ON THE PERIMETER DEVIATION OF A CONVEX DISC FROM A POLYGON(*)

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SOMMARIO. - *Nel piano siano C_1 e C_2 due insiemi compatti e convessi. Indichiamo con $\rho^P(C_1, C_2)$ la distanza tra loro nella metrica L_1. Si denota con P_n un qualunque poligono convesso di n vertici al massimo. Fissato un convesso C, esiste un poligono $P_n = P_n(C)$ minimante la distanza $\rho^P(C, P_n)$. In questo lavoro studiamo alcune proprietà di tale $P_n(C)$. Se l'insieme C ha il perimetro p, si prova che*

$$\rho^P(C, P_n(C)) \leq p\left(1 - \frac{2n}{\pi}\, arcsen(\frac{1}{2}\, sen\frac{\pi}{n})\right) .$$

L'uguaglianza vale se C è un cerchio.

SUMMARY. - *Let C_1 and C_2 be two compact convex subsets of the plane. We denote by $\rho^P(C_1, C_2)$ the distance between C_1 and C_2 determined by the L_1 metric. Let P_n be any convex polygon with at most n vertices. Given a convex set C, there is a polygon $P_n = P_n(C)$ minimizing the distance $\rho^P(C, P_n)$. In this paper we study some properties of $P_n(C)$. If the set C has the perimeter p, we prove that*

$$\rho^P(C, P_n(C)) \leq p\left(1 - \frac{2n}{\pi}\, \arcsin(\frac{1}{2}\, \sin\frac{\pi}{n})\right) .$$

Equality holds if C is a circle.

Dedicated to Professor Rodolfo Permutti in sincere friendship.

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1. Introduction.

Let $\mathcal{C}^2$ denote the class of all compact convex and non-empty subsets of the Euclidean plane. We shall measure the distance between two sets $K_1, K_2 \in \mathcal{C}^2$ by their *perimeter deviation*

$$\rho^P(K_1, K_2) = 2p([K_1, K_2]) - p(K_1) - p(K_2)\,, \tag{1}$$

where $[K_1, K_2]$ denotes the convex hull of $K_1 \cup K_2$, and $p(K)$ the perimeter of the set K (see [1]). The distance function ρ^P was originally introduced by D. E. McClure and R. A. Vitale [8] as the L_1-metric in the space of support functions of elements of the class $\mathcal{C}^2$. Various properties of L_m metrics were discussed by R. A. Vitale [11] and, for $m = 1$, by the author [1].

For any $n \geq 3$, we mean by a *convex n-gon* a compact convex polygon with at most n sides, possibly a line segment or a point. We shall write $\mathcal{P}_n$ for the class of all convex n-gons. A regular polygon with precisely n sides will be called a *regular n-gon.*

In the present paper we shall consider the approximation of a set $C \in \mathcal{C}^2$ by a convex n-gon P in the sense of perimeter deviation. The mutual position of C and P will not be restricted in any way. Let a set $C \in \mathcal{C}^2$ and an integer $n \geq 3$ be given. From the Bolzano-Weierstraß theorem the existence of some convex n-gon $P_n(C)$ can be deduced such that

$$\rho^P(C, P_n(C)) = \inf \rho^P(C, P_n)\,, \tag{2}$$

where the infimum is taken over all $P_n \in \mathcal{P}_n$. For simplicity we write

$$\rho^P(C, P_n(C)) = \rho_n(C)\,. \tag{3}$$

In a recent paper [2] the minimum of $\rho_n(C)$ is determined, extended over all sets C of given perimeter and area. Here we shall establish an inequality in the other direction.

THEOREM. *Let C be a compact convex set of perimeter p and let $n \geq 3$. Then*

$$\rho_n(C) \leq p\left(1 - \frac{2n}{\pi}\, arcsin\,(\frac{1}{2}\, sin\,\frac{\pi}{n})\right) \tag{4}$$

with equality if C is a circle.

By two results of R. Schneider [9], [10] together with the paper [3], the least perimeter deviation between a convex set C of given perimeter and a convex n-gon contained in C (containing C) attains its maximum if and only if C is a circle. We conjecture that also in (4) equality holds only for the circle.

For the Hausdorff distance or the symmetric difference, a precise inequality like (4) seems to be not known. The paper [4] gives a comprehensive review of approximation problems for convex bodies.

2. Some lemmas.

Let $C \in \mathcal{C}^2$ and $n \geq 3$ be given. The following lemmas give some information on the polygon $P_n(C) = P$ and the function $\rho_n(C)$ defined by (2) and (3), respectively. To prove the theorem we need only Lemma 10, 11 and 12 and Corollary 5.

LEMMA 1. *If C has non-empty interior, then C and P have interior points in common.*

Proof. Suppose that P is a segment or a point. We choose some point $U \in C \setminus P$ and consider the n-gon $P' = [P, U]$. Since $p(P') > p(P)$ and $p([C, P']) = p([C, P])$ we have $\rho^P(C, P') < \rho^P(C, P)$ which contradicts condition (2). Thus P has interior points. Therefore, we can assume without loss of generality that

$$(5) \qquad C \not\subset P, \quad P \not\subset C\,.$$

Observe that by (2)

$$(6) \qquad p([C, P]) \leq p([C, P'])$$

for any $P' \in \mathcal{P}_n$ with $p(P') = p(P)$. Now, (5) and (6) are just the assumptions (ii) and (iii) for a number of lemmas in [2], p. 102. Lemma 5 of this series is equivalent to our Lemma 1. $\diamondsuit$

LEMMA 2. *When C is not a convex n-gon, then P has precisely n vertices.*

Proof. Suppose that P has less than n vertices. If $C \not\subset P$, consider the n-gon $P' = [P, U]$, where U is some point from $C \setminus P$.

If $C \subset P$, let P' be the n-gon obtained from P by cutting off some vertex lying outside C. In both cases we have $\rho^P(C, P') < \rho^P(C, P)$, a contradiction to (2). ◇

In the Lemmas 3 to 9 we make the general assumption that C has non-empty interior. Let us denote the distinct vertices of P in the anti-clockwise sense by $A_1, \ldots, A_m$, where $3 \leq m \leq n$, and put $A_{m+1} = A_1$. We write $A \vee B$ for the line joining the points A and B.

LEMMA 3. *For any two distinct vertices of P we have*

$$[C, P] \cap A_i A_j = [C, P] \cap (A_i \vee A_j) \,. \tag{7}$$

Proof. If this were not true we could choose a point $U \in ([C, P] \cap (A_i \vee A_j)) \setminus A_i A_j$. The set $P' = [P, U]$ is a convex n-gon satisfying $p(P') > p(P)$ and $p([C, P']) = p([C, P])$. But this contradicts condition (2). ◇

COROLLARY 1. *No vertex of P lies in the interior of $[C, P]$.*

LEMMA 4. *The segment joining any two distinct vertices of P intersects C.*

Proof. Again, we can assume that (5) and (6) are satisfied. The assertion now follows from Lemma 8 of [2] and the above Lemma 3. ◇

Let A be a point outside C or on the boundary of C. Let $S(C, A)$ be the intersection of the closed half-planes containing C and bounded by lines passing through A. Obviously, $S(C, A)$ is a convex cone (possibly a half-plane) which will be called the *projection cone of C with apex A*.

LEMMA 5. *For $i = 1, \ldots, m$, we have*

$$P \subset S(C, A_i) \,.$$

Proof. For a vertex A_i outside C the assertion follows from Lemma 4. Let us now assume that a vertex, say A_2, is on the boundary of C. It suffices to show that $A_1 \in S(C, A_2)$. Suppose, on the contrary, that A_1 is in the exterior of $S(C, A_2)$, so that $C \cap A_1A_2 = \{A_2\}$. By Lemma 3, we see that $C \cap (A_1 \vee A_2) = \{A_2\}$. The cone $S(C, A_1)$ is bounded by the half-lines A_1A_2 and A_1T, where T is a point on the boundary of C. Note that $P \subset S(C, A_1)$. Let P' be the convex n-gon obtained from P by replacing the vertex A_1 by some point $A_1' \in A_1T$ sufficiently close to A_1. The required contradiction to condition (2) follows from

$$\begin{aligned}
&\rho^P(C, P') - \rho^P(C, P) \\
&= 2\,|TA_1'| + 2\,|A_1'A_2| - |A_mA_1'| - |A_1'A_2| \\
&\quad -2\,|TA_1| - 2\,|A_1A_2| + |A_mA_1| + |A_1A_2| \\
&= |A_1'A_2| - |A_1A_2| - 2\,|A_1'A_1| + |A_mA_1| - |A_mA_1'| \\
&= |A_1'A_2| - |A_1A_2| - |A_1'A_1| \\
&\quad + |A_mA_1| - |A_mA_1'| - |A_1'A_1| < 0 .
\end{aligned}$$

◇

LEMMA 6. *Let A_i be a vertex of P on the boundary of C. Then the bisector s of the outer angle of P at A_i supports C.*

Proof. If this were not true, we could displace the vertex A_i along s into the interior of C. This process leaves $[C, P]$ unchanged and enlarges the perimeter of P. Thus $\rho^P(C, P)$ would be reduced, which is impossible. ◇

Let A_1 be a vertex of P in the exterior of C, and let b be the bisector of the angle $A_mA_1A_2$. Let T and U denote boundary points of C such that $A_1 \vee T$ and $A_1 \vee U$ are support lines of C.

LEMMA 7. *If A_1 is exterior to C, then b bisects the angle TA_1U.*

Proof. Let φ be the angle between b and A_1T, and let ψ be the angle between A_1U and b. We suppose, contrary to the assertion, that

$$\varphi < \psi . \tag{8}$$

Then A_1A_2 intersects the interior of C. We displace A_1 on the normal to b through a small distance x into the point A_1' such that A_1'

und U are in the same half-plane bounded by b. For the n-gon $P' = A_1'A_2 \ldots A_m$ we have

$$p(P') > p(P) \tag{9}$$

which implies that $A_1' \notin [C, P]$. The two support lines of C passing through A_1' intersect the boundary of C at two points T' on the arc $\widehat{TU}$ and U' on the complementary arc. Let T'' be the point of intersection of $A_1 \vee T$ and $A_1' \vee T'$, and write $|A_1U| = u$, $|A_1T''| = v$. Since $|\widehat{UU'}| \geq |UU'|$ and $|\widehat{T'T}| \leq |T'T''| + |T''T|$ we obtain

$$\begin{aligned}
&p([C, P']) - p([C, P]) \\
&= |A_1'U'| - |A_1U| - |\widehat{UU'}| + |A_1'T'| + |\widehat{T'T}| - |A_1T| \\
&\leq |A_1'U| - |A_1U| + |A_1'T''| - |A_1T''| \\
&= \sqrt{x^2 + u^2 - 2xu\sin\psi} - u + \sqrt{x^2 + v^2 + 2xv\sin\varphi} - v\,.
\end{aligned}$$

Hence

$$\begin{aligned}
&\frac{1}{x}\left(p([C, P']) - p([C, P])\right) \\
&\leq \frac{x - 2u\sin\psi}{\sqrt{x^2 + u^2 - 2xu\sin\psi} + u} + \frac{x + 2v\sin\varphi}{\sqrt{x^2 + v^2 + 2xv\sin\varphi} + v}\,.
\end{aligned} \tag{10}$$

Since v is greater than a positive constant, we see that the right side of (10) tends to $\sin\varphi - \sin\psi$ as $x \to 0$. In view of (8) we conclude that $p([C, P']) < p([C, P])$ for sufficiently small x. The combination with (9) yields $\rho^P(C, P') < \rho^P(C, P)$, a contradiction to (2). ◊

Retaining the notation of Lemma 7 we prove

LEMMA 8. *If A_1 is exterior to C and $\angle TA_1U = 2\alpha$, $\angle A_mA_1A_2 = 2\beta$, then*

$$\frac{\cos\alpha}{\cos\beta} = \frac{1}{2}\,. \tag{11}$$

Proof. Let us suppose that

$$\frac{\cos\alpha}{\cos\beta} < \frac{1}{2}\,. \tag{12}$$

Lemma 7 says that the angles $A_mA_1A_2$ and TA_1U have a common bisector b. By (12), the segments A_1A_m and A_1A_2 intersect the interior of C. We displace A_1 along b through a small distance x into the point A_1' in the exterior of P. It will be shown that the n-gon $P' = A_1'A_2 \dots A_m$ satisfies

$$\rho^P(C,P') < \rho^P(C,P) \tag{13}$$

which is impossible.

Let T' and U' denote two boundary points of C such that $A_1' \vee T'$ and $A_1' \vee U'$ are support lines. Then we obtain

$$\begin{aligned}&\rho^P(C,P') - \rho^P(C,P)\\ &= 2\mid A_1'T' \mid -2\mid A_1T \mid -2\,|\widehat{TT'}| - \mid A_1'A_m \mid + \mid A_1A_m \mid\\ &\quad +2\mid A_1'U' \mid -2\mid A_1U \mid -2\,|\widehat{UU'}| - \mid A_1'A_2 \mid + \mid A_1A_2 \mid .\end{aligned}$$

To prove (13) it will suffice to show that

$$2\mid A_1'T' \mid -2\mid A_1T \mid -2\,|\widehat{TT'}| - \mid A_1'A_m \mid + \mid A_1A_m \mid < 0 . \tag{14}$$

Denoting the intersection of $A_1 \vee T$ and $A_1' \vee T'$ by T'' and writing $\mid A_1T'' \mid = v$ we find

$$\begin{aligned}\mid A_1'T' \mid - \mid A_1T \mid - |\widehat{TT'}| &\geq \mid A_1'T' \mid - \mid A_1T \mid - \mid TT'' \mid - \mid T''T' \mid\\ &= \mid A_1'T'' \mid - \mid A_1T'' \mid\\ &= \sqrt{x^2+v^2+2xv\cos\alpha} - v .\end{aligned}$$

Hence

$$\frac{1}{x}\left(\mid A_1'T' \mid - \mid A_1T \mid - |\widehat{TT'}|\right) \geq \frac{x+2v\cos\alpha}{\sqrt{x^2+v^2+2xv\cos\alpha}+v} . \tag{15}$$

Putting $\mid A_1T \mid = t$ we obtain

$$\begin{aligned}\mid A_1'T' \mid - \mid A_1T \mid - |\widehat{TT'}| &\leq \mid A_1'T' \mid - \mid A_1T \mid - \mid TT' \mid\\ &\leq \mid A_1'T \mid - \mid A_1T \mid\\ &= \sqrt{x^2+t^2+2xt\cos\alpha} - t ,\end{aligned}$$

so that

$$(16)\quad \frac{1}{x}\left(|\, A_1'T' \,| - |\, A_1T \,| - |\widehat{TT'}|\right) \le \frac{x + 2t\cos\alpha}{\sqrt{x^2 + t^2 + 2xt\cos\alpha} + t}\,.$$

Combining (15) and (16) and observing that $v \ge t$, where t does not depend on x, we conclude that

$$(17)\qquad \lim \frac{1}{x}\left(|\, A_1'T' \,| - |\, A_1T \,| - |\widehat{TT'}|\right) = \cos\alpha$$

as x tends to 0.

Using the notation $|\, A_1A_m \,| = c$ we get

$$\frac{1}{x}(-\,|\, A_1'A_m \,| + |\, A_1A_m \,|) = -\frac{x + 2c\cos\beta}{\sqrt{x^2 + c^2 + 2xc\cos\beta} + c}\,,$$

whence

$$(18)\qquad \lim \frac{1}{x}(-\,|\, A_1'A_m \,| + |\, A_1A_m \,|) = -\cos\beta\,.$$

From (17), (18) and (12) we deduce that (14) and (13) are satisfied for sufficiently small x.

Similarly, by displacing A_1 into the interior of P it can be shown that also the supposition

$$\frac{\cos\alpha}{\cos\beta} > \frac{1}{2}$$

leads to a contradiction to (2). We omit the details of the proof. This completes the proof of Lemma 8. ◇

COROLLARY 2. *If the vertex A_1 is in the exterior of C, then A_mA_1 and A_1A_2 intersect the interior of C.*

COROLLARY 3. *If C is not a convex n-gon, then P does not contain C.*

LEMMA 9. *If the boundary of C contains the vertex A_1 of P, then A_1 lies on more than one support line of C.*

Proof. Suppose on the contrary that only one support line s is passing through A_1. By Lemma 6, the normal to s bisects the angle $A_m A_1 A_2$. Thus we can proceed as in the proof of Lemma 8 by showing that inequality (14) continues to hold if $T = A_1$. Using the notation $|A_1T'| = w$, $\angle T'A_1A_1' = \gamma$, $\angle A_1A_1'T' = \delta$ we have

$$\begin{aligned} |A_1'T'| - |\widehat{A_1T'}| &\le |A_1'T'| - |A_1T'| \\ &= \sqrt{w^2 + x^2 - 2wx\cos\gamma} - w\,. \end{aligned}$$

Hence

$$\begin{aligned} \frac{1}{x}\left(|A_1'T'| - |\widehat{A_1T'}|\right) &\le \frac{x - 2w\cos\gamma}{\sqrt{w^2 + x^2 - 2wx\cos\gamma} + w} \\ &< 2\frac{x - w\cos\gamma}{w} = 2\cot\delta\,\sin\gamma\,. \end{aligned} \tag{19}$$

Since s is the unique support line of C at A_1, it is easy to see that $\lim \delta = \pi/2$ as $x \to 0$. From (19) it follows that

$$\lim \frac{1}{x}\left(|A_1'T'| - |\widehat{A_1T'}|\right) = 0 \tag{20}$$

as $x \to 0$. The desired inequality (14) is a consequence of (18) and (20). ◇

Lemma 2, Corollary 2 and Lemma 9 imply

COROLLARY 4. *If all boundary points of C are regular, then P has precisely n vertices, all of which are exterior to C and all sides of P intersect the interior of C.*

A similar remark applies to the approximation of a strictly convex set in the symmetric difference metric; see [5], p. 220.

Let C be a circle with centre O. From Corollary 4 and Lemma 7 we gather that P is circumscribed about a circle with centre O, and from Lemma 8 we infer that P is inscribed in another circle with centre O. Thus P is a regular n-gon. Exploiting relation (11) we finally obtain

COROLLARY 5. *If K_0 is a circle of perimeter p, then $P_n(K_0)$ is a regular n-gon concentric with K_0 and satisfying (11). Furthermore,*

$$\rho_n(K_0) = p\left(1 - \frac{2n}{\pi}\arcsin(\frac{1}{2}\sin\frac{\pi}{n})\right). \tag{21}$$

Corollary 5 is a particular case of Theorem 2 in [2], p. 117.

In the proof of our theorem we shall make use of the Blaschke symmetrization of a convex set. Let H be a hyperplane in d-dimensional Euclidean space, and let K be a compact convex set. If $\overline{K}$ denotes the reflection of K in H and $+$ indicates Minkowski addition, we say that

$$K^* = \frac{1}{2}(K + \overline{K})$$

is obtained from K by *Blaschke symmetrization about the hyperplane* H. Observe that K^* is a compact convex set which is symmetrical about H. Various properties of this process are discussed in Hadwiger's book [7], p. 260. We point out that $W(K^*) = W(K)$ and W denotes the mean width. In particular, when $d = 2$, then

$$p(K^*) = p(K), \tag{22}$$

which means that Blaschke symmetrization leaves the perimeter of a plane convex set unchanged.

The next lemma states that Blaschke symmetrization does not reduce the least perimeter deviation between a given convex set and some convex n-gon.

LEMMA 10. *Let C be a compact convex set and let $n \geq 3$ be given. If C^* is obtained from C by Blaschke symmetrization, then*

$$\rho_n(C^*) \geq \rho_n(C). \tag{23}$$

Proof. We may assume that C has non-empty interior; otherwise $\rho_n(C) = 0$. Also the Blaschke symmetral of C

$$C^* = \frac{1}{2}(C + \overline{C}) \tag{24}$$

has inner points. There exists a convex n-gon P such that

$$\rho^P(C^*, P) = \rho_n(C^*).$$

We denote both the vertices of P and their vectors by $z_1, \dots, z_m$, where $m \leq n$.

We proceed to show that there are two points z_1' and z_1'' such that

$$2z_1 = z_1' + z_1'', \tag{25}$$

$$2[C^*, z_1] = [C, z_1'] + [\overline{C}, z_1'']. \tag{26}$$

When $z_1 \in C^*$, we can find points $z_1' \in C$ and $z_1'' \in \overline{C}$ satisfying (25) and (26). Thus we may suppose that $z_1 \notin C^*$. Let s and t be the support lines of C^* containing z_1, and let v and w be outer normal vectors of s and t, respectively. Let s', t' and s'', t'' denote the support lines with the same normal vectors and associated with C and $\overline{C}$, respectively. The points z_1' and z_1'' defined by

$$z_1' = s' \cap t', \quad z_1'' = s'' \cap t'' \tag{27}$$

are in the exterior or on the boundary of C and $\overline{C}$, respectively. For any two points $x' \in C \cap s'$ and $x'' \in \overline{C} \cap s''$ we have

$$\langle z_1' - x',\ v\rangle = \langle z_1'' - x'',\ v\rangle = 0,$$

so that

$$\langle \frac{1}{2}(z_1' + z_1'') - \frac{1}{2}(x' + x''),\ v\rangle = 0. \tag{28}$$

Because $\frac{1}{2}(x' + x'') \in C^* \cap s$, we can note

$$\langle z_1 - \frac{1}{2}(x' + x''),\ v\rangle = 0. \tag{29}$$

Combining (28) and (29) we obtain

$$\langle z_1 - \frac{1}{2}(z_1' + z_1''),\ v\rangle = 0$$

and similarly

$$\langle z_1 - \frac{1}{2}(z_1' + z_1''),\ w\rangle = 0.$$

Since v and w are linearly independent vectors, relation (25) is proved.

For a unit vector u let $h(X,u)$ denote the support function of the set X in direction u. We shall prove (26) by showing that

$$2h([C^*,z_1],u) = h([C,z_1'],u) + h\left([\overline{C},z_1''],u\right) . \tag{30}$$

(i) If u is a normal vector of $[C^*,z_1]$ at z_1, then u is also a normal vector of $[C,z_1']$ at z_1' and of $[\overline{C},z_1'']$ at z_1''. In this case we have

$$h([C^*,z_1],u) = h(\{z_1\},u)$$

and similarly for the sets $[C,z_1']$ and $[\overline{C},z_1'']$. Thus (30) follows from (25).

(ii) For any other u

$$h([C^*,z_1],u) = h(C^*,u)$$

holds and similarly for $[C,z_1']$ and $[\overline{C},z_1'']$. Now (30) is a consequence of (24).

Applying this construction to $[C^*,z_1]$, $[C,z_1']$, $[\overline{C},z_1'']$ and z_2 in place of C^*, C, $\overline{C}$ and z_1 and starting from (26) in place of (24) we can find two points z_2', z_2'' such that

$$2z_2 = z_2' + z_2'' ,$$

$$2[C^*,[z_1,z_2]] = [C,[z_1',z_2']] + \left[\overline{C},[z_1'',z_2'']\right] .$$

Repeated application of this procedure produces two sets of m points, $\{z_1',\dots,z_m'\}$ and $\{z_1'',\dots,z_m''\}$, where

$$2z_i = z_i' + z_i'' \quad (i=1,\dots,m), \tag{31}$$

$$2[C^*,P] = [C,P'] + [\overline{C},P''] \tag{32}$$

and $P' = \operatorname{conv}\{z_1',\dots,z_m'\}$, $P'' = \operatorname{conv}\{z_1'',\dots,z_m''\}$. By (31), the n-gons P' and P'' satisfy

$$P \subset \frac{1}{2}(P'+P'') . \tag{33}$$

From (24), (32) and (33) we deduce

$$\begin{aligned}\rho_n(C^*) &= \rho^P(C^*,P) = 2p([C^*,P]) - p(C^*) - p(P)\\ &\geq p([C,P']) + p\left([\overline{C},P'']\right)\\ &\quad - \frac{1}{2}p(C) - \frac{1}{2}p(\overline{C}) - \frac{1}{2}p(P') - \frac{1}{2}p(P'')\\ &= \frac{1}{2}\rho^P(C,P') + \frac{1}{2}\rho^P(\overline{C},P'') \geq \rho_n(C),\end{aligned}$$

as required. ◇

LEMMA 11. *Let K be a compact convex and non-empty subset of $E^d (d \geq 2)$, and let O be a fixed point. Let S denote the class of convex sets, each obtained from K by a finite number of Blaschke symmetrizations about hyperplanes passing through O. The class S contains an infinite subsequence that converges in the Hausdorff metric to the ball K_0 with centre O and $W(K_0) = W(K)$.*

We omit the proof since it is very similar to Hadwiger's proof of the "1. Kugelungstheorem"; see [6], p. 26 or [7], p. 170.

LEMMA 12. *For given $n \geq 3$, the function $\rho_n(C)$ is continuous in the Hausdorff metric.*

Proof. Let C_m $(m = 1, 2, \ldots)$ and C_0 be sets from $\mathcal{C}^2$ such that $C_m \to C_0$ as $m \to \infty$. There exist convex n-gons $P_n(C_0) = Q_0$ and $P_n(C_m) = Q_m$, for $m = 1, 2, \ldots$, such that

$$\rho^P(C_0, Q_0) = \rho_n(C_0), \quad \rho^P(C_m, Q_m) = \rho_n(C_m).$$

From the inequalities

$$\begin{aligned}\rho^P(C_0,Q_0) &\leq \rho^P(C_0,Q_m) \leq \rho^P(C_0,C_m) + \rho^P(C_m,Q_m)\\ &\leq \rho^P(C_0,C_m) + \rho^P(C_m,Q_0)\\ &\leq 2\rho^P(C_0,C_m) + \rho^P(C_0,Q_0)\end{aligned}$$

and $\lim \rho^P(C_0, C_m) = 0$ we conclude that $\lim \rho^P(C_m, Q_m) = \rho^P(C_0, Q_0)$. Hence $\lim \rho_n(C_m) = \rho_n(C_0)$, as required. ◇

3. Proof of the theorem. Let C be a compact convex set with perimeter p, and let O be a fixed point. By Lemma 11, there is an infinite sequence of convex sets (C_m), $m = 1, 2, \ldots$, each obtained from C by repeated Blaschke symmetrization about lines through O, such that

$$C_m \to K_0 \quad (m \to \infty), \tag{34}$$

where K_0 is the circle with centre O and perimeter p. Lemma 10 implies that

$$\rho_n(C) \leq \rho_n(C_m) \tag{35}$$

for $m = 1, 2 \ldots$, and from (34), (35) and Lemma 12 it follows that

$$\rho_n(C) \leq \lim \rho_n(C_m) = \rho_n(K_0).$$

The reference to Corollary 5 completes the proof. ◊

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