

# APPROXIMATE POLYHEDRAL RESOLUTIONS WITH IRREDUCIBLE BONDING MAPPINGS (\*)

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SOMMARIO. - *Gli spazi normali finitistici sono caratterizzati come spazi che ammettono risoluzioni approssimate (surgettive) consistenti in poliedri finito-dimensionali. Gli spazi normali aventi dimensione  $\leq n$  sono caratterizzati come spazi che ammettono risoluzioni irriducibili approssimate consistenti in poliedri aventi dimensioni  $\leq n$ .*

SUMMARY. - *Normal finitistic spaces are characterized as spaces which admit irreducible (surjective) approximate resolutions consisting of finite-dimensional polyhedra. Normal spaces having dimension  $\leq n$  are characterized as spaces admitting irreducible approximate resolutions consisting of polyhedra having dimension  $\leq n$ .*

## 1. Preliminaries.

Recently, S. Mardešić and N. Uglešić proved that every mapping of a normal space into an arbitrary polyhedron can be approximated by an irreducible mapping into some of its subpolyhedra ([5]). This important and interesting result allows us to give a much better characterization of normal finitistic spaces than the one in [7]. Irreducible mappings are very convenient for the construction of surjective approximate resolutions of a normal space ([3], [8]), and were successfully used for solving some problems in the theory of commutative inverse systems. Till now only irreducible mappings into compact polyhedra were considered, where irreducible approximation is achieved in a few simple steps. Here we shall construct an approximate resolution of a normal finitistic space, which consists of finite-dimensional polyhedra and whose projections and bonding maps are irreducible.

We shall use the same terminology and notion as in ([6]). A *normal* or *numerable* (open) *covering* of a (topological) space  $X$  is an open covering  $\mathcal{U}$

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of  $X$  which admits a subordinate partition of unity. The set of all normal coverings of  $X$  is denoted by  $Cov(X)$ . If  $\mathcal{U}, \mathcal{V} \in Cov(X)$  and  $\mathcal{V}$  refines  $\mathcal{U}$ , we write  $\mathcal{V} \leq \mathcal{U}$ . For two maps  $f, g : Y \rightarrow X$  which are  $\mathcal{U}$ -near (for every  $y \in Y$  there exists a  $U \in \mathcal{U}$  with  $f(y), g(y) \in U$ ), we write  $(f, g) \leq \mathcal{U}$ . The order of  $\mathcal{U}$ , denoted  $ord(\mathcal{U})$ , is the largest integer  $n$  such that  $\mathcal{U}$  contains  $n$  elements with nonempty intersection, or  $\infty$  if no such integer exists. We say that  $\dim X \leq n$  if, for any  $\mathcal{U} \in Cov(X)$ , there is a  $\mathcal{V} \in Cov(X)$  such that  $\mathcal{V} \leq \mathcal{U}$  and  $ord(\mathcal{V}) \leq n + 1$ .

A space  $X$  is called *finitistic* if for each normal covering  $\mathcal{U} \in Cov(X)$  there are a positive integer  $n$  and a normal covering  $\mathcal{V} \in Cov(X)$  such that  $\mathcal{V} \leq \mathcal{U}$  and  $ord(\mathcal{V}) \leq n$ . This means that for each  $\mathcal{U} \in Cov(X)$  of a finitistic space  $X$  there exists a refinement  $\mathcal{V} \in Cov(X)$  such that  $|N(\mathcal{V})|$  is a finite-dimensional polyhedron, where  $N(\mathcal{V})$  denotes the nerve of the covering  $\mathcal{V}$ . By the definition of finitistic spaces, it is clear that every compact space and every finite-dimensional space is finitistic. Finitistic spaces need not be finite-dimensional; any compact infinite-dimensional space provides an example.

Let  $X$  be a topological space,  $K$  a simplicial complex and let  $f, g : X \rightarrow |K|$  be mappings into the geometric realization  $|K|$ , endowed with the  $CW$ -topology. We say that  $g$  is a  $K$ -modification of  $f$  if for every point  $x \in X$  and every (closed) simplex  $\sigma \in K$ ,  $f(x) \in \sigma$  implies  $g(x) \in \sigma$ . If  $K'$  is a subdivision of  $K$  and  $g : X \rightarrow |K'|$  is a  $K'$ -modification of  $f : X \rightarrow |K'| = |K|$ , then  $g$  is also a  $K$ -modification of  $f$ .

We say that a mapping  $f : X \rightarrow |K|$  is  $K$ -irreducible if for every  $K$ -modification  $g$  of  $f$  one has  $g(X) = |K|$ . Since  $f$  is its own  $K$ -modification, every  $K$ -irreducible map  $f$  is onto. A mapping  $f : X \rightarrow P$  into a polyhedron is called *irreducible* if it is  $K$ -irreducible for some triangulation  $K$  of  $P$ . Note that every irreducible map  $f : X \rightarrow P$  is onto.

An *approximate inverse system*  $\mathcal{X}$  is a collection  $\{X_\lambda, \mathcal{U}_\lambda, p_{\lambda\lambda'}, \Lambda\}$  consisting of

- i) a preordered indexing set  $\Lambda = (\Lambda, \leq)$  (it needs not be antisymmetric), which is directed and unbounded (i.e. has no maximal element),
- ii) for each  $\lambda \in \Lambda$ ,  $X_\lambda$  is a topological space and  $\mathcal{U}_\lambda \in Cov(X_\lambda)$ ,
- iii) for any two related indices  $\lambda \leq \lambda'$ ,  $p_{\lambda\lambda'} : X_{\lambda'} \rightarrow X_\lambda$  is a (continuous) map ( $p_{\lambda\lambda} = id_{X_\lambda}$  is the identity map on  $X_\lambda$ ).

Furthermore, the following three condition must be satisfied:

(A1) for any three related indices  $\lambda \leq \lambda' \leq \lambda''$ ,

$$(p_{\lambda\lambda'}p_{\lambda'\lambda''}, p_{\lambda\lambda''}) \leq \mathcal{U}_\lambda ;$$

(A2) for each  $\lambda \in \Lambda$  and each  $\mathcal{U} \in \text{Cov}(X_\lambda)$ , there exists a  $\lambda' \geq \lambda$  such that

$$(p_{\lambda\lambda_1}p_{\lambda_1\lambda_2}, p_{\lambda\lambda_2}) \leq \mathcal{U}, \text{ whenever } \lambda_2 \geq \lambda_1 \geq \lambda' ;$$

(A3) for each  $\lambda \in \Lambda$  and each  $\mathcal{U} \in \text{Cov}(X_\lambda)$ , there exists a  $\lambda' \geq \lambda$  such that

$$\mathcal{U}_{\lambda''} \leq p_{\lambda\lambda''}^{-1}\mathcal{U} = \{p_{\lambda\lambda''}^{-1}(U) : U \in \mathcal{U}\}, \text{ whenever } \lambda'' \geq \lambda' .$$

An *approximate mapping*  $p = \{p_\lambda : \lambda \in \Lambda\} : X \rightarrow \mathcal{X} = \{X_\lambda, \mathcal{U}_\lambda, p_{\lambda\lambda'}, \Lambda\}$  of a topological space  $X$  into an approximate inverse system  $\mathcal{X}$  is a family of maps  $p_\lambda : X \rightarrow X_\lambda$ ,  $\lambda \in \Lambda$ , such that the following condition holds:

(AS) For any  $\lambda \in \Lambda$  and any  $\mathcal{U} \in \text{Cov}(X_\lambda)$ , there exists a  $\lambda' \geq \lambda$  such that  $(p_{\lambda\lambda''}p_{\lambda''\lambda}, p_\lambda) \leq \mathcal{U}$ , for every  $\lambda'' \geq \lambda'$ .

Let  $POL$  denote the class of all polyhedra (endowed with the CW-topology).

An *approximate resolution of a space*  $X$  is an approximate mapping  $p = \{p_\lambda : \lambda \in \Lambda\} : X \rightarrow \mathcal{X} = \{X_\lambda, \mathcal{U}_\lambda, p_{\lambda\lambda'}, \Lambda\}$  of  $X$  into an approximate system  $\mathcal{X}$  satisfying the following two conditions:

(R1) For any  $P \in POL$ ,  $\mathcal{V} \in \text{Cov}(P)$  and mapping  $f : X \rightarrow P$  there is a  $\lambda \in \Lambda$  such that, for every  $\lambda' \geq \lambda$ , there exists a mapping  $g : X_{\lambda'} \rightarrow P$  satisfying  $(gp_{\lambda'}, f) \leq \mathcal{V}$ .

(R2) For every  $P \in POL$  and  $\mathcal{V} \in \text{Cov}(P)$  there is a  $\mathcal{V}' \in \text{Cov}(P)$  such that for any  $\lambda \in \Lambda$  and any two maps  $g, g' : X_\lambda \rightarrow P$ , for which  $(gp_\lambda, g'p_\lambda) \leq \mathcal{V}'$ , there exists a  $\lambda' \geq \lambda$  such that  $(gp_{\lambda\lambda'}, g'p_{\lambda\lambda'}) \leq \mathcal{V}$ , for any  $\lambda'' \geq \lambda'$ .

Let  $p = \{p_\lambda : \lambda \in \Lambda\} : X \rightarrow \mathcal{X}$  be a polyhedral resolution (commutative or approximate). Then  $p$  is called *irreducible* if all bonding maps  $p_{\lambda\lambda'}$  and all projections  $p_\lambda$  are irreducible. Note that each irreducible resolution has surjective bonding maps and surjective projections.

## 2. Irreducible resolutions of finitistic spaces.

**PROPOSITION 2.1.** *Let  $X$  be a normal space and  $\mathcal{U} \in \text{Cov}(X)$ . Then there exist a subcomplex  $K \leq N(\mathcal{U})$  of the nerve  $N(\mathcal{U})$  of  $\mathcal{U}$  and a canonical map  $f : X \rightarrow |K|$  of  $\mathcal{U}$ , which is  $K$ -irreducible.*

*Proof.* Since  $\mathcal{U}$  is a normal covering of  $X$ , there exists a canonical mapping  $\Phi : X \rightarrow |N(\mathcal{U})|$  of  $\mathcal{U}$ . If  $\Phi$  is already  $N(\mathcal{U})$ -irreducible, we put  $K = N(\mathcal{U})$  and  $f = \Phi$ . If not, there exist a subcomplex  $K \leq N(\mathcal{U})$  and a  $N(\mathcal{U})$ -modification  $f : X \rightarrow |K|$  of  $\Phi$ , which is  $K$ -irreducible ([5], Corollary 1.) So, we only need to prove that  $f$  is a canonical mapping of  $\mathcal{U}$ , i.e.  $f^{-1}(st(U, N(\mathcal{U}))) \subseteq U$ , for each  $U \in \mathcal{U} = N(\mathcal{U})^\circ$ . Let  $x \in f^{-1}(st(U, N(\mathcal{U})))$ . Then  $f(x) \in st(U, N(\mathcal{U}))$ , which implies that there exists a simplex  $\sigma \in N(\mathcal{U})$   $\sigma = [U_0 = U, U_1 \dots U_k]$  such that  $f(x) \in \sigma$ . Since  $f$  is a  $N(\mathcal{U})$ -modification of  $\Phi$ , there exists a simplex  $\tau = [U_0 = U, U_1, \dots, U_k, U_{k+1}, \dots, U_n]$ ,  $n \geq k$ , such that  $\Phi(x) \in \tau$ . Note that  $\Phi(x) \notin \tau \cap \partial st(U, N(\mathcal{U}))$ , for otherwise  $f(x) \in \partial st(U, N(\mathcal{U}))$ . So  $\Phi(x) \in st(U, N(\mathcal{U}))$ , which implies  $x \in \Phi^{-1}(st(U, N(\mathcal{U}))) \subseteq U$ . Consequently,  $f^{-1}(st(U, N(\mathcal{U}))) \subseteq U$ , which shows that  $f$  is canonical for  $\mathcal{U}$ .

The proof of the next lemma is obtained by appropriate changes in the proof of Lemma 3.4 of [7].

**LEMMA 2.2.** *Let  $X$  be a normal finitistic space, let  $P_1, \dots, P_n$  be polyhedra, let  $f_1 : X \rightarrow P_1, \dots, f_n : X \rightarrow P_n$  be mappings and let  $\mathcal{U}_1 \in Cov(P_1), \dots, \mathcal{U}_n \in Cov(P_n)$  be open coverings. Then there exist a finite-dimensional polyhedron  $P$ , an irreducible map  $f : X \rightarrow P$  and PL-mappings  $p_1 : P \rightarrow P_1, \dots, p_n : P \rightarrow P_n$  such that  $(f_i, p_i f) \leq \mathcal{U}_i$ , for  $i = 1, \dots, n$ . Moreover, if for a given  $i$  the polyhedron  $P_i$  is finite-dimensional and  $f_i$  is irreducible, then the corresponding mapping  $p_i$  is also irreducible. In that case it is possible to choose a triangulation  $K_i$  of  $P_i$  such that  $f_i$  and  $p_i$  are  $K_i$ -irreducible.*

*Proof.* For each  $i = 1, \dots, n$  choose a triangulation  $K_i$  of  $P_i$  so fine that the covering  $\bar{\mathcal{S}}_i$  formed by all the closures of the members of  $\mathcal{S}_i = \{st(v, K_i) : v \in K_i^0\} \in Cov(P_i)$  refines  $\mathcal{U}_i$  i.e.  $\bar{\mathcal{S}}_i \leq \mathcal{U}_i$  ([4], Theorem 4, Appendix 1). If for a given  $i$ ,  $P_i$  is finite-dimensional and  $f_i$  is irreducible, let  $M_i$  be a triangulation of  $P_i$  such that  $f_i$  is  $M_i$ -irreducible. Let  $L_i$  be a common subdivision of  $M_i$  and  $K_i$ . Put now  $\mathcal{S}_i = \{st(v, L_i) : v \in L_i^0\} \in Cov(P_i)$ . Note that  $f_i$  is  $L_i$ -irreducible and also  $\bar{\mathcal{S}}_i \leq \mathcal{U}_i$ . In order to simplify notations rename the triangulation  $L_i$  again by  $K_i$ . Now choose a normal  $\mathcal{V} \in Cov(X)$  such that  $\mathcal{V} \leq f_i^{-1}(\mathcal{S}_i)$ ,  $i = 1, \dots, n$ . Since  $X$  is a finitistic space there exist an integer  $n$  and a normal covering  $\mathcal{U} \in Cov(X)$  such that  $\mathcal{U} \leq \mathcal{V}$  and  $\text{ord}(\mathcal{U}) \leq n$ . Then, by Proposition 2.1 there exist a finite-dimensional polyhedron  $P = |K|$  and a  $K$ -irreducible

mapping  $f : X \rightarrow |K| \subseteq |N(\mathcal{U})|$ , which is a canonical mapping of  $\mathcal{U}$ .

Now, we define mappings  $\pi_i : N(\mathcal{U})^0 = \mathcal{U} \rightarrow K_i^0$ ,  $i = 1, \dots, n$ , in the following way. To a vertex  $U \in N(\mathcal{U})^0$  we assign a vertex  $v_i = \pi_i(U) \in K_i^0$  such that  $U \subseteq f_i^{-1}(st(v_i, K_i))$ ,  $i = 1, \dots, n$ .

CLAIM 1. For each  $i = 1, \dots, n$ ,  $\pi_i : N(\mathcal{U})^0 \rightarrow K_i^0$  is a simplicial mapping.

Let  $U_1, \dots, U_m$  be vertices of  $N(\mathcal{U})^0$ , which span a simplex of  $N(\mathcal{U})$ . Then  $U_1 \cap \dots \cap U_m \neq \emptyset$  and therefore,  $\emptyset \neq U_1 \cap \dots \cap U_m \subseteq f_i^{-1}(st(\pi_i(U_1), K_i^0)) \cap \dots \cap f_i^{-1}(st(\pi_i(U_m), K_i^0)) = f_i^{-1}(st(\pi_i(U_1), K_i^0) \cap \dots \cap st(\pi_i(U_m), K_i^0))$ . However, this implies  $st(\pi_i(U_1), K_i^0) \cap \dots \cap st(\pi_i(U_m), K_i^0) \neq \emptyset$ , which shows that the vertices  $\pi_i(U_1), \dots, \pi_i(U_m)$  indeed span a simplex of  $K_i$ .

For each  $i = 1, \dots, n$  the mapping  $\pi_i$  induces a mapping  $|\pi_i| : |N(\mathcal{U})| \rightarrow |K_i|$ . Put  $p_i = |\pi_i| : P \rightarrow P_i$ . Note that each  $p_i : P \rightarrow P_i$  is a PL-mapping.

CLAIM 2. For each  $i = 1, \dots, n$ ,  $p_i f : X \rightarrow |K_i|$  is a  $K_i$ -modification of  $f_i$ .

Let  $x \in X$  be an arbitrary point of  $X$  and  $\sigma = [v_1, \dots, v_k] \in K_i$  a simplex of  $K_i$  such that  $f_i(x) \in \sigma$ . We need to prove that  $p_i f(x) \in \sigma$ . Let  $U_1, \dots, U_s$  be all the members of the covering  $\mathcal{U}$ , which contain  $x$ , i.e.  $x \in U_1 \cap \dots \cap U_s$ . Then  $f(x) \in \tau = [U_1, \dots, U_s] \in K \cap N(\mathcal{U})$  and  $p_i f(x) \in [p_i(U_1), \dots, p_i(U_s)] = [\pi_i(U_1), \dots, \pi_i(U_s)]$ . Put  $\pi_i(\{U_1, \dots, U_s\}) = \{w_1, \dots, w_t\} \subseteq K_i^0$ ,  $t \leq s$ , and therefore,  $p_i f(x) \in [w_1, \dots, w_t]$ . Since  $x \in U_1 \cap \dots \cap U_s \subseteq f_i^{-1}(st(\pi_i(U_1), K_i^0)) \cap \dots \cap f_i^{-1}(st(\pi_i(U_s), K_i^0))$ , we obtain  $f_i(x) \in st(w_1, K_i^0) \cap \dots \cap st(w_t, K_i^0)$ . Now we have  $f_i(x) \in \sigma \cap st(w_1, K_i^0) \cap \dots \cap st(w_t, K_i^0)$  and therefore,  $[v_\ell, \dots, v_k] \cap st(w_j, K_i^0) \neq \emptyset$ , for each  $j = 1, \dots, t$ . But this implies that each  $w_j$  is some  $v_\ell$ , i.e.  $\{w_1, \dots, w_t\} \subseteq \{v_1, \dots, v_k\}$ . This means that  $[w_1, \dots, w_t]$  is a face of  $[v_1, \dots, v_k]$  and therefore  $p_i f(x) \in \sigma$ .

CLAIM 3.  $(p_i f, f_i) \leq \mathcal{U}_i$ .

Because of Claim 2, for each  $x \in X$  there exists a simplex  $\sigma \in K_i$  such that  $p_i f(x), f_i(x) \in \sigma$ . Then  $p_i f(x), f_i(x) \in st(v, K_i^0)$ , where  $v$  is any vertex of  $\sigma$ . Since  $\bar{\mathcal{S}}_i \leq \mathcal{U}_i$ , there exists a  $U \in \mathcal{U}_i$  such that  $p_i f(x), f_i(x) \in U$ .

CLAIM 4. If  $f_i$  is  $K_i$ -irreducible, then  $p_i$  is  $K_i$ -irreducible.

We shall first prove that  $p_i f$  is  $K_i$ -irreducible if  $f_i$  is  $K_i$ -irreducible. Let  $h : X \rightarrow |K_i|$  be any  $K_i$ -modification of  $p_i f$ . Then  $h$  is also a  $K_i$ -modification of  $f_i$ . Since  $f_i$  is  $K_i$ -irreducible, we conclude that  $h$  is a surjection. This proves that  $p_i f$  is  $K_i$ -irreducible. Now, let  $h : |K| \rightarrow |K_i|$  be any  $K_i$ -modification of  $p_i$ . Then  $h f$  is  $K_i$ -modification of  $p_i f$ . Since  $p_i f$  is  $K_i$ -irreducible, we conclude that  $h f$  is surjective and thus,  $h$  must also be surjective. This completes the proof of Lemma 2.2.

REMARK 2.3. Let  $X$  be a normal space with dimension  $\dim X \leq m$ . Then it is possible to achieve that the polyhedron  $P$  in Lemma 2.2 has  $\dim P \leq m$ .

Using Lemma 2.2 in the construction, described in [1], we get the following theorem, which is an improvement of Theorem 3.5 of [7].

THEOREM 2.4. *Let  $X$  be a normal space. Then the following statements are equivalent.*

- i)  $X$  is finitistic.
- ii)  $X$  admits an approximate irreducible resolution  $p = \{p_\lambda : \lambda \in \Lambda\} : X \rightarrow \mathcal{X} = \{X_\lambda, \mathcal{U}_\lambda, p_{\lambda\lambda'}, \Lambda\}$  such that all  $X_\lambda$  are finite-dimensional polyhedra, all bonding maps are PL and  $\Lambda$  is cofinite.
- iii)  $X$  admits an approximate resolution  $p = \{p_\lambda : \lambda \in \Lambda\} : X \rightarrow \mathcal{X} = \{X_\lambda, \mathcal{U}_\lambda, p_{\lambda\lambda'}, \Lambda\}$  such that all  $X_\lambda$  are finite-dimensional spaces.

In the special case, when  $X$  is a normal space with  $\dim X \leq m$ , we obtain the following theorem.

THEOREM 2.5. *Let  $X$  be a normal space. Then the following statements are equivalent.*

- i)  $X$  has  $\dim X \leq m$ .
- ii)  $X$  admits an approximate irreducible resolution  $p = \{p_\lambda : \lambda \in \Lambda\} : X \rightarrow \mathcal{X} = \{X_\lambda, \mathcal{U}_\lambda, p_{\lambda\lambda'}, \Lambda\}$  such that all  $X_\lambda$  are polyhedra with  $\dim X_\lambda \leq m$ , all bonding maps are PL and  $\Lambda$  is cofinite.

- iii)  $X$  admits an approximate resolution  $p = \{p_\lambda : \lambda \in \Lambda\} : X \rightarrow \mathcal{X} = \{X_\lambda, \mathcal{U}_\lambda, p_{\lambda\lambda'}, \Lambda\}$  such that all  $X_\lambda$  are spaces with  $\dim X_\lambda \leq m$ .

REMARK 2.6. Theorem 2.5 is a generalization of Theorem 1 of [2] (where a similar statement was proved for compact Hausdorff spaces) and an improvement of Theorem 1 of [9], for normal finite-dimensional spaces (see also [1], §5.). Note that Theorem 1 of [2] was not proved using irreducible mappings, and the constructed approximate resolution does not have surjective bonding maps. This defect could be avoided using the construction of irreducible representations in the sense of [8].

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