ordering on $P_Q(C)$ given by

$$X_1 \leq X_2 \Leftrightarrow X_1 \text{ is homeomorphic to a topological subspace of } X_2$$

still allows us to 'realise' every qoset on at most α elements within the family of subspaces of the topological space $(C, \tau(\uparrow))$.

It might be considered more interesting to obtain analogous conclusions about chains endowed with less trivial topologies, such as the real line with

its usual metric topology. The formulation here presented does indeed permit this, as we now show.

Accordingly let us take C as the real line (naturally topologised), Q the set of rationals, \mathcal{I} the family of open intervals and \mathcal{F} the collection of continuous injections from subsets of C into C . It is clear that conditions (i) to (v) are satisfied by these choices; we confirm also (vi).

LEMMA. *Let $f : \mathbb{Q} \rightarrow \mathbb{R}$ be a continuous injection. Then either f has \mathfrak{c} non-extension-points or every open interval I includes \mathfrak{c} unique-extension-points of f .*

Proof. Multi-extension-points cannot of course occur for continuous functions with T_2 co-domain, so essentially our task is to show that the set Q_X of Q -extension-points for f is “small”. For each rational number k put

$$E_k = \{j \in \mathbb{R} \setminus \mathbb{Q} : j \text{ is a } Q\text{-extension-point of } f \text{ and } f^*(j) = k\}.$$

If q is rational and $f(q) \neq k$ then we can find $\epsilon > 0$ such that

$$x \in \mathbb{Q} \cap (q - \epsilon, q + \epsilon) \Rightarrow |f(x) - k| > \frac{1}{2}|f(q) - k| > 0$$

which implies that no point of $(q - \epsilon, q + \epsilon)$ can belong to E_k ; bearing in mind that f is one-to-one on \mathbb{Q} , this means that $\overline{E_k}$ contains at most one rational number, so E_k is nowhere-dense in \mathbb{R} .

Choose an enumeration

$$\mathbb{Q} = \{k_1, k_2, k_3, k_4, \dots\}$$

of the rationals. Within I choose two disjoint (non-degenerate) closed intervals $I_o = [a_o, b_o], I_1 = [a_1, b_1]$ disjoint from E_{k_1} ; within $\text{int } I_o = (a_o, b_o)$ and $\text{int } I_1 = (a_1, b_1)$ choose pairs of disjoint intervals I_{00}, I_{01} and I_{10}, I_{11} disjoint from E_{k_2} ; within the interiors of each of these four, choose a pair of intervals disjoint from E_{k_3} , and so on. The usual ‘nested interval’ argument serves to produce, for each of the 2^{\aleph_0} possible sequences of zeros and ones, a distinct point of I which cannot belong to any E_{k_i} , that is, which is excluded from Q_X . Thus I contains \mathfrak{c} non- or unique-extension-points of f . (This way of determining the cardinality of the complement of a first-category set is taken from Hobson [2], page 136. We should like to thank Professor D.H. Armitage for invaluable conversations in this area.)

Thus, we have:

THEOREM. *Let the family of subspaces of the real continuum be quasi-ordered by homeomorphic embeddability. Within this family we may find*

order-isomorphic copies of every quasi-ordered set on 2^{\aleph_0} (or fewer) points.

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