CUBICAL MONADS AND THEIR SYMMETRIES (*)

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SOMMARIO. - *Si presenta un'impostazione dell'algebra omotopica basata su un endofuntore cilindro I, e più precisamente sulla nozione di diade* $(I, \partial^-, \partial^+, e, g^-, g^+)$, *o monade cubica. Questo quadro di base può essere arricchito di simmetrie, come l'inversione* $r : I \to I$ *e l'interscambio* $s : I^2 \to I^2$ *del caso topologico classico, e anzi di simmetrie generalizzate, applicabili anche, ad esempio, agli oggetti cubici e alle algebre graduate differenziali. Le due monadi associate ad una diade, il cono inferiore e il cono superiore, sono ottenute mediante collasso di una base del cilindro; le simmetrie sono importanti per il loro studio.*

SUMMARY. - *This work is concerned with a setting for homotopical algebra based on a cylinder endofunctor I, and more precisely on the notion of diad* $(I, \partial^-, \partial^+, e, g^-, g^+)$, *or cubical monad. This basic frame can be enriched with symmetries, as the reversion* $r : I \to I$ *and interchange* $s : I^2 \to I^2$ *for the classical topological case; or with generalised symmetries, applying also, for instance, to cubical objects or differential graded algebras. The two monads associated to a diad, lower cone and upper cone, are obtained by collapsing one base of the cylinder; symmetries are relevant for the study of their properties.*

Introduction.

A *dioid*, or cubical monoid, is here a set equipped with two monoid structures, such that the unital element of each is absorbant for the other; every lattice with 0 and 1 is so.

The "category-theoretical version" produces the notion of *diad* (or cubical monad, or I-category) in the same way as the diagrammatical axioms for a monoid can be turned into the axioms for a monad. However, the as-

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sociativity of the two operations is not assumed for a diad, as its relevance in the present context is limited.

An I-category is thus a category $\mathbf{A}$ equipped with a *cylinder* endofunctor $I : \mathbf{A} \to \mathbf{A}$, two *faces* (or units) ∂^-, $\partial^+ : 1 \to I$, a *degeneracy* $e : I \to 1$ and two *connections* (or operations) g^-, $g^+ : I^2 \to I$ satisfying rather obvious coherence axioms (1.5). It is a "homotopy system" in the sense of Kan [Ka], but enriched with connections. The prime example, of course, is the cylinder endofunctor for topological spaces, $I(X) = X \times [0,1]$, with connections defined by the lattice operations of join and meet in [0, 1]. The dual notion, *codiad* or cubical comonad or $\mathbf{P}$-category, has again a well-known representative for topological spaces, the path endofunctor $P(X) = X^{[0,1]}$ right adjoint to I. Finally, in an IP-category, one has an adjunction $I \dashv P$ transforming one structure into the other; and in a symmetrical monoidal closed category, every dioid determines such an IP-structure (1.9; 5.3-5).

Important enrichments of this setting concern symmetries, corresponding to *involutive* and *commutative* dioids. For the topological cylinder, we have thus a *reversion* $r : I \to I$ (exchanging the lower structure (∂^-, g^-) with the upper one) and an *interchange* $s : I^2 \to I^2$ (exchanging $I\partial^\varepsilon$ with $\partial^\varepsilon I$ and invariant under the connections). This also happens for chain complexes (5.1). For the P-categories of cubical objects or differential graded algebras (2.1-3) we need more general notions, a *generalised reversion* (R, r) and a *generalised interchange* (S, s), consisting of involutive endofunctors R, S with natural transformations $r : RP \to PR$, $s : SPSP \to PSPS$ (2.4-5).

For a cubical object X, one obtains RX by exchanging the lower and upper structure, SX by reversing the indexing ($i' = n+1-i$, in degree n). For cochain algebras, RA is the opposite algebra while $SA = A$.

A few homotopical properties of an I-category are briefly considered in ch. 3, 4 and will be further developed elsewhere. In the presence of pushouts and a terminal object, every I-category has two associated monads, the *lower cone* (C^-, ∂, g) and the *upper cone* (C^+, ∂, g), obtained by collapsing, respectively, the upper or lower base of the cylinder (3.5, 3.7); their operations are induced by the non-collapsed connections of I. An interchange supplies the homotopical invariance of I, C^- and C^+ (3.3, 3.8), while a reversion produces an isomorphism between the lower and upper cone, C^- and C^+ (3.9). More generally, under a *generalised* interchange (S, s) the functors I, C^- and C^+ are invariant *up to* $\bar{I}$*-homotopy*, where $\bar{I} = SIS$ is the secondary cylinder; under a *generalised* reversion the *re-*

versed lower cone $RC^{-}A$ is isomorphic to the upper cone $C^{+}RA$ of the *reversed* object.

This work inserts in the study of categories with a cylinder endofunctor (or "homotopy system"), originated by Kan [Ka] and pursued also by Kamps [Km1, 3] and Baues [Ba]. It is also related with the notion of category with a "generalised homotopy system" [Km2] or h-category [G1, G2], since a *semidiad* $(I, \partial^{-}, \partial^{+}, e)$ is the same as an h-category with homotopy-corepresenters (3.2). But here the second-order homotopy properties, to be investigated in the sequel, are to follow from connections and symmetries rather than from a vertical composition of homotopies as in [G1, G2], which can fail in important cases like differential graded algebras.

Connections in cubical objects are investigated in works of Brown-Higgins [BH1-3]. A recent result, analogous to the classical Moore's one for simplicial groups [Mo], proves that a cubical group with lower connections is automatically Kan (Tonks [To]).

Relations can also be found with the use of monads for homology and derived functors (cf. [BB] and references therein), and more closely with the cone-setting for *additive* homotopical algebra introduced by Rodriguez [Rd]. Note that in the additive case the cone endofunctor describes the whole homotopical structure, since homotopies $f \simeq g$ are determined by *nullhomotopies* $0 \simeq g - f$, which are *co-represented* by the cone (3.5). Note also that the associativity axiom is not required in [Rd], but there is an interchange $s : C^2 \to C^2$ exchanging the faces $C\partial$ and ∂C of the double cone C^2.

The links among adjunctions, (associative) monads and their algebras are well-known and important (see Mac Lane's text [Ml]). Such topics seem not have a counterpart here.

A formal argument in this sense can be based on the following result (I thank D. Bourn for pointing it out to me): The category $\mathbf{A}^T$ of algebras for a monad T over $\mathbf{A}$ can be obtained as the lax limit in $\mathbf{CAT}$ of a diagram naturally formed of the data of the monad itself ([Bo, Gr]). But the lax limit of the diagram so obtained from a diad is trivial ($\mathbf{A}$ itself), because of the presence of the degeneracy e.

Outline. Chapter 1 introduces dioids and diads, the dual notions and the selfdual case of IP-categories. Ch. 2 is concerned with symmetries. Homotopies defined by the I-structure and the associated lower- and upper-cone monads are considered in ch. 3, together with their relations with symmetries. Finally, ch. 4 treats homotopies in IP-categories and the cone-arc adjunction for the pointed case, while ch. 5 briefly sketches the IP-structures of chain complexes, categories and reflexive graphs.

1. Dioids and Diads.

A dioid is a set equipped with two monoid structures, so that the unital element of each is absorbant for the other. The "category-theoretical version" produces the notion of a *diad*, or cubical monad, or I-category.

1.1. Dioids.

A *dioid* will be an algebra $X = (|X|, 1, 0, \ . \ , *)$ such that

a) $(1, \cdot)$ and $(0, *)$ are two monoid structures over the same set $|X|$,

b) 0 is absorbant for . and 1 is absorbant for $*$: $x.0 = 0 = 0.x$, $x * 1 = 1 = 1 * x$.

A dioid is commutative if both its binary operations are so. Given a dioid X, the *opposite dioid* is obtained by exchanging the two monoid structures

$$RX = X^{op} = (|X|, 0, 1, *, .) \ , \tag{1}$$

a procedure which will be referred to as *op-duality*.

An *involutive dioid* is equipped with an involution $r : |X| \to |X|$ exchanging the two monoid structures

$$r(1) = 0, \qquad r(x.y) = rx * ry \ , \tag{2}$$

or, in other words, with a dioid-isomorphism $r : X \to X^{op}$. Then X can be equivalently described as $X = (|X|, 1, r, .)$, where $(1, .)$ is a monoid structure, r is a set-theoretical involution and the element $0 = r(1)$ is absorbant for the product. The second operation is reconstructed as $x * y = r(rx.ry)$.

The category **Did** of dioids and *homomorphisms* is obvious; it is complete and cocomplete. Its terminal object is the one-element dioid $\top = \{*\}$, the only case where the zeroary operations coincide; the initial object is the two-element dioid $S^0 = \{0, 1\}$. $R : \mathbf{Did} \to \mathbf{Did}$ is an involutive endofunctor; restricted to involutive dioids, R is isomorphic to the identity via r.

Every unital ring A has an associated structure of involutive dioid

(3) $(|A|, 1, r, .),\ r(x) = 1 - x,\ x * y = r(rx.ry) = x + y - x.y\ .$

In particular the real field $\mathbb{R}$ has an involutive subdioid consisting of the interval [0, 1]; and these operations $., *$ on [0, 1] are often used in homotopy theory.

1.2. Lattices and idempotent dioids.

However, we shall currently use a different structure on [0, 1], which will be called the *standard-interval* involutive $\vee$-dioid

(1) $I = ([0,1], 0, r, \vee), \quad r(t) = 1 - t, \quad t \vee t' = \max(t, t')\ ,$

its opposite being the standard interval involutive
$\wedge$-dioid $I^{op} = ([0,1], 1, r, \wedge)$.

More generally, any lattice X (with 0 and 1) has an associated $\vee$-dioid $X = (|X|, 0, 1, \vee, \wedge)$ carrying the same information. As well known, the dioids one obtains in this way are characterized by the fact that their binary operations are commutative, idempotent and satisfy the absorption laws.

Similarly, any *involutive lattice* X (equipped with an involutive anti-automorphism) has an associated involutive $\vee$-dioid $X = (|X|, 0, r, \vee)$. X is a *complemented lattice* if moreover $x \vee rx = 1$, $x \wedge rx = 0$, and a Boolean algebra if its two operations are mutually distributive.

Among the above examples, $\{*\}$ and S^0 are Boolean dioids. The *n-cube* involutive $\vee$-dioid I^n, i.e. the cartesian power of I in **Did** (or in the category of involutive lattices)

(2) $$I^n = ([0,1]^n, 0, r^n, \vee), \quad 0 = (0, \ldots 0),$$
$$r^n(t_1, \ldots t_n) = (1 - t_1, \ldots 1 - t_n),$$

(3) $$(t_1, \ldots t_n) \vee (s_1, \ldots s_n) = (t_1 \vee s_1, \ldots t_n \vee s_n),$$

is a *non*-complemented distributive involutive lattice (for $n \geq 1$). I^n, endowed with the euclidean topology, is a *topological* involutive dioid, meaning that the involution and join are continuous.

A spectral measure for the Banach space W can be seen as a dioid-homorphism $E : X^{op} \to BW$ defined over the involutive $\wedge$-dioid $X^{op} =$

$(|X|, 1, r, \wedge)$ of a Boolean algebra X, with values in the involutive dioid $(|BW|, 1, r, .)$ associated to the unital ring of bounded endo-operators of W.

1.3. Non-idempotent examples.

Besides the dioids associated to non-idempotent rings (1.1.3), a standard non-idempotent example (again on the compact real interval) is the *dioidal half-line*, i.e. the Alexandroff compactification $H^1 = [0, +\infty]$ of the additive monoid $[0, +\infty[$, with the extended sum and the involution $r(x) = x^{-1}$. The opposite operation

$$x * y = (x^{-1} + y^{-1})^{-1} , \tag{1}$$

can be called *inverse sum*, or also *harmonic sum* (since $1/n * 1 = 1/(n+1)$).

$\mathbb{P}^1\mathbb{R} = \mathbb{R} \cup \{\infty\}$ and $\mathbb{P}^1\mathbb{C} = \mathbb{C} \cup \{\infty\}$ have a similar structure of involutive dioid. But H^1 is a topological dioid, as the sum is proper over $[0, +\infty[$, while $\mathbb{P}^1\mathbb{R}$ and $\mathbb{P}^1\mathbb{C}$ are not.

The dioid H^1 has an interpretation referring to electric circuits of pure resistors. Interpret its elements as *resistances*, "+" as their *series combination*, "$*$" as their *parallel combination* (where conductances, the inverses of resistances, are to be added), "0" as the *perfect conductor*, "$+\infty$" as the *perfect insulator*. Similarly, the dioid $\mathbb{P}^1\mathbb{C}$ formalizes the calculus of *impedances* (and their inverses, *admittances*) for networks of resistors, inductors and capacitors in steady sinusoidal state. The operation $*$ of H^1 is also of use in geometrical optics, where $p * q = f$ is the well-known formula relating corresponding points for a lens of focal length f.

More generally, one can consider the *dioidal n-orthant* $H^n = \mathbb{R}^n_+ \cup \{\infty\}$ as the compactification of the additive monoid $\mathbb{R}^n_+$, with the extended sum and the involution, $r(x) = x/||x||^2$. It is a topological dioid, since the sum is proper over $\mathbb{R}^n_+$. Note that H^n is not a cartesian power of H^1, and H^2 is not a subdioid of $\mathbb{P}^1\mathbb{C}$ (whose inverse sum is given by the involution $r(z) = z^{-1} = \bar{z}/|z|^2$).

Finally, if M is any topological (additive) monoid, equipped with a continuous involution r over the subspace $|M| - \{0\}$, one gets a topological involutive dioid $X = M \cup \{\infty\}$ through one-point compactification, provided that the sum $M \times M \to M$ is a proper map and that $r(x) \to \infty$ for $x \to 0$.

1.4. Diagrammatic description.

Actually, we are not interested here in the algebraic theory of dioids, but in its category-theoretical transposition. A dioid can be described as a diagram (1) in the (cartesian) monoidal category **Set**

$$(1) \qquad \{*\} = X^0 \;\underset{e}{\overset{\partial^\varepsilon}{\rightleftarrows}}\; X \;\overset{g^\varepsilon}{\longleftarrow}\; X^2 \qquad (\varepsilon = 0, 1 \text{ or } -, +)$$

such that some diagrams, expressing the algebraic axioms, commute. And a topological dioid is such a diagram in the cartesian category **Top**. One can similarly define a *dioid-object* in any monoidal category (various examples are considered in 5.3-5). It should be noted that the map $e : X \to X^0$ is determined by X (and redundant) in the cartesian case, where the neutral element $X^0 = \{*\}$ is the terminal object, but not generally.

1.5. Diads.

A *diad*, or *cubical monad*, or *I-structure* over the category **A** will be a similar setting in the category End**A** of endofunctors of **A** and natural transformations, with respect to the monoidal structure given by the composition of such endofunctors. However, we do not require the associativity axiom, which seems not of basic interest here.

In other words, a diad is a collection $(I, \partial^-, \partial^+, e, g^-, g^+)$ of an endofunctor $I : A \to A$ (the *cylinder* endofunctor) with natural transformations, respectively called *lower face* or *lower unit* (∂^- or ∂^0), *upper face* (∂^+ or ∂^1), *degeneracy* (e), *connections* or *main operations* (g^-, g^+)

$$(1) \qquad id\ \mathbf{A} = 1 = I^0 \;\underset{e}{\overset{\partial^\varepsilon}{\rightleftarrows}}\; I \;\overset{g^\varepsilon}{\longleftarrow}\; I^2 \quad (\varepsilon = 0, 1 \text{ or } -, +)$$

making the following diagrams commutative ($\varepsilon \neq \eta$)

$$(2) \qquad \begin{array}{ccc} 1 & \xrightarrow{\partial^\varepsilon} & I \\ & \diagdown\!\!\diagdown & \downarrow{\scriptstyle e} \\ & & 1 \end{array} \qquad\qquad (3) \qquad \begin{array}{ccccc} I & \xleftarrow{Ie} & I^2 & \xrightarrow{eI} & I \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle g^\varepsilon} & & \downarrow{\scriptstyle e} \\ 1 & \xleftarrow{e} & I & \xrightarrow{e} & 1 \end{array}$$

$$(4)\quad \begin{array}{ccccc} 1 & \xrightarrow{I\partial^\varepsilon} & I^2 & \xleftarrow{\partial^\varepsilon I} & I \\ & \diagdown\!\!\diagdown & \downarrow{\scriptstyle g^\varepsilon} & /\!\!/ & \\ & & I & & \end{array} \qquad (5)\quad \begin{array}{ccccc} I & \xrightarrow{I\partial^\varepsilon} & I^2 & \xleftarrow{\partial^\varepsilon I} & I \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle g^\eta} & & \downarrow{\scriptstyle e} \\ 1 & \xrightarrow{\partial^\varepsilon} & I & \xleftarrow{\partial^\varepsilon} & 1 \end{array}$$

A *semi-diad* (or *I0-category*) has reduced data $(I, \partial^\varepsilon, e)$ satisfying (2).

An *involutive* diad is further equipped with a *reversion* $r : I \to I$ exchanging the lower and upper structure, while a *symmetrical* diad has an *interchange* $s : I^2 \to I^2$ exchanging $I\partial^\varepsilon$ with $\partial^\varepsilon I$ and invariant under the connections. The formal definitions will be given in ch. 2. An *associative* diad verifies the associativity axiom, $g^\varepsilon.Ig^\varepsilon = g^\varepsilon.g^\varepsilon I : I^3 \to I$.

1.6. The topological cylinder.

The main example of a symmetrical involutive (and associative) diad is the standard topological cylinder $I : \mathbf{Top} \to \mathbf{Top}$, $I(X) = X \times I$ (with operations $\vee$, $\wedge$)

(1)	$\partial^- : X \to X \times I,$	$x \mapsto (x, 0),$
(1′)	$\partial^+ : X \to X \times I,$	$x \mapsto (x, 1),$
(2)	$e : X \times I \to X,$	$(x, t) \mapsto x,$
(3)	$\vee : X \times I^2 \to X \times I,$	$(x, t, t') \mapsto (x, t \vee t'),$
(3′)	$\wedge : X \times I^2 \to X \times I,$	$(x, t, t') \mapsto (x, t \wedge t'),$
(4)	$r : X \times I \to X \times I,$	$(x, t) \mapsto (x, 1 - t),$
(4′)	$s : X \times I^2 \to X \times I^2,$	$(x, t, t') \mapsto (x, t', t).$

Note that our endofunctor is $- \times I$, and all the above natural transformations derive from the mere structure of the standard interval I as a commutative involutive dioid in **Top** (1.2.1). Similarly, any dioid-object in any category with finite products (resp. monoidal category) produces an associative *cartesian* (resp. *monoidal*) diad.

For instance, the real field $\mathbb{R}$, with the dioid-structure considered in 1.1.3, defines a cartesian symmetrical involutive diad in the category of differentiable manifolds, which determines the diffeohomotopies.

The category $\mathbf{Top}^\top$ of pointed topological spaces has a similar, non-cartesian structure, with $I(X) = (X \times [0, 1])/\{0_X\} \times [0, 1]$; it is a pointed structure, in the sense that I preserves the zero-object. And it is produced by the standard interval dioid I of **Top**, through the action of **Top** over $\mathbf{Top}^{\top}$ by *smash product*

$$(5)\quad \mathbf{Top}^{\top} \times \mathbf{Top} \to \mathbf{Top}^{\top},\ (X, A) \mapsto X \wedge A = (X \times A)/(\{0_X\} \times A)\ .$$

Similarly, any dioid-object in a monoidal category **A** produces an associative **A**-*monoidal* diad in any category equipped with an action of **A**.

1.7. Codiads.

Dually, a *codiad*, or *cubical comonad*, or *P-category* $(P, \partial^{\varepsilon}, e, g^{\varepsilon})$ consists of an endofunctor $P : \mathbf{A} \to \mathbf{A}$ (the path endofunctor) with natural transformations

$$(1)\qquad id\mathbf{A} = 1 = P^0 \underset{e}{\overset{\partial^{\varepsilon}}{\leftleftarrows\!\!\rightarrow}} P \overset{g^{\varepsilon}}{\rightrightarrows} P^2$$

satisfying the dual conditions. A *semi-codiad* (or P0-*category*) $(P, \partial^-, \partial^+, e)$ has just a path endofunctor with faces and degeneracy verifying $\partial^{\varepsilon} e = 1$.

The standard example of a symmetrical involutive codiad is the path functor $P : \mathbf{Top} \to \mathbf{Top}$ of topological spaces, right adjoint to the cylinder endofunctor

(2) $P(X) = X^I$ (with compact-open topology),
(3) $\partial^- : PX \to X,\quad \partial^- \lambda = \lambda(0),\quad \partial^+ : PX \to X,\quad \partial^+ \lambda = \lambda(1),$
(4) $e : X \to PX,\quad (ex)(t) = x,$
(5) $g^- : PX \to P^2X,\quad (g^- \lambda)(t, t') = \lambda(t \vee t'),$
(5′) $g^+ : PX \to P^2X,\quad (g^+ \lambda)(t, t') = \lambda(t \wedge t'),$
(6) $r : PX \to PX,\quad (r\lambda)(t) = \lambda(1 - t),$
(6′) $s : P^2X \to P^2X,\quad (s\alpha)(t, t') = \alpha(t', t).$

This codiad is also produced by the standard dioid I, through the internal hom-functor given by the compact-open topology on the space X^A of continuous functions from A to X

$$(7)\qquad \mathbf{Top}^{op} \times \mathbf{Top} \to \mathbf{Top},\ (A, X) \mapsto X^A\ ,$$

and the canonical isomorphism $X^{I \times I} \cong (X^I)^I$. However, since **Top** is not cartesian-closed, some restriction on the domain of the contravariant variable A may be useful (1.9).

Again, $\mathbf{Top}^\top$ has a pointed codiad, with $P(X) = X^I$, based at the constant path at 0_X. It is produced by the standard dioid I of **Top**, through the functor

$$\mathbf{Top}^{op} \times \mathbf{Top}^\top \to \mathbf{Top}^\top, \quad (A, X) \mapsto X^A \ , \tag{8}$$

where X^A is based at the constant map at 0_X.

1.8. IP-categories.

Clearly, in **Top** and $\mathbf{Top}^\top$, the adjunction of the cylinder and path functors is consistent with their natural transformations, so that the I- and P-structure determine each other. An IP0-*category* is a category equipped with adjoint endofunctors $I \dashv P$ and I0, P0-structures

$$I : \mathbf{A} \to \mathbf{A}, \quad P : \mathbf{A} \to \mathbf{A}, \quad u : 1 \to PI, \quad v : IP \to 1 \tag{1}$$

$$(vI.Iu = 1, Pv.uP = 1),$$

$$\delta^\varepsilon : 1 \to I, \quad \eta : I \to 1, \quad \eta\delta^\varepsilon = 1, \tag{2}$$

$$\partial^\varepsilon : P \to 1, \quad e : 1 \to P, \quad \partial^\varepsilon e = 1, \tag{2'}$$

which are consistent with the adjunction (an adjoint map is denoted by a "prime")

$$eA = (\eta A)' = P\eta A.uA : A \to PIA \to PA, \tag{3}$$

$$\eta A = (eA)' = vA.IeA : IA \to IPA \to A, \tag{3'}$$

$$\partial^\varepsilon A = vA.\delta^\varepsilon PA : PA \to IPA \to A, \tag{4}$$

$$\delta^\varepsilon A = \partial^\varepsilon IA.uA : A \to PIA \to IA. \tag{4'}$$

More simply, if not symmetrically, an IP0-category can be presented as an I0-category where the endofunctor I has a right adjoint P; then the latter is determined (up to isomorphism) and, defining e and ∂^ε via (3), (4) it is easy to prove that we get natural transformations verifying (2′), (3′), (4′). Dually, one can start from the P-structure.

The cubical categories constructed from I or P are canonically isomorphic; an n-morphism $A \Rightarrow B$ is equivalently determined by a map $I^n A \to B$ or by the corresponding map in the adjunction, $A \to P^n B$.

Similarly, an *IP-category* is a category equipped with adjoint endofunctors $I \dashv P$ and consistent I- and P-structures. We have thus connections for I and P, satisfying their axioms (1.5; 1.7) and consistent with the adjunction

$$\gamma^{\varepsilon} : I^2 \to I, \quad g^{\varepsilon} : P \to P^2, \tag{5}$$

$$g^{\varepsilon}A = P^2vA.(\gamma^{\varepsilon})'PA : PA \to P^2IPA \to P^2A, \tag{6}$$

$$\gamma^{\varepsilon}A = (g^{\varepsilon})'IA.I^2uA : I^2A \to I^2PIA \to IA. \tag{6'}$$

1.9. Monoidal closed IP-structures.

Let **A** be a symmetrical monoidal closed category [EK, Ke], with tensor product $-\otimes-$ and internal hom-functor $[-,-]$; write E the identity of $\otimes$.

A dioid-object I of **A** (1.4)

$$E \underset{e}{\overset{\partial^\varepsilon}{\rightleftarrows}} I \overset{g^\varepsilon}{\longleftarrow} I \otimes I \tag{1}$$

produces an IP-structure over **A**, which will be called *monoidal closed* (and *cartesian closed* if the tensor product is the categorical product)

$$IA = A \otimes I, \quad PA = [I, A] \ , \tag{2}$$

where, for instance, $\partial^\varepsilon A : A \to IA$ is defined by $A \otimes \partial^\varepsilon : A \otimes E \to A \otimes I$, through the composition

$$\partial^\varepsilon A = (A \to A \otimes E \to A \otimes I = IA) \ . \tag{3}$$

Some examples are considered in 5.3-5.

The standard IP-structure on **Top** is of a slightly more general nature, as **Top** is not cartesian closed; it can be thought to be produced by the standard dioid I in (**Top**, $\times$), through the adjunctions $X \times K \dashv Y^K$, with K variable in a suitable full subcategory **K** (containing the standard interval and closed under finite products), e.g. compact, or locally compact Hausdorff spaces. Similarly the standard IP-structure on $\mathbf{Top}^\top$ is produced by the same dioid I, through the functors

$$\mathbf{Top}^\top \times \mathbf{K} \to \mathbf{Top}^\top, \ (X, K) \mapsto X \wedge K = (X \times K)/([0_X] \times K), \tag{4}$$

$$\mathbf{K}^{op} \times \mathbf{Top}^\top \to \mathbf{Top}^\top, \quad (K, Y) \mapsto [K, Y] = Y^K \ , \tag{5}$$

and the adjunctions $X \times K \dashv Y^K$, with K variable in **K**.

2. Other Examples and Symmetries.

Two well known and important P-categories are considered, cubical objects (with connections) and differential graded algebras, also in order to motivate our definitions of generalised symmetries. In the first case, the associativity of cubical comonads is of interest.

2.1. Cubical objects.

A *cubical object* $X = ((X_n), (\partial^\varepsilon_{ni}), (e_{ni}), (g^\varepsilon_{ni}))$ is here assumed to be equipped with faces $(\partial^\varepsilon_{ni})$ and degeneracies (e_{ni}), and also with *lower* and *upper connections*

$$g^\varepsilon_{ni} : X_{n+1} \to X_n \quad (i = 1, \ldots, n;\ \varepsilon = 0, 1) \tag{1}$$

verifying the obvious axioms (see in [BH1] the axioms for the lower connections (g^-_{ni})).

Formally, this inclusion is prescribed by our definition of diad and more consistent with the notion of simplicial object. As a matter of fact, cubical groups (with lower connections, at least) are Kan [To].

For a category **C**, the category Cub **C** of cubical objects over **C** has thus an *associative* cubical comonad $(P; \partial^\varepsilon, e, g^\varepsilon)$ where P shifts the cubical object X "one degree down", and the components of the natural transformations are obtained from the maps of X made "superfluous" from the shifting

$$P : Cub\ \mathbf{C} \to Cub\ \mathbf{C}, \tag{1}$$

$$P((X_n), (\partial^\varepsilon_{ni}), (e_{ni}), (g^\varepsilon_{ni})) = \tag{2}$$

$$= ((X_{n+1}),\ (\partial^\varepsilon_{n+1,i+1}), (e_{n+1,i+1}), (g^\varepsilon_{n+1,i+1})),$$

$\partial^\varepsilon X : PX \to X, \quad (\partial^\varepsilon X)_n = \partial^\varepsilon_{n+1,1} : X_{n+1} \to X_n,$ (3) $eX : X \to PX, \quad (eX)_n = e_{n+1,1} : X_n \to X_{n+1},$ (4) $g^\varepsilon X : P^2X \to PX, \quad (g^\varepsilon X)_n = g^\varepsilon_{n+1,1} : X_{n+2} \to X_{n+1}$.(5)

With this structure, Cub **C** is the *cofree* associative P-category generated by **C** (with respect to the forgetful functor | - | from associative

P-categories to categories). The counit of the adjunction is

$$V_{\mathbf{C}} : |Cub\ \mathbf{C}| \to \mathbf{C}, \quad V_{\mathbf{C}}((X_n), (\partial^{\varepsilon}_{ni}), (e_{ni}), (g^{\varepsilon}_{ni})) = X_0, \tag{6}$$

$while the unit of the adjunction, for an associative P-category \mathbf{A}, is the P-functor turning every object$

$A of \mathbf{A} into the cubical object with components P^n A$ and maps produced by the natural transformations of the codiad

$$U_A = P_* : \mathbf{A} \to Cub|\mathbf{A}|, \quad P_*(A) = ((P^n A), (\partial^{\varepsilon}_{ni}), (e_{ni}), (g^{\varepsilon}_{ni})) \ . \tag{7}$$

Clearly, there is no reversion $r : P \to P$ exchanging the lower and upper transformations of P, and no interchange $s : P^2 \to P^2$ exchanging $P\partial^{\varepsilon}$ with $\partial^{\varepsilon} P$.

But we do have an *external reversion* and an *external interchange* (2.4-5) surrogating them and consisting of involutive endofunctors R, S: $Cub\ \mathbf{C} \to Cub\ \mathbf{C}$. The former exchanges the lower and upper maps of a cubical object, giving $RP = P\mathrm{R}$ and $R\partial^- = \partial^+ R$. The second reverses the indices i according to the transformation $i' = n+1-i$ and verifies $SPSP = PSPS$ and $SPS.\partial^{\varepsilon} = \partial^{\varepsilon}.SPS$; it produces a secondary path-functor $\bar{P} = SPS$ which discards $\partial^{\varepsilon}_{n+1,n+1}$ instead of $\partial^{\varepsilon}_{n+1,1}$.

2.2. Differential graded algebras.

Let K be a commutative unital ring. We sketch now, here and in the next section, the homotopical properties of the category **Dga** of *dg-algebras* (differential graded unital K-algebras), as a P-category with interchange and a *generalised reversion*. The proofs, which are not given here, consist of standard but long calculations.

An object $A = ((A^n), (\partial^n))$ is a positive cochain complex of K-modules (indexed over $\mathbb{Z}$, with $A^n = 0$ for $n < 0$) with a product of graded K-algebra consistent with the differential $\partial^n : A^n \to A^{n+1}$

$$\partial(x.y) = \partial x.y + (-1)^{|x|} x.\partial y \ . \tag{1}$$

The path endofunctor $P : \mathbf{Dga} \to \mathbf{Dga}$ is described as follows, with $\varepsilon = (-1)^{|a|}$

$$(PA)^n = A^n \oplus A^{n-1} \oplus A^n, \quad 1 = (1, 0, 1) \in A^0 \oplus A^{-1} \oplus A^0, \tag{2}$$

$$(a, h, b).(c, k, d) = (ac, hd + \varepsilon ak, bd),$$

(3) $$\partial(a, h, b) = (\partial a, -a + b - \partial h, \partial b),$$

(4) $$\partial^-(a, h, b) = a, \quad \partial^+(a, h, b) = b,$$

$$\delta(a, h, b) = h \quad (\delta : \partial^- \to \partial^+ : PA \to A) \ .$$

The P0-structure is given by the above transformations $\partial^-, \partial^+ : P \to 1$, together with

(5) $$e : 1 \to P, \quad e(a) = (a, 0, a) \ .$$

2.3. The second order structure for Dga.

A generic element $\xi = (a, h, b; u, z, v; c, k, d)$ of the second order path-object

(1) $$(P^2A)^n =$$

$$(A^n \oplus A^{n-1} \oplus A^n) \oplus (A^{n-1} \oplus A^{n-2} \oplus A^{n-1}) \oplus (A^n \oplus A^{n-1} \oplus A^n) \ ,$$

will be represented as a square diagram, so that its faces $\partial^\varepsilon P$ and $P\partial^\varepsilon$ respectively appear as horizontal or vertical edges

(2) $$\begin{array}{ccc} c & \xrightarrow{k} & d \\ {\scriptstyle u}\uparrow & z & \uparrow{\scriptstyle v} \\ a & \xrightarrow[h]{} & b \end{array} \qquad \begin{array}{ll} \partial^- P(\xi) = (a, h, b), & \partial^+ P(\xi) = (c, k, d), \\ P\partial^-(\xi) = (a, u, c), & P\partial^+(\xi) = (b, v, d). \end{array}$$

With the above representation, g^- and g^+ are given by

(3) $$\begin{array}{ccc} b & \xrightarrow{0} & b \\ {\scriptstyle h}\uparrow & 0 & \uparrow{\scriptstyle 0} \\ a & \xrightarrow[h]{} & b \end{array}$$ (4) $$\begin{array}{ccc} a & \xrightarrow{h} & b \\ {\scriptstyle 0}\uparrow & 0 & \uparrow{\scriptstyle h} \\ a & \xrightarrow[0]{} & a \end{array}$$

(5) $$g^-(a, h, b) = (a, h, b; h, 0, 0; b, 0, b),$$

(6) $$g^+(a, h, b) = (a, 0, a; 0, 0, h; a, h, b) \ .$$

The interchange $s : P^2A \to P^2A$ is obtained through a reflection with respect to the "main diagonal" (in bold character), with a sign-change in the middle term

$$(7) \qquad \begin{array}{ccc} c & \xrightarrow{k} & \mathbf{d} \\ {\scriptstyle u}\uparrow & \mathbf{z} & \uparrow{\scriptstyle v} \\ \mathbf{a} & \xrightarrow[h]{} & b \end{array} \quad \mapsto \quad \begin{array}{ccc} b & \xrightarrow{v} & \mathbf{d} \\ {\scriptstyle h}\uparrow & -\mathbf{z} & \uparrow{\scriptstyle k} \\ \mathbf{a} & \xrightarrow[u]{} & c \end{array}$$

$$s(\mathbf{a}, h, b;\ u, \mathbf{z}, v;\ c, k, \mathbf{d}) = (\mathbf{a}, u, c;\ h, -\mathbf{z}, k;\ b, v, \mathbf{d}).$$

Finally, it can be proved that a reversion $r : P \to P$ turning ∂^- into ∂^+ can only exist under very restrictive conditions on the ring K. But we always have an opposite dg-algebra $RA = A^{op}$ over the same graded module

$$(8) \qquad a^*.b^* = (b.a)^*, \quad \partial(a^*) = (-1)^{|a|}(\partial a)^*\ ,$$

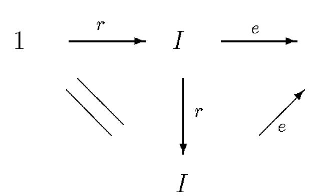
and a *generalised reversion* (R, r)

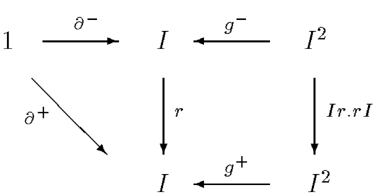
$$(9) \qquad r : (PA)^{op} = P(A^{op}), \quad r(a, h, b)^* = (b^*, (-1)^{|a|}h^*, a^*)$$

which turns the lower structure of $(PA)^{op}$ into the upper structure of $P(A^{op})$.

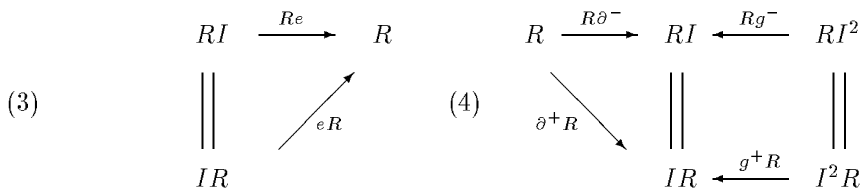
2.4. Reversion.

Coming back to the general theory, an *involutive* diad I is equipped with a reversion $r : I \to I$ exchanging the lower structure (∂^-, g^-) with the upper one (∂^+, g^+)

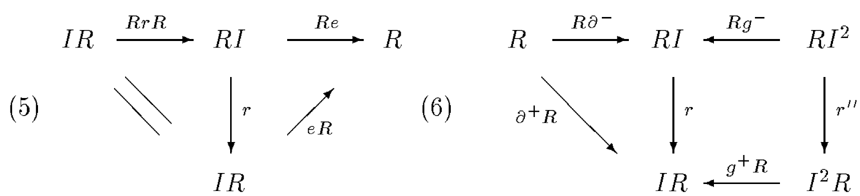
(1)


(2)


An *external reversion* (as -dually- in the P-category Cub $\mathbf{C}$, the procedure of exchanging the lower and upper maps of a cubical object), consists instead of an involutive (covariant) endofunctor $R : \mathbf{A} \to \mathbf{A}$ such that $RI = IR$ and



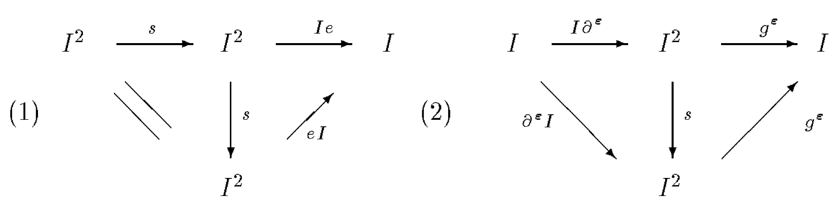
For **Dga** it is necessary to generalise both these notions. A *generalised reversion* $(R, r : RI \to IR)$ consists thus of an involutive endofunctor $R : \mathbf{A} \to \mathbf{A}$ with a natural transformation r verifying the following axioms (note that r is iso, with $r^{-1} = RrR$)



where $r'' = Ir.rI : (RI^2 \to IRI \to I^2R)$ is the reversion of the double cylinder. For an I0-category one should discard the diagrams concerning the connections.

2.5. Interchange.

A *symmetrical* diad I is equipped with an (internal) *interchange* $s : I^2 \to I^2$ exchanging the faces $I\partial^\varepsilon$ with $\partial^\varepsilon I$ and invariant under the connections



In a *symmetrical involutive diad* the reversion has to be consistent with the symmetry: $s.Ir = rI.s$.

An *external interchange* (such as, in Cub $\mathbf{C}$, to reverse the index i) consists of an involutive endofunctor $S : \mathbf{A} \to \mathbf{A}$ such that $SISI = ISIS$ and

$$(3) \qquad SISe = eSIS : SISI = ISIS \to SIS \ ,$$

$$SIS\partial^\varepsilon = \partial^\varepsilon SIS : SIS \to SISI = ISIS \ .$$

A *generalised interchange* can now be defined as a pair $(S, s : SISI \to ISIS)$ consisting of an involutive endofunctor $S : \mathbf{A} \to \mathbf{A}$ and a natural transformation s verifying the following axioms (again, s is iso, with $s^{-1} = SsS$)

$$(4) \quad \begin{array}{ccccc} ISIS & \xrightarrow{SsS} & SISI & \xrightarrow{SISe} & I \\ & \searrow\!\!\!= & \downarrow s & \nearrow eSIS & \\ & & ISIS & & \end{array} \qquad (5) \quad \begin{array}{ccc} I & \xrightarrow{S\partial^\varepsilon SI} & SISI \\ & \searrow IS\partial^\varepsilon S & \downarrow s \\ & & ISIS \end{array}$$

As in the case of cubical objects, it is of interest to note that $\mathbf{A}$ can be provided with a *secondary semi-diad* $\bar{I}$, and that the axiom (5) can be rewritten in the two equivalent forms of (7)

$$(6) \qquad \bar{I} = SIS, \qquad \bar{e} = SeS : \bar{I} \to 1, \qquad \bar{\partial}^\varepsilon = S\partial^\varepsilon S : 1 \to \bar{I} \ ,$$

$$(7) \qquad s\bar{\partial}^\varepsilon I = I\bar{\partial}^\varepsilon : I \to I\bar{I}, \qquad (SsS)\partial^\varepsilon \bar{I} = \bar{I}\partial^\varepsilon : \bar{I} \to \bar{I}I \ .$$

We do not have -here- examples not belonging to the simpler situations above; nevertheless this generalised notion consents a unified treatment of their consequences.

2.6. IP-categories and symmetries.

Let $\mathbf{A}$ be an IP-category (1.8), with structural adjunction $I \dashv P$, $u : 1 \to PI$, $v : IP \to 1$. We just treat the case of strict symmetries, existing in various examples we consider here: $\mathbf{Top}$, $\mathbf{Top}^\top$, chain or cochain complexes (5.1); the argument extends to the generalised ones.

A (strict) reversion $\rho : I \to I$ for the cylinder (2.4) produces a reversion $r : P \to P$ for the path-endofunctor, and conversely

(1) $$r = (P \xrightarrow{uP} PIP \xrightarrow{P\rho P} PIP \xrightarrow{Pv} P), \quad \rho = (I \xrightarrow{Iu} IPI \xrightarrow{IrI} IPI \xrightarrow{vI} I).$$

Similarly, there is a biunivocal correspondence between interchanges $\sigma : I^2 \to I^2$, $s : P^2 \to P^2$ for the cylinder (2.5) and for the path-endofunctor, through the composed adjunction $I^2 \to P^2$

(2) $$s = (P^2 \to P^2I^2P^2 \xrightarrow{P^2\sigma P^2} P^2I^2P^2 \to P^2),$$

(2′) $$\sigma = (I^2 \to I^2P^2I^2 \xrightarrow{I^2sI^2} I^2P^2I^2 \to I^2)\ .$$

3. Homotopies and Cone.

We briefly treat the homotopy structure defined by a cubical monad, the associated lower and upper cone monads and some consequences of the symmetries on the homotopical properties of the cylinder and cone functors.

3.1. I0-categories and homotopies.

Let **A** be an I0-category, with semidiad

(1) $$\partial^\varepsilon : 1 \rightleftarrows I : e, \qquad e\partial^\varepsilon = 1\ .$$

A *homotopy* $\alpha : f \to g : A \to B$, between parallel maps f, g is given by a map α with

(2) $$\alpha : IA \to B, \qquad \alpha\partial^- = f, \qquad \alpha\partial^+ = g\ .$$

Every map f has a vertical identity $1_f : f \to f$ (*represented* by $f.eA = eB.If : IA \to B$) and there is a "reduced" horizontal composition $\circ$ of cells and maps (also written by juxtaposition)

(3) $$k \circ \alpha \circ h = k.\alpha.Ih : kfh \to kgh : A' \to B',$$

for $h : A' \to A$ and $k : B \to B'$.

The *homotopy relation* (for $f, g : A \to B$)

(4) $f \simeq g$ if there exists a homotopy $f \to g$ *or* $f \to g$.

is reflexive and symmetrical, generally non transitive, but consistent with composition *of maps*

(5) $f \simeq g$ implies $kfh \simeq kgh$ (where $h : A' \to A,\ k : B \to B'$) .

A *homotopy equivalence* $u : A \to B$ is a map admitting some $v : B \to A$ such that $vu \simeq 1$ and $uv \simeq 1$. Since this notion need not be stable under composition, a finite composition $u_n \dots u_2.u_1$ of homotopy equivalences will be called a *composed homotopy equivalence.*

Under op-duality, i.e. exchanging ∂^- and ∂^+ in the semidiad, the vertical domain and codomain of homotopies are inverted, while the other notions above are invariant.

If the semidiad has a generalised interchange $(S, s : SISI \to ISIS)$, the secondary semidiad $\bar{I}$ considered in 2.5.6 produces a notion of $\bar{I}$-homotopy

(6) $\alpha : SIS(A) \to B, \quad \alpha.S\partial^- S = f, \quad \alpha.S\partial^+ S = g$.

3.2. h-categories.

The above definitions of homotopies, vertical identities and reduced horizontal composition make **A** into a "category with a generalised homotopy system" [Km2; def. 2.1], or *h-category* [G1, G2].

This structure can be formally described as a category enriched over reflexive graphs (5.5). Concretely, it consists of a category, of cells between its maps (called homotopies) and of a reduced horizontal composition of cells and maps verifying obvious axioms of identities and associativity:

(1) $$1_B \circ \alpha \circ 1_A = \alpha, \quad k \circ 1_f \circ h = 1_{kfh},$$

$$k' \circ (k \circ \alpha \circ h) \circ h' = (k'k) \circ \alpha \circ (hh') .$$

On the other hand, given an h-category **A**, define the cylinder object IA of A to be the *corepresenter of homotopies* from A. In other words, it comes equipped with a cell ι

$$
(2) \qquad
\begin{array}{ccc}
A & \xrightarrow{\;1\;} & A \\
{\scriptstyle 1}\downarrow & \swarrow{\scriptstyle \iota} & \downarrow{\scriptstyle \partial^-} \\
A & \xrightarrow[\partial^+]{} & IA
\end{array}
\qquad \iota : \partial^- \to \partial^+ : A \to IA
$$

verifying the universal property: For every every cell $\varphi : f = g : A \to X$ there is precisely one map $F : IA \to X$ such that $F \circ \iota = \varphi$ (and therefore $F\partial^- = f$, $F\partial^+ = g$). This also means that the square (2) is an h-pushout, or standard homotopy pushout, or co-comma square for the h-structure.

Note that ι is epi on maps: $F \circ \iota = G \circ \iota$ implies $F = G$. The map $e : IA \to A$ is obtained by representing the vertical identity of 1_A.

It is now easy to verify that *an I0-category is the same as an h-category where every object has a cylinder* or, equivalently, where homotopies can be corepresented by maps.

3.3. Proposition. Homotopies and Symmetries.

Let A be an I0-category. a) In the presence of a generalised interchange $(S, s : SISI \to ISIS)$, as defined in 2.5, the cylinder functor I is invariant *up to* $\bar{I}$*-homotopy* (3.1.6); hence up to homotopy in the strict case ($S = 1$). b) In the presence of a generalised reversion $(R, r : RI \to IR)$, as defined in 2.4, every homotopy $\alpha : f_0 \to f_1$ produces a *reversed* homotopy $\alpha^r : Rf_1 \to Rf_0$; hence, from f_1 to f_0 in the strict case ($R = 1$).

Proof. a) Take an $\bar{I}$-homotopy $\alpha : \bar{I}(A) \to B$, with $\alpha.\bar{\partial}^\varepsilon = f_\varepsilon$. Then the following diagram commutes

$$
(1) \qquad
\begin{array}{ccccc}
\bar{I}IA & \xrightarrow{\;s\;} & I\bar{I}A & \xrightarrow{\;I\alpha\;} & IB \\
{\scriptstyle \bar{\partial}^\varepsilon I}\uparrow & & \uparrow{\scriptstyle I\bar{\partial}^\varepsilon} & & \uparrow{\scriptstyle If_\varepsilon} \\
IA & = & IA & = & IA
\end{array}
$$

because of the axiom 2.5.7. The diagram shows precisely that the composed map $I\alpha.s$ in the upper row is an $\bar{I}$-homotopy from If_0 to If_1.

b) The reversed homotopy $\alpha^r : Rf_1 \to Rf_0$ is represented by the upper row of the diagram

$$\begin{array}{ccccc} IRA & \xrightarrow{RrR} & RIA & \xrightarrow{R\alpha} & RB \\ \uparrow{\scriptstyle \partial^\eta R} & & \uparrow{\scriptstyle R\partial^\varepsilon} & & \uparrow{\scriptstyle Rf_\varepsilon} \\ RA & = & RA & = & RA \end{array} \qquad (\eta \neq \varepsilon)\ . \tag{2}$$

3.4. I-homotopical categories.

It is well known that the basic topics of *right* homotopical algebra (concerning homotopy *co*kernels) can be established in categories with cylinder functor and pushouts. It is useful to distinguish between two levels, corresponding to *semidiads* and *diads*, and respectively supplying *first* or *second* order homotopy properties.

An *I-semihomotopical* category will thus be an I0-category $(\mathbf{A}, I, \partial^\varepsilon, e)$ with terminal object $\top$ and pushouts. More particularly, an *I-homotopical* category is an I-category $(\mathbf{A}, I, \partial^\varepsilon, e, g^\varepsilon)$ with terminal object $\top$ and I-preserved pushouts. The assumption of a (generalised) interchange and a (generalised) reversion is useful (see 3.8-9), but the latter can be avoided by working with both the lower and upper cone associated to I.

Let $\mathbf{A}$ be always I-semihomotopical. The object $\top$ is automatically 2-*terminal* – for every object A there is exactly one cell $A \to \top$, the identity of $t = tA : A \to \top$. A map $a : \top \to X$ is called a *point* of X, while a map $f = a.tA : A \to X$ factoring through $\top$ is called a constant or $\top$-*null* map.

Every morphism $f : A \to B$ has a *cokernel* Cok f, the pushout of the map $tA : A \to \top$ along f, equipped with two structural maps c, c'' (verifying $cf = c''.tA$)

$$c = \text{cok}\ f : B \to \text{Cok}\ f, \qquad c'' : \top \to \text{Cok}\ f\ .$$

$\mathbf{A}$ is *pointed* if the terminal object $\top$ is also initial; then it is also written 0 and called a zero object; note that $\mathbf{A}$, having pushouts and initial object, is finitely cocomplete. More particularly, $\mathbf{A}$ is *I-pointed* if $\top$ is initial and preserved by I (or, equivalently, 2-initial), as it happens in $\mathbf{Top}^\top$.

3.5. The cone endofunctor.

The I0-structure $(I, \partial^\varepsilon, e)$ of the I-semihomotopical category $\mathbf{A}$ produces a semimonad $(C, \partial : 1 \to C)$, where $C = C^- = \Delta$ will be called the associated *lower cone* functor

(1) $$CA = \mathrm{Cok}\ (\partial^+ : A \to IA), \quad \delta : IA \to CA,\ d : \top \to CA$$

(2) $$\partial A = \delta\partial^- : A \to CA\ .$$

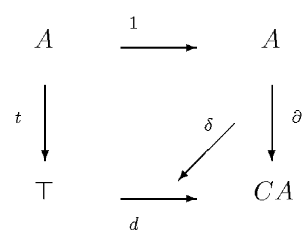
In particular, as $t : \top \to \top$ is iso, also $\delta\top : I\top \to C\top$ is so.

The name of *lower* cone comes from the fact that the structure (i.e. the face ∂, but also the connection g in the I-homotopical case considered below) is induced by the lower structure of I.

Actually, we get a *based* semimonad (C, ∂, d), where $d : \top \to CA$ (the *vertex* of the cone) is a natural transformation from the constant functor $A \mapsto \top$ to C. If $\mathbf{A}$ is pointed, d is the zero map and can be ignored.

From the homotopical viewpoint CA is the *corepresenter of (lower) nullhomotopies*. In other words, equipped with the maps ∂, d and the cell δ

(3)



$$\partial : A \to CA, \quad d : \top \to CA\ ,$$

$$\delta : \partial \to d.tA : A \to CA$$

it verifies the universal property: For every point $x : \top \to X$ and every cell $\varphi : f \to xt : A \to X$ there is precisely one map $g : CA \to X$ such that $g\partial = f$, $gd = x$ and $g\delta = \varphi$.

Op-dually, we also have an *upper cone* semimonad $(C^+, \partial : 1 \to C^+)$, where

(4) $$C^+(A) = \nabla A = Cok(\partial^- : A \to IA)\ .$$

3.6. Lemma: The double cone.

a) Let **A** be I-semihomotopical, with I-preserved pushouts. For every object A the double (lower) cone $C^2A = CCA$ can be identified to the colimit (c, c_1, c_2) of the upper row of the diagram (1), where

$$
\text{(1)} \qquad
\begin{array}{ccccccccc}
I\top & \xleftarrow{It} & IA & \xrightarrow{I\partial^+} & I^2A & \xleftarrow{\partial^+ I} & IA & \xrightarrow{e} & A \\
 & & & \searrow^{c_1} & \downarrow{\scriptstyle c} & \swarrow_{c_2} & & & \\
 & & & & C^2A & & & &
\end{array}
$$

$$
\text{(2)} \qquad c = \delta C.I\delta = C\delta.\delta I, \quad c_1 = \delta C.Id = Cd.\delta\ ,
$$

$$
c_2 = dC.tA = c_1.\partial^+.tA.
$$

b) If **A** is I-pointed (has an I-preserved zero object), the maps c_1 and c_2 are just zero-maps and C^2A is the *joint cokernel* of the upper faces of the double-cylinder $I\partial^+, \partial^+I : IA \to I^2A$, through the epimorphism $c : I^2A \to C^2A$.

Note that the last formula in (2) shows that c_2 is determined by c_1, so that the pair (c, c_1) is jointly epi.

Proof. It suffices to prove a). The colimit (1) is produced by the three pushouts of diagram (3)

$$
\text{(3)} \qquad
\begin{array}{ccccc}
 & & IA & \xrightarrow{e} & A \\
 & & \downarrow{\scriptstyle \partial^+ I} & & \downarrow \\
IA & \xrightarrow{I\partial^+} & I^2A & \longrightarrow & \bullet \\
{\scriptstyle It}\downarrow & & \downarrow{\scriptstyle I\delta} & & \downarrow \\
I\top & \xrightarrow{Id} & ICA & \longrightarrow & X
\end{array}
$$

(4)
$$\begin{array}{ccccc}
A & \xrightarrow{\partial^+} & IA & \xrightarrow{e} & A \\
\downarrow & & \downarrow{\scriptstyle \delta} & & \downarrow \\
\top & \xrightarrow{d} & CA & \longrightarrow & \top \\
{\scriptstyle \partial^+}\downarrow & & {\scriptstyle \partial^+ C}\downarrow & & \downarrow{\scriptstyle dC} \\
I\top & \xrightarrow{Id} & ICA & \xrightarrow{\delta C} & C^2A
\end{array}$$

In order to calculate X, consider now the diagram (4); the left upper square is a pushout, by definition of CA; then, also the right upper one is so (their pasting is trivially a pushout); finally, also the right lower square is a pushout, by definition of $C(CA)$. The right rectangle is thus a pushout; as the composed middle column of (4) coincides with the composed middle column of (3), we have proved that $X = C^2A$; also the formulas (2) can be read over the diagrams (3)-(4); use the dotted part of (4) for the last equality in (2).

3.7. Theorem: from diads to monads.

a) Let **A** be I-homotopical. Then the cubical (associative) monad $(I, \partial^\varepsilon, e, g^\varepsilon)$ produces a *based* monad (C, ∂, g, d), where the connection $g : C^2 \to C$ is induced by g^- (as specified below, in (3)).

The transformation d is absorbant for g, in the sense that the following diagram commutes

(1)
$$\begin{array}{ccccc}
\top & \xrightarrow{dC} & C^2A & \xleftarrow{Cd} & C\top \\
& {\scriptstyle d}\searrow & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle t} \\
& & CA & \xleftarrow{d} & \top
\end{array}$$

b) If **A** is I-pointed and has an interchange $s : I^2 \to I^2$, there is an induced involutive *interchange* transformation $s : C^2 \to C^2$, exchanging the faces $C\partial$ and ∂C.

The relevance of the pointed-case assumption appears clearly from the proof. Concretely, it is easy to see that in **Top** the interchange $s : I^2 \to I^2$

has no induced transformation over the double-cone C^2.

Proof. a) Use the above description of the double cone C^2A as a colimit (3.6). The operation g^- gives a commutative diagram (2), because of the "absorbant element" axiom

$$\text{(2)}\qquad \begin{array}{ccccccccc} I\top & \xleftarrow{It} & IA & \xrightarrow{I\partial^+} & I^2A & \xleftarrow{\partial^+I} & IA & \xrightarrow{e} & A \\ \downarrow t & & \downarrow e & & \downarrow g^- & & \downarrow e & & \downarrow t \\ \top & \longleftarrow & A & \xrightarrow{\partial^+} & IA & \xleftarrow{\partial^+} & A & \rightarrow & \top \end{array}$$

$$\text{(3)}\qquad \begin{array}{ccccc} I\top & \xrightarrow{c_1} & C^2A & \xleftarrow{c} & I^2A \\ \downarrow t & & \downarrow g & & \downarrow g^- \\ \top & \xrightarrow{d} & CA & \xleftarrow{\delta} & IA \end{array}$$

and the colimit is our operation $g : C^2A \to CA$, determined by the commutative diagram (3).

It is now easy to see that (C, ∂, g, d) is indeed a based monad, deducing the properties of (∂, g) from the analogous ones of (∂^-, g^-). In particular, the properties of d in the diagram (1) follow from the following calculations (recall that δT is iso (3.5), whence cancellable)

$$\text{(4)}\qquad g.dC = g.c_1.\partial^+ = dt.\partial^+ = d \ ,$$

$$\text{(5)}\qquad g.Cd.\delta\top = g.c_1 = d.t(I\top) = d.t(C\top).\delta\top \ .$$

b) In the pointed case, the interchange $s : I^2 \to I^2$ of the diad

$$\text{(6)}\qquad \begin{array}{ccc} I^2A & \xrightarrow{c} & C^2A \\ \downarrow s & & \downarrow s \\ I^2A & \xrightarrow{c} & C^2A \end{array}$$

induces, by 3.6 b), an involutive transformation $s : C^2 \to C^2$ determined by the diagram (6). Using the formula $c = \delta C.I\delta$ of 3.6.2

$$\text{(7)}\qquad (s.C\partial).\delta = s.\delta C.I\partial = s.\delta C.I\delta.I\partial^- = s.c.I\partial^- = c.s.I\partial^- =$$
$$c.\partial^- I = \delta C.I\delta.\partial^- I = \delta C.\partial^- C.\delta = \partial C.\delta \ .$$

3.8. Homotopical invariance of cone.

Let **A** be I-homotopical, with a generalised interchange $(S, s : SISI \to ISIS)$. We prove now various properties of the cone endofunctor, which depend on s, and culminate in its invariance up to $\bar I$-homotopy (and up to homotopy for $S = 1$).

a) First we show that there is a canonical transformation $s' : \bar IC \to C\bar I$ induced by $s : \bar II \to I\bar I$. The commutative diagram (1), together with the fact that $\bar I$ preserves pushouts (applied to its upper row) produces in fact a colimit morphism $s'A : \bar ICA \to C\bar IA$, determined by (2)

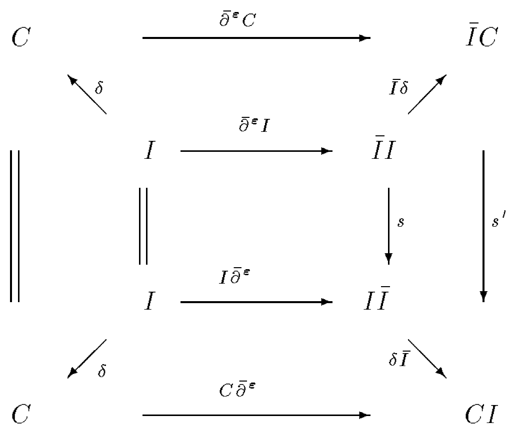
$$
(1)\quad \begin{array}{ccccc}
\bar I\top & \xleftarrow{\bar It} & \bar IA & \xrightarrow{\bar I\partial^+} & \bar IIA \\
\downarrow{\scriptstyle t} & & \| & & \downarrow{\scriptstyle s} \\
\top & \xleftarrow{s} & \bar IA & \xrightarrow{\partial^+\bar I} & I\bar IA
\end{array}
\qquad
(2)\quad \begin{array}{ccccc}
\bar I\top & \xrightarrow{\bar Id} & \bar ICA & \xleftarrow{\bar I\delta} & \bar IIA \\
\downarrow{\scriptstyle t} & & \downarrow{\scriptstyle s'} & & \downarrow{\scriptstyle s} \\
\top & \xrightarrow{d\bar I} & C\bar IA & \xleftarrow{\delta\bar I} & I\bar IA
\end{array}
$$

b) This morphism s' turns the maps $\bar\partial^\varepsilon C$ into the maps $C\bar\partial^\varepsilon$; in other words, the diagram (3) below is commutative (for $\varepsilon = 0, 1$)

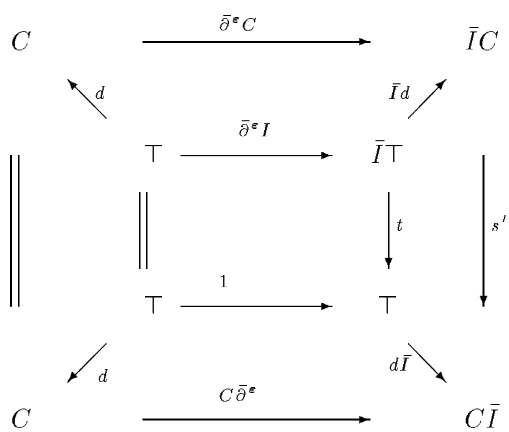
$$
(3)\quad \begin{array}{ccc}
CA & \xrightarrow{\bar\partial^\varepsilon C} & \bar ICA \\
\| & & \downarrow{\scriptstyle s'} \\
CA & \xrightarrow{C\bar\partial^\varepsilon} & C\bar IA
\end{array}
$$

The thesis, $s'.\bar\partial^\varepsilon C = C\bar\partial^\varepsilon$, follows from the cancellation property of the colimit cocone $(\delta,\ d)$ ending in CA. First, the diagram (4) below, where each elementary quadrilateral is already known to be commutative, proves that $s'.\bar{\partial}^\varepsilon C.\delta = C\bar{\partial}^\varepsilon.\delta$; similarly (5) shows that $s'.\bar{\partial}^\varepsilon C.d = C\bar{\partial}^\varepsilon.d$

(4)



(5)



c) Finally, the cone-functor C is invariant up to $\bar{I}$-homotopy. Indeed, take $\alpha : \bar{I}A \to B$ with $\alpha\bar{\partial}^\varepsilon = f_\varepsilon$. Then the following diagram, commutative because of b), shows that the upper row $\Phi = C\alpha.s'A : \bar{I}CA \to CB$

represents an $\bar{I}$-homotopy $Cf_0 \to Cf_1$

$$
(6) \qquad \begin{array}{ccccc}
\bar{I}CA & \xrightarrow{s'} & C\bar{I}A & \xrightarrow{C\alpha} & CB \\
{\scriptstyle \bar{\partial}^\varepsilon C}\uparrow & & {\scriptstyle C\bar{\partial}^\varepsilon}\uparrow & \nearrow{\scriptstyle Cf_\varepsilon} & \\
CA & = & CA & &
\end{array}
$$

d) For **Top**, the canonical transformation $s' : IC \to CI$ defined in a) is not iso. But, if **A** is pointed (3.4), all the vertical arrows of diagram (1) are isomorphisms, and so is s'.

e) Similarly one can show that the suspension endofunctor $\Sigma A = \mathrm{Cok}(\partial : A \to CA)$ is invariant up to $\bar{I}$-homotopy.

3.9. The reversion of cones.

Given an I-homotopical category with generalised reversion $(R, r : RI \to IR)$, there is a canonical *cone-reversion* isomorphism $r^C : RC^-A \to C^+RA$ consistent with the embedding of bases

$$
(1) \qquad (RA \xrightarrow{\partial^+ R} RC^-A) = (RA \xrightarrow{R\partial^-} RC^-A \xrightarrow{r^C} C^+RA)
$$

Indeed, the automorphism R preserves pushouts and $\top$; thus, the commutative diagram (2) here below has, for colimit, the commutative diagram (3), where r^C is iso (since r is so)

$$
(2) \qquad \begin{array}{ccccc}
R\top & \xleftarrow{t} & RA & \xrightarrow{R\partial^+} & RIA \\
\| & & \| & & \downarrow{\scriptstyle r} \\
\top & \longleftarrow & RA & \xrightarrow{\partial^- R} & IRA
\end{array}
$$

$$
(3) \qquad \begin{array}{ccccc}
R\top & \xrightarrow{Rd} & RC^-A & \xleftarrow{R\delta} & RIA \\
\| & & \downarrow{\scriptstyle r^C} & & \downarrow{\scriptstyle r} \\
\top & \xrightarrow{dR} & C^+RA & \xleftarrow{\delta R} & IRA
\end{array}
$$

This isomorphism can be extended to the lower and upper mapping cones of a map f, which is of interest in order to simplify the Puppe sequence of f itself (to be studied in a subsequent paper).

4. P-homotopical and IP-homotopical Categories.

IP-semihomotopical categories are characterised as h-categories with cylinder and path-objects. In the pointed case, the cone-arc adjunction is dealt with.

4.1. P-homotopical categories.

Dually, a *P-semihomotopical* (resp. *P-homotopical*) category **A** is equipped with a semicodiad (resp. a codiad $(P, \partial^\varepsilon, e, g^\varepsilon)$), has an initial object $\bot$ and pullbacks (resp. preserved by P). It is pointed (P-pointed) if $\bot$ is also terminal (and preserved by P).

Dga is a non-pointed P-homotopical category; $\bot$ is the ring K of scalars, while $\top$ is the null dg-algebra (an absolute terminal object). Cub **C** is P-homotopical, provided that **C** is finitely complete, with initial object.

In a P-semihomotopical category, the associated semicomonad (E, ∂) consists of the (lower) *co-cone*, or arc-object $EA = E^-A$

(1) $$EA = \text{Ker }(\partial^+ : PA \to A), \quad \delta : EA \to PA, \quad d : EA \to \bot \ ,$$

(2) $$\partial = \partial^- \delta : EA \to A \ ,$$

which is the pullback of $iA : \bot \to A$ along $\partial^+ : PA \to A$. Through the nullhomotopy

(3) $$\delta : \partial \to iA.d : EA \to A \ ,$$

EA is the universal nullhomotopy representer of A (representing all homotopies into A whose vertical codomain is $\bot$-*null*, i.e. factors through $\bot$, via $iA : \bot \to A$).

4.2. IP0-categories and h-categories.

Analogously to the characterization 3.2 for I0-categories, an IP0-category is the same as an h-category where every object has a cylinder-object IA and a path-object PA, or also where homotopies can be both corepresented and represented.

We only write down the main part of the proof: In an h-category with cylinder and path-objects, the functors I and P are canonically adjoint.

The structural homotopies $\iota^A : A \Rightarrow IA$ and $\iota_A : PA \Rightarrow A$ are distinguished by upper or lower indices. For every object A, the canonical homotopy $\iota^A : A \Rightarrow IA$ takes values in IA, hence factors uniquely through the universal cell $\iota_{IA} : PIA \Rightarrow IA$, yielding the unit $u : 1 \to PI$ of the adjunction; the counit $v = (v_A : IPA \to A)_A$ is obtained similarly

$$u_A : A \to PIA, \ \ \iota_{IA} \circ u_A = \iota^A, \tag{1}$$

$$v_A : IPA \to A, \ \ v_A \circ \iota^{PA} = \iota_A, \tag{2}$$

$$(3) \qquad \begin{array}{ccc} A & \overset{\iota^A}{\Longrightarrow} & IA \\ & \searrow^{u_A} & \Uparrow \iota_{IA} \\ & & PIA \end{array} \qquad\qquad (4) \qquad \begin{array}{ccc} PA & \overset{\iota_A}{\Longrightarrow} & A \\ \iota^{PA} \Downarrow & \nearrow_{v_A} & \\ IPA & & \end{array}$$

The naturality of the transformation u over a morphism $f : A \to B$ follows from the diagram (5)

$$(5) \qquad \begin{array}{ccccc} A & \xrightarrow{u_A} & PIA & \overset{\iota_{IA}}{\Longrightarrow} & IA \\ \downarrow f & & \downarrow PIf & & \downarrow If \\ B & \xrightarrow{u_B} & PIB & \overset{\iota_{IB}}{\Longrightarrow} & IB \end{array}$$

$$(6) \qquad \begin{array}{ccccc} IA & \xrightarrow{Iu_A} & IPIA & \xrightarrow{v_{IA}} & IA \\ \Uparrow \iota^A & & \Uparrow \iota^{PIA} & & \| \\ A & \xrightarrow{u_A} & PIA & \overset{\iota_{IA}}{\Longrightarrow} & IA \end{array}$$

where the outer rectangle commutes, by definition of u and of If, the right-hand square commutes by definition of $P(If)$ and the homotopy ι_{IB} is epi on maps (3.2). Analogously, v is natural.

Last, we check one of the coherence conditions, $v_{IA}.Iu_A = 1_{IA}$. The diagram (6) above is commutative by definition of Iu_A (left square) and of v_{IA} (right square), whence

$$(v_{IA}.Iu_A) \circ \iota^A = \iota_{IA} \circ u_A = \iota^A = 1_{IA} \circ \iota^A \ , \tag{7}$$

and the thesis follows now from the cancellation property of ι^A.

4.3. IP-homotopical categories.

An *IP-semihomotopical* (resp. *IP-homotopical*) category is an IP0- (resp IP-) category (1.8) which is finitely complete and cocomplete. Note that I, as a left-adjoint, automatically preserves $\bot$ and pushouts, while P preserves $\top$ and pullbacks.

The IP-semihomotopical category **A** is *pointed* if $\bot = \top = 0$ (automatically preserved by I and P).

Top is IP-homotopical, non pointed. Its initial object $\bot = \emptyset$ is absolute (each map with values in it is iso). It follows that the left homotopical structure is trivial: $EA = \emptyset$ for every A. Instead $\mathbf{Top}^{\top}$ is pointed IP-homotopical; EA consists of the space of *arcs* of A (paths whose end is the base-point 0_A) and the functor E is right adjoint to the cone functor (see 4.4 below).

Note that the zero object is necessary to get this adjunction $C \dashv E$. More generally, if **A** is I-semihomotopical, with initial object $\bot$ *preserved* by its cone functor C (as it necessarily happens if C has a right adjoint), then **A** is pointed. In fact, the structural map $d : \top \to C\bot = \bot$ yields an arrow $\top \to \bot$ which is necessarily reciprocal to the unique map $\bot \to \top$.

4.4. The cone-arc adjunction.

Let **A** be IP-semihomotopical and *pointed*: Then, the $\top$-null maps and the $\bot$-null maps coincide, so that the functors C and E respectively corepresent and represent the same nullhomotopies.

It is thus easy to see that there is a canonical adjunction $C \dashv E$ between the (lower) cone and arc functors, which can be obtained as the adjunction $I \to P$ in 4.2 (again the structural homotopies $\delta^A : A \to CA$ and $\delta_A : EA \to A$ are distinguished by upper or lower indices)

(1) $$u_A : A \to ECA, \quad \delta_{CA} \circ u_A = \delta^A \ ,$$

(2) $$\begin{array}{ccc} A & \overset{\delta^A}{\Longrightarrow} & CA \\ & {\scriptstyle u_A}\searrow & \Uparrow {\scriptstyle \delta_{CA}} \\ & & ECA \end{array}$$

(3) $$\begin{array}{ccc} EA & \overset{\delta_A}{\Longrightarrow} & A \\ {\scriptstyle \delta^{EA}} \Downarrow & \nearrow {\scriptstyle v_A} & \\ CEA & & \end{array}$$

5. Some other examples.

Finally, we briefly sketch the IP-structure of chain complexes, small categories and small reflexive graphs, highlighting the relations with the monoidal closed structures.

5.1. Chain complexes.

Let **D** be an additive category.

The canonical projections and injections of biproducts in **D** are written pr_i and in_i. A map

(1) $$f : \oplus A_j \to \oplus B_i \qquad (i = 1, \ldots m;\ j = 1, \ldots n)$$

of components $f_{ij} = pr_i f in_j$ will be written in the following "concrete" form

(2) $$f(x_1, \ldots x_n) = (\Sigma f_{1j} x_j, \ldots \Sigma f_{mj} x_j),$$

which allows one to calculate "on variables" as in concrete categories of modules, but can be formally justified by setting $x_j = pr_j : \oplus A_j \to A_j$.

Let $C_*\mathbf{D}$ denote the category of unbounded chain complexes $A = ((A_n), (\partial_n))$, indexed over $\mathbb{Z}$, with the usual morphisms (of degree zero). $C_*\mathbf{D}$ has a well-known h-structure (3.2), which makes it into an IP-category with strict reversion and interchange. The calculus is a simplified version of the case of dg-algebras (2.2-3).

A homotopy $\alpha : f \to g : A \to B$ is defined by a sequence of **D**-maps (α_n) so that

(3) $$\alpha = (f, g, (\alpha_n : A_n \to B_{n+1})), \quad -f_n + g_n = \alpha_{n-1}\partial_n + \partial_{n+1}\alpha_n,$$

with obvious horizontal composition and vertical identities

(4) $$k\alpha h = (kfh,\ kgh,\ (k\alpha_n h)), \qquad 1_f = (f, f, (0))\ .$$

We often write $\alpha(a)$ instead of $\alpha_n(a)$, for a variable in A_n. Homotopies can be represented and corepresented, so that we have the cylinder and path endofunctors

(5) $$(IA)_n = A_n \oplus A_{n-1} \oplus A_n,$$
$$\partial(a, h, b) = (\partial a - h, -\partial h, \partial b + h),$$

(5′) $\delta^-(a) = (a, 0, 0)$, $\delta^+(a) = (0, 0, a)$,
$\iota^A(a) = (0, a, 0) \quad (\iota^A : \delta^- \to \delta^+ : A \to IA)$,

(6) $(PA)_n = A_n \oplus A_{n+1} \oplus A_n$,
$\partial(a, h, b) = (\partial a, \quad -a + b - \partial h, \partial b)$,

(6′) $\partial^-(a, h, b) = a, \quad \partial^+(a, h, b) = b$,
$\iota_A(a, h, b) = h \quad (\iota_A : \partial^- \to \partial^+ : PA \to A)$,

linked by a canonical adjunction $I \dashv P$ (4.2)

(7) $u : 1 \to PI$,
$u_n : A_n \to (A_n \oplus A_{n-1} \oplus A_n) \oplus (A_{n+1} \oplus A_n \oplus A_{n+1}) \oplus$
$(A_n \oplus A_{n-1} \oplus A_n)$,

$u(a) = (a, 0, 0; 0, a, 0; 0, 0, a)$,

(8) $v : IP \to 1$,
$v_n : (A_n \oplus A_{n+1} \oplus A_n) \oplus (A_{n-1} \oplus A_n \oplus A_{n-1}) \oplus$
$(A_n \oplus A_{n+1} \oplus A_n) \to A_n, \; v(a, h, b; x, e, y; c, k, d) = a + e + d.$

The IP0-structure is given by the faces $\delta^-, \delta^+, \partial^-, \partial^+$ of (5′), (6′), together with the degeneracies

(9) $\varepsilon : I \to I, \; \varepsilon(a, h, b) = a + b, \qquad e : I \to P, \quad e(a) = (a, 0, a).$

As in the case of dg-algebras (2.3.1), the second order path-object P^2A has components

(10) $$(P^2A)_n =$$

$$(A_n \oplus A_{n+1} \oplus A_n) \oplus (A_{n+1} \oplus A_{n+2} \oplus A_{n+1}) \oplus (A_n \oplus A_{n+1} \oplus A_n).$$

The connections g^-, g^+, the strict reversion r and the strict interchange s of P are given by

(11) $$g^-(a, h, b) = (a, h, b; h, 0, 0; b, 0, b),$$

$$g^+(a, h, b) = (a, 0, a; 0, 0, h; a, h, b) \;,$$

(12) $$r : PA \to PA, \quad r(a, h, b) = (b, -h, a)$$

(13) $$s : P^2A \to P^2A,$$

$$s(a, h, b; u, z, v; c, k, d) = (a, u, c; h, -z, k; b, v, d)\ .$$

The connections γ^-, γ^+ and the symmetries ρ, σ of the cylinder I can now be derived from the adjunction (through the formulas 1.8.6 and 2.6.1-2).

$C_*\mathbf{D}$ always has homotopy pullbacks and homotopy pushouts, which are easily constructed from the biproducts of $\mathbf{D}$ (extending the construction of PA and IA, respectively). On the other hand, clearly, $C_*\mathbf{D}$ has finite limits (resp. colimits) iff $\mathbf{D}$ has them; in this case $C_*\mathbf{D}$ is P-homotopical (resp. I-homotopical), by 4.3.

5.2. Positive chain complexes.

The subcategory $C.\mathbf{D}$ of *positive* chain complexes (with $A_n = 0$ for $n < 0$) has again homotopies defined as above. They are produced by a diad I' which is the restriction of the cylinder functor I for unbounded complexes considered above

(1) $$I' : C.\mathbf{D} \to C.\mathbf{D}, \quad (I'A)_n = A_n \oplus A_{n-1} \oplus A_n\ .$$

Assume now that $\mathbf{D}$ has kernels (and therefore all finite limits). Then our h-structure is defined by an IP-structure, with path functor $P' : C.\mathbf{D} \to C.\mathbf{D}$

(2) $$(P'A)_n = A_n \oplus A_{n+1} \oplus A_n\ (n > 0)\ ,$$

$$(P'A)_0 = \mathrm{Ker}\ (\partial_0^{PA} : A_0 \oplus A_1 \oplus A_0 \to A_0)\ .$$

Indeed, in this hypothesis, the embedding $U : C.\mathbf{D} \to C_*\mathbf{D}$ has a reflector F and a coreflector G

(3) $$F : C_*\mathbf{D} \to C.\mathbf{D}, \quad (FA)_n = A_n\ or\ 0, \quad \text{for } n \geq 0\ or\ n < 0\ ,$$

(4) $$G : C_*\mathbf{D} \to C.\mathbf{D},$$

$$(GA)_n = A_n\ or\ \mathrm{Ker}\partial_0\ or\ 0, \text{for } n > 0\ or\ n = 0\ or\ n < 0\ ,$$

so that the adjunctions $I \dashv P$ (in $C_*\mathbf{D}$) and $F \dashv U \dashv G$

(5) $$C.\mathbf{D}\ \underset{G}{\overset{U}{\rightleftarrows}}\ C_*\mathbf{D}\ \underset{P}{\overset{I}{\rightleftarrows}}\ C_*\mathbf{D}\ \underset{U}{\overset{F}{\rightleftarrows}}\ C_*\mathbf{D}$$

produce a composed adjunction $I' \dashv P'$ (in $C.\mathbf{D}$), with $I' = FIU, P' = GPU$.

5.3. Complexes of modules.

Let now K be a commutative unital ring and $\mathbf{D} = K$-**Mod** the category of K-modules, with the usual tensor product $\otimes = \otimes_K$ and internal hom-functor $\mathrm{Hom} = \mathrm{Hom}_K$. In this case the IP-structures of $C_*\mathbf{D}$ and $C.\mathbf{D}$ are *monoidal closed* (1.9).

Indeed, the category $C_*\mathbf{D}$ of unbounded chain complexes (of K-modules) has a classical symmetrical monoidal closed structure [EK, p. 558]

$$(1) \qquad (A \otimes B)_n = \oplus_p (A_p \otimes B_{n-p}),$$

$$\partial(a \otimes b) = (\partial a) \otimes b + (-1)^{|a|} a \otimes (\partial b) \ ,$$

$$(2) \qquad (\mathrm{Hom}(A, B))_n = \Pi_p \ \mathrm{Hom}(A_p, B_{n+p}),$$

$$(\partial f)x = \partial(fx) - (-1)^{|f|} f(\partial x) \ ,$$

whose identity is the complex K (concentrated in degree zero).

We obtain an "interval" I by setting $I = I(K)$; it is a complex concentrated in degrees 0 and 1

$$(3) \qquad I_0 = K \oplus K, \quad I_1 = K, \quad \partial_1(\lambda) = (-\lambda, \lambda) \ ,$$

and it is easy to verify that the cylinder and path functor of $C_*\mathbf{D}$ (5.1) are given by

$$(4) \qquad I(A) \cong I \otimes A, \quad P(A) \cong \mathrm{Hom}(I, A) \ .$$

Further the object $I = I(K)$ has a dioid-structure in $(C_*\mathbf{D}, \otimes)$, coming from the diad I and the fact that $I^2(K) = I(I(K)) \cong I \otimes I$. And this dioid determines the whole IP-structure of $C_*\mathbf{D}$, according to the general procedure for monoidal closed categories (1.9).

The same argument applies to positive chain complexes of modules, through the appropriate monoidal closed structure. This can be derived from the reflector F and coreflector G (5.2); thus the new tensor product is still expressed by (1), while the new hom is positive and has a different formula in degree zero

$$(\mathrm{Hom}\ (A, B))_0 = \mathrm{Ker}(\partial_0 : (\Pi_p \mathrm{Hom}(A_p, B_p) \to \Pi_p \mathrm{Hom}(A_p, B_{p-1})).$$

5.4. Categories.

The category **Cat** of small categories is cartesian closed, with $[X, Y] = Y^X$ the category of functors $X \to Y$ and their natural transformations. The identity of the tensor product is the ordinal $\mathbf{1} = \{0\}$. The ordinal category $\mathbf{2} = \{0, 1\}$ is a dioid-object in $(\mathbf{Cat}, \times)$

$$(2) \qquad \mathbf{1} \underset{e}{\overset{\partial^\varepsilon}{\rightleftarrows}} \mathbf{2} \overset{g^\varepsilon}{\leftarrow} \mathbf{2} \times \mathbf{2} \qquad\qquad (3) \qquad \begin{array}{ccc} (0,1) & \to & (1,1) \\ \uparrow & \nearrow & \uparrow \\ (0,0) & \to & (1,0) \end{array}$$

where $\mathbf{2} \times \mathbf{2}$ is the order-category displayed in (3), while the functors ∂^ε and g^ε are determined by their action on the objects, as follows

$$(4) \qquad \partial^\varepsilon(0) = \varepsilon, \quad g^-(i, j) = i \vee j, \quad g^+(i, j) = i \wedge j \quad (\varepsilon, i, j = 0, 1).$$

The dioid $\mathbf{2}$ is commutative, i.e. the "cartesian interchange" $s : \mathbf{2} \times \mathbf{2} \to \mathbf{2} \times \mathbf{2}$ $(s(i, j) = (j, i))$ makes the diagrams 2.5.1-2 commute. And it has an obvious generalised reversion $r : \mathbf{2} \to \mathbf{2}^{op} (r(i) = 1 - i)$, based on the duality involution $(-)^{op} : \mathbf{Cat} \to \mathbf{Cat}$. Our dioid produces thus a cartesian closed IP-structure (1.9) on **Cat**, $IX = X \times 2$ and $PX = X^{\mathbf{2}}$ (the category of maps of X and commutative squares), with strict interchange and generalised reversion. A *homotopy* for this structure, given by a functor $\alpha : IX \to Y$ (or by the corresponding $\alpha' : X \to PY$), amounts to a natural transformation $\alpha : f_0 \to f_1 : X \to Y$ between two functors $f_\varepsilon : X \to Y (f_\varepsilon = \alpha.\partial^\varepsilon X)$; the reversed homotopy is obviously $\alpha^{op} : f_1^{op} \to f_0^{op} : X^{op} \to Y^{op}$.

5.5. Reflexive graphs.

Consider the category $\mathbf{Cub}_1$ of (small) reflexive graphs, or 1-truncated cubical sets, or 1-truncated simplicial sets. To fix the notation, an object is a diagram in Set

$$(1) \qquad X_0 \underset{e}{\overset{\partial^\varepsilon}{\leftleftarrows\!\!\to}} X_1 \qquad \partial^- e = 1 = \partial^+ e \quad (\varepsilon = -, +)$$

consisting of a set of *vertices* X_0, a set of *arrows* (or edges) X_1, the *domain* and *codomain* mappings ∂^-, ∂^+, the *degeneracy* mapping e.

This category $\mathbf{Cub}_1$ has a monoidal closed structure. The *internal hom-functor* $[X, Y]$ is given by the reflexive graph consisting of morphisms of reflexive graphs $X \to Y$ with their transformations. The *tensor product* $X \otimes Y$ is the subgraph of $X \otimes Y$ containing all the objects $(x, y) \in X_0 \times Y_0$ and only those arrows $(u, v) \in X_1 \times Y_1$ such that either u or v is degenerated (an identity).

Consider now the ordinal $\mathbf{2} = \{0,1\}$ as a reflexive graph, and a dioid-object in $(\mathbf{Cub}_1, \otimes)$; the description is the same as above (5.4.2-4), excepting the fact that the reflexive graph $\mathbf{2} \otimes \mathbf{2}$ has *four* non-degenerate arrows (as in 5.4.3, without the diagonal).

One obtains thus an IP-structure on $\mathbf{Cub}_1$, with $IX = X \otimes \mathbf{2}$ and $PX = [\mathbf{2}, X]$; to assign a homotopy $\alpha : IX \to Y$ (or the adjoint one, $\alpha' : X \to PY$) is here equivalent to give a transformation $\alpha : f_0 = f_1 : X \to Y$ between two morphisms $f_\varepsilon : X \to Y$ of reflexive graphs. The symmetries proceed as above.

Cat has also a monoidal closed, non cartesian, structure (called the "funny" structure in Street [St]) where $[X, Y]$ is the category of functors $X \to Y$, with their *transformations* (of graph-morphisms, without requiring the naturality condition) while $X \otimes Y$ is the category generated by the tensor product $|X| \otimes |Y|$ of the underlying reflexive graphs (as above), under the obvious relations coming from the composition in X and Y

– $(g, ey).(f, ey) = (gf, ey)$, if f, g are composable arrows of X and y is an object of Y,

– $(ex, g).(ex, f) = (ex, gf)$, if f, g are composable arrows of Y and x is an object of X.

Again, the ordinal category $\mathbf{2} = \{0,1\}$ produces a monoidal closed IP-structure; a homotopy $\alpha : IX \to Y$ amounts now to a (possibly *non*-natural) transformation of functors $\alpha : f_0 \to f_1 : X \to Y$.

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