IMPROPER AFFINE SPHERES AND EUCLIDEAN MINIMAL SURFACES (*)

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SOMMARIO. - *In questo lavoro si da un risultato sulle ipersuperficie minimali. Si comparano le ipersuperfici minimali con le ipersuperfici affini.*

SUMMARY. - *In this paper one obtains a result on minimal surfaces. One compares minimal surfaces with affine surfaces.*

A wide class of affine surfaces is formed by the affine maximal surfaces. This class contains the improper affine spheres as a subclass. We consider the elliptic case, i.e. we assume that the Berwald-Blaschke metric is everywhere positive definite.

In this paper we prove

THEOREM A. *Suppose two improper affine spheres are given as differentiable graphs over a plane region. Then they induce in a quite natural way an Euclidean minimal surface.*

Before proving Theorem A let us consider the following.

Suppose Ω is a region in the plane and that a surface $\Sigma : \Omega \to E_3$ is given by a differentiable function z as a graph over Ω

$$\Sigma(x,y) = \begin{pmatrix} x \\ y \\ z(x,y) \end{pmatrix} \tag{1}$$

Denote by d the determinant of the Hessian of z

$$d = z_{xx}\, z_{yy} - z_{xy}^2\ . \tag{2}$$

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$\Sigma : \Omega \to E_3$ is an affine maximal surface if $z : \Omega \to R$ satisfies the Euler-Lagrange equation

$$d\{z_{xx}d_{yy} + z_{yy}d_{yy} - 2z_{xy}d_{xy}\} = \\ = \frac{7}{4}\{z_{xx}d_y^2 + z_{yy}d_x^2 - 2z_{xy}d_{xy}\} . \tag{3}$$

$\Sigma : \Omega \to E_3$ is an improper affine sphere with affine normal parallel to the third axis if $z : \Omega \to R$ satisfies the Monge-Ampère equation

$$d = z_{xx}z_{yy} - z_{xy}^2 . = 1 . \tag{4}$$

$\Sigma : \Omega \to E_3$ is an Euclidean minimal surface if $z : \Omega \to R$ satisfies

$$(1 + z_x^2)z_{yy} - 2z_x z_y z_{xy} + (1 + z_y^2)z_{xx} = 0 . \tag{5}$$

Every improper affine sphere is an affine maximal surface, i.e. every solution of (4) solves (3).

THEOREM B. *Suppose Ω is a region in the plane and $z : \Omega \to R$ is a solution of the Monge-Ampère equation* (4). *Then $z : \Omega \to R$ induces a holomorphic function $f : \Gamma \to R$, thus an improper affine sphere is in a natural way related to a holomorphic function.*

The basic idea in the following proof is from E. Heinz and was used by Jörgens in [JÖ].

Proof. $z : \Omega \to R$ induces the differential

$$dz = M(x, y)dx + B(x, y)dy , \tag{6}$$

where $M(x, y), B(x, y)$ denote the first partial derivatives

$$M(x, y) = z_x,\ B(x, y) = z_y . \tag{7}$$

Define new coordinates (a, b) by

$$a = x, b = B(x, y) . \tag{8}$$

Because $z : \Omega \to R$ solves (4) $B_y = z_{yy} \neq 0$. Thus there is a function τ such that

$$y = \tau(x, B(x, y)) . \tag{9}$$

Define the function μ by

$$\mu(a,b) = M(a,\tau(a,b)) \ . \tag{10}$$

Then the function f with

$$f(a+ib) = \mu(a,b) + i\tau(a,b) \tag{11}$$

is holomorphic.

Now suppose one is given two improper affine spheres $\Sigma : \Omega \to E_3$, $\Pi : \Omega \to E_3$ defined by differentiable functions $z : \Omega \to E_3$, $w : \Omega \to E_3$ are graphs over a region Ω

$$\Sigma(x,y) = \begin{pmatrix} x \\ y \\ z(x,y) \end{pmatrix} \ . \tag{12}$$

$$\Pi(x,y) = \begin{pmatrix} x \\ y \\ w(x,y) \end{pmatrix} \tag{13}$$

Then by the above method two holomorphic functions f, g are induced. Now using the Weierstrass representation for Euclidean minimal surfaces f, g induce holomorphic functions ϕ_1, ϕ_2, ϕ_3 with

$$\phi_1 = \frac{1}{2}f(1-g^2),\ \phi_2 = \frac{1}{2}if(1+g^2),\ \phi_3 = fg \ .$$

The Euclidean minimal surface $x = (x_1 x_2, x_3)$ is given by

$$x_k = Re\{\int \phi_k d\xi\},\ \xi = a+ib,\ k = 1,2,3 \ .$$

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